Exercise 0.1 (Wave equation in $\mathbf{R}^2$). Define the open sector $U \subset \mathbf{R}^2$ and the differential operator $\Box$ on $\mathbf{R}^2$ by

$$U = \{ (x_1, x_2) \in \mathbf{R}_+ \times \mathbf{R} \mid |x_2| < x_1 \} \qquad \text{and} \qquad \Box = D_1^2 - D_2^2.$$

Furthermore, consider an arbitrary C^∞ function $\phi : \mathbf{R}^2 \to \mathbf{R}$ with compact support. In two different ways we will prove the following identity:

$$(\star) \qquad \int_U \Box\,\phi(x)\,dx = 2\phi(0).$$

For the first proof, define $\Psi \in \mathrm{Mat}(2, \mathbf{R})$ by $\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

(i) Show that $\Psi \in \mathbf{SO}(2, \mathbf{R})$ and verify that Ψ is the rotation in $\mathbf{R}^2$ by the angle $-\frac{\pi}{4}$ about the origin. Deduce that $\Psi : \mathbf{R}^2 \to \mathbf{R}^2$ is a C^∞ diffeomorphism with the property $U = \Psi(V)$ where $V = \mathbf{R}_+^2$. Prove that $D\Psi(y) = \Psi$ and compute $\det D\Psi(y)$, for all $y \in \mathbf{R}^2$.

(ii) Write $\phi \circ \Psi = \widetilde{\phi} : \mathbf{R}^2 \to \mathbf{R}$ and use the chain rule to prove the following identity of mappings $\mathbf{R}^2 \to \mathrm{Lin}(\mathbf{R}, \mathbf{R}^2)$:

$$\begin{pmatrix} \widetilde{D_1\phi} \\ \widetilde{D_2\phi} \end{pmatrix} = \begin{pmatrix} D_1\phi \\ D_2\phi \end{pmatrix} \circ \Psi = \Psi \begin{pmatrix} D_1\widetilde{\phi} \\ D_2\widetilde{\phi} \end{pmatrix}; \qquad \text{conclude}$$

$$\widetilde{D_i\phi} = \frac{1}{\sqrt{2}}\big((-1)^{i-1}D_1 + D_2\big)\widetilde{\phi} \qquad (1 \le i \le 2).$$

Next apply the latter identity with ϕ replaced by $D_i\phi$, with $1 \le i \le 2$ respectively, and deduce

$$(\Box\,\phi) \circ \Psi = 2D_1D_2\widetilde{\phi} : \mathbf{R}^2 \to \mathbf{R}.$$

Which theorem is needed in the proof of the last identity?

(iii) On the basis of parts (i) and (ii) as well as the Fundamental Theorem of Integral Calculus on $\mathbf{R}$ show that the identity in $(\star)$ applies.

(iv) Use the Differentiation Theorem to prove that a solution u of the *inhomogeneous wave equation* $\Box\,u = \phi$ in $\mathbf{R}^2$ is given by

$$u : \mathbf{R}^2 \to \mathbf{R} \qquad \text{with} \qquad u(x) = \int_U \phi(x - \xi)\,d\xi.$$

In the subsequent parts (v) through (vii) we give a second, independent, proof of $(\star)$ by means of Green's Integral Theorem. To this end, consider the vector veld

$$f = S\,\mathrm{grad}\,\phi = \begin{pmatrix} D_2\phi \\ D_1\phi \end{pmatrix} : \mathbf{R}^2 \to \mathbf{R}^2, \qquad \text{with} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Mat}(2, \mathbf{R}).$$

(v) Prove the identity $\mathrm{curl}\, f = \Box\,\phi$ of functions on $\mathbf{R}^2$.

(vi) Show that a positive parametrization $y : \mathbf{R} \to \partial U$ is given by

$$y(s) = \begin{pmatrix} \operatorname{sgn}(s)\, s \\ -s \end{pmatrix} \qquad (s \in \mathbf{R}),$$

where sgn denotes the sign function. Next, verify

$$Dy(s) = \begin{pmatrix} \operatorname{sgn}(s) \\ -1 \end{pmatrix} \qquad \text{en} \qquad S\,Dy(s) = -\operatorname{sgn}(s)\,Dy(s) \qquad (s \in \mathbf{R} \setminus \{0\}),$$

and conclude on account of the chain rule

$$-\operatorname{sgn}(s)\frac{d(\phi \circ y)}{ds}(s) = \langle f \circ y,\, Dy\rangle(s) \qquad (s \in \mathbf{R} \setminus \{0\}).$$

(vii) Use the compactness of the support of ϕ to show that the identity from Green's Integral Theorem applies in case of the unbounded open set U and the vector field f, and conclude on the basis of this identity and parts (v) and (vi) that ($\star$) follows.

Solution of Exercise 0.1

(i) We have

$$\Psi^t\Psi = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = I \qquad \text{and} \qquad \det \Psi = \frac{1}{2}(1+1) = 1.$$

Accordingly $\Psi \in \mathbf{SO}(2, \mathbf{R})$ and therefore it is of the form $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, that is, $\cos\alpha = -\sin\alpha = \frac{1}{2}\sqrt{2}$, hence $\alpha = -\frac{\pi}{4}$. In particular, $\Psi \in \operatorname{Aut}(\mathbf{R}^2)$, which implies that $\Psi : \mathbf{R}^2 \to \mathbf{R}^2$ is a C^∞ diffeomorphism. $D\Psi(y) = \Psi$ follows from $\Psi \in \operatorname{End}(\mathbf{R}^2)$, and so $\det D\Psi(y) = 1$, for all $y \in \mathbf{R}^2$.

(ii) The chain rule, transposition and the orthogonality of Ψ, successively, imply

$$D(\phi \circ \Psi) = (D\phi) \circ \Psi\, D\Psi, \quad \Longrightarrow \quad \operatorname{grad}\widetilde{\phi} = (D\Psi)^t(\operatorname{grad}\phi) \circ \Psi,$$

$$\Longrightarrow \qquad \widetilde{\operatorname{grad}\phi} = (\operatorname{grad}\phi) \circ \Psi = \big((D\Psi)^t\big)^{-1}\operatorname{grad}\widetilde{\phi} = \Psi \operatorname{grad}\widetilde{\phi}.$$

As a consequence we obtain, for $1 \le i \le 2$,

$$\widetilde{D_i^2\phi} = \frac{1}{\sqrt{2}}((-1)^{i-1}D_1 + D_2)\widetilde{D_i\phi} = \frac{1}{2}\big((-1)^{i-1}D_1 + D_2\big)^2\widetilde{\phi},$$

$$\Longrightarrow \quad (\Box\,\phi) \circ \Psi = \frac{1}{2}\big((D_1 + D_2)^2 - (-D_1 + D_2)^2\big)\widetilde{\phi} = 2D_1D_2\widetilde{\phi},$$

where we used Theorem 2.7.2 on the equality of mixed partial derivatives.

(iii) In fact, the Change of Variables Theorem 6.6.1 and Theorem 6.4.5 imply

$$\int_U \Box\,\phi(x)\,dx = \int_{\Psi(V)} \Box\,\phi(x)\,dx = \int_V (\Box\,\phi) \circ \Psi(y)|\det D\Psi(y)|\,dy$$

$$= 2\int_{\mathbf{R}_+}\int_{\mathbf{R}_+} D_1(D_2\widetilde{\phi})(y_1, y_2)\,dy_1\,dy_2 = -2\int_{\mathbf{R}_+} D_2\widetilde{\phi}(0, y_2)\,dy_2$$

$$= 2\widetilde{\phi}(0) = 2\phi\big((\Psi(0)\big) = 2\phi(0).$$

(iv) On the strength of the Differentiation Theorem 2.10.4 we have, for all $x \in \mathbf{R}^2$,

$$\Box\, u(x) = \int_U \Box_x\, \phi(x-\xi)\, d\xi = \int_U (\Box\, \phi)(x-\xi)\, d\xi = \phi(x-0) = \phi(x).$$

(v) In the notation of Formula (8.20) and Lemma 8.3.10.(iii) we have

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = J\,\overline{}.$$

Since $J \in \mathbf{SO}(2, \mathbf{R})$

$$\operatorname{curl} f = \operatorname{div}(J^t f) = \operatorname{div}(J^t J\, \overline{\operatorname{grad} \phi}) = \operatorname{div}(\overline{\operatorname{grad} \phi}) = (D_1^2 - D_2^2)\phi = \Box\, \phi.$$

(vi) Differentiation gives the formula for $Dy(s)$ upon noticing that sgn is a locally constant function. $\nu\big(y(s)\big) = -\begin{pmatrix} 1 \\ \operatorname{sgn}(s) \end{pmatrix}$, and accordingly

$$\det(\nu \circ y \mid Dy)(s) = \begin{vmatrix} -1 & \operatorname{sgn}(s) \\ -\operatorname{sgn}(s) & -1 \end{vmatrix} = 2 > 0.$$

Therefore $y : \mathbf{R} \to \partial U$ is a positive parametrization. We have

$$S Dy(s) = \begin{pmatrix} -1 \\ \operatorname{sgn}(s) \end{pmatrix} = -\operatorname{sgn}(s) \begin{pmatrix} \operatorname{sgn}(s) \\ -1 \end{pmatrix} = -\operatorname{sgn}(s) Dy(s).$$

We now obtain by means of the chain rule and $S^t = S$, for $s \in \mathbf{R} \setminus \{0\}$,

$$\begin{aligned} \frac{d(\phi \circ y)}{ds}(s) &= D\phi\big(y(s)\big)Dy(s) = -\operatorname{sgn}(s)\langle \operatorname{grad}\, \phi\big(y(s)\big),\ S Dy(s)\, \rangle \\ &= -\operatorname{sgn}(s)\langle\, (S \operatorname{grad} \phi) \circ y(s),\ Dy(s)\, \rangle = -\operatorname{sgn}(s)\langle\, f \circ y,\ Dy\, \rangle(s). \end{aligned}$$

(vii) On the basis of Green's Integral Theorem 8.3.5 and the compact support of ϕ we find

$$\begin{aligned} \int_U \Box\, \phi(x)\, dx &= \int_U \operatorname{curl} f(x)\, dx = \int_{\partial U} \langle\, f(y), d_1 y\, \rangle = \int_{\mathbf{R}} \langle\, f \circ y, Dy\, \rangle(s)\, ds \\ &= -\operatorname{sgn}(s) \int_{\mathbf{R}} \frac{d(\phi \circ y)}{ds}(s)\, ds = \int_{-\infty}^0 \frac{d(\phi \circ y)}{ds}(s)\, ds \\ &\quad - \int_0^\infty \frac{d(\phi \circ y)}{ds}(s)\, ds = \phi\big(y(0)\big) - \big(-\phi\big(y(0)\big)\big) = 2\phi(0). \end{aligned}$$