Review of "Distributions – Theory and Applications," by J.J. Duistermaat and J.A.C. Kolk

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From early times, people doing PDE have been motivated to consider derivatives of functions that possess little or no smoothness. For example, one wants to treat functions of the form

$$u(t,x) = f(t+x) + g(t-x) \tag{1}$$

as solutions to the wave equation $u_{tt} - u_{xx} = 0$, even when f and g are quite rough (perhaps having jump discontinuities). As another example, shock wave solutions to

$$u_t + f(u)_x = 0 \tag{2}$$

are considered, when u is piecewise C^1 with a jump across a curve $x = \gamma(t)$; one imposes across γ the Rankine-Hugoniot condition $s[u] = [f]$, where $s = dx/dt$, $[u]$ is the jump of u, and $[f]$ the jump of $f(u)$ across γ. In such cases, these functions are interpreted as weak solutions, satisfying, respectively,

$$\begin{aligned} \iint (\varphi_{tt} - \varphi_{xx})u \, dx \, dt = 0, \\ \iint (-\varphi_t u - \varphi_x f(u)) \, dx \, dt = 0, \end{aligned} \tag{3}$$

for all smooth, compactly supported "test functions" φ.

Making use of functional analysis, which had developed nicely over the first half of the 20th century, L. Schwartz [6] presented a beautiful and systematic theory of distributions, which among other things incorporates such notions of weak solutions.

A distribution on an open set $\Omega \subset \mathbb{R}^n$ is defined as a continuous linear functional

$$u : C_0^\infty(\Omega) \longrightarrow \mathbb{C}, \tag{4}$$

where $C_0^\infty(\Omega)$ consists of smooth functions with compact support in Ω. Here, continuity means that $u(f_j) = \langle f_j, u\rangle \to \langle f, u\rangle$ if $f_j \to f$ in $C_0^\infty(\Omega)$, i.e., if there exists a compact $K \subset \Omega$ such that all f_j are supported in K and $f_j \to f$, with all derivatives, uniformly. For example, if u is continuous, or more generally merely locally integrable on Ω, it defines an element of $\mathcal{D}'(\Omega)$, via

$$\langle f, u\rangle = \int_\Omega f(x)u(x)\, dx, \quad f \in C_0^\infty(\Omega). \tag{5}$$

Now one can apply $\partial_j = \partial/\partial x_j$ to any element of $\mathcal{D}'(\Omega)$, as follows:

$$\langle f, \partial_j u\rangle = -\langle \partial_j f, u\rangle. \tag{6}$$

In case $u \in C^1(\Omega)$, this becomes

$$\langle f, \partial_j u\rangle = -\int \frac{\partial f}{\partial x_j} u(x)\, dx, \tag{7}$$

which, by integration by parts, agrees with the classical definition of $\partial_j u = \partial u/\partial x_j$. To give an example of the application of (6), let $H(x) = 1$ for $x > 0$, 0 for $x < 0$. Then

$$\langle f, \partial_x H\rangle = -\int_0^\infty f'(x)\, dx = f(0), \tag{8}$$

so

$$\partial_x H = \delta, \quad \text{where} \quad \langle f, \delta\rangle = f(0) \tag{9}$$

defines δ as the "Dirac delta function." To take another example, again for $x \in \mathbb{R}$,

$$\begin{aligned} \langle f, \partial_x \log|x|\rangle &= -\int f'(x)\, \log|x|\, dx \\ &= \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \Bigl(\frac{f(x)}{x} - \frac{f(-x)}{x}\Bigr)\, dx \\ &= \Bigl\langle f, \mathrm{PV}\,\frac{1}{x}\Bigr\rangle, \end{aligned} \tag{10}$$

where the last identity defines the distribution $\mathrm{PV}(1/x)$. The key here is to excise $[-\varepsilon, \varepsilon]$ from the first integral in (10), apply ordinary integration by parts ($\partial_x \log|x| = 1/x$ on $\mathbb{R} \setminus 0$) and pass to the limit. A similar attack works on

$$G_n(x) = |x|^{2-n} \ \ (n \geq 3), \quad G_2(x) = \log|x| \ (n = 2), \tag{11}$$

on $\mathbb{R}^n$. These functions are harmonic on $\mathbb{R}^n \setminus 0$. If we write

$$\langle f, \Delta G_n \rangle = \int \Delta f(x)\, G_n(x)\, dx, \tag{12}$$

excise $\{x \in \mathbb{R}^n : |x| \le \varepsilon\}$, apply Green's theorem and take $\varepsilon \to 0$, we get

$$\Delta G_n = C_n \delta \ \text{ on } \ \mathbb{R}^n, \quad C_n = -(n-2)\text{Area}(S^{n-1}) \ (n \neq 2), \quad C_2 = 2\pi. \tag{13}$$

The Fourier transform, given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\xi}\, dx \tag{14}$$

for $f \in L^1(\mathbb{R}^n)$, provides a powerful tool for analyzing PDE, since it intertwines $\partial/\partial x_j$ and multiplication by $-i\xi_j$. The Fourier inversion formula is that $\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = I$, where $\mathcal{F}^*$ is defined as in (14), with $e^{-ix\cdot\xi}$ replaced by $e^{ix\cdot\xi}$. To prove this, it is convenient to have a function space invariant under the action of $\mathcal{F}$, not $L^1(\mathbb{R}^n)$, since $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$. Schwartz produced a beautiful theory of Fourier analysis using the space

$$\begin{aligned} &\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : p_N(f) < \infty, \ \forall N\}, \\ &p_N(f) = \sup_x\, (1 + |x|^2)^N \sum_{|\alpha| \le N} |D^\alpha f(x)|. \end{aligned} \tag{15}$$

It is readily verified that $\mathcal{F}, \mathcal{F}^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. One way to get the inversion formula is to sneak up on it:

$$\begin{aligned} \mathcal{F}^*\mathcal{F}f(x) &= \lim_{t \searrow 0}\, (2\pi)^{-n/2} \int e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= \lim_{t \searrow 0} \int p(t, x-y) f(y)\, dy. \end{aligned} \tag{16}$$

To get this last line, replace $\hat{f}(\xi)$ by its integral definition and switch order of integration. One has a classical Gaussian integral to calculate, and the result is

$$p(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}. \tag{17}$$

It is then an easy task to show that $\int p(t, x-y) f(y)\, dy \to f(x)$ as $t \searrow 0$, for each $f \in \mathcal{S}(\mathbb{R}^n)$.

The next step is to extend the Fourier transform and Fourier inversion formula from $\mathcal{S}(\mathbb{R}^n)$ to its dual space $\mathcal{S}'(\mathbb{R}^n)$, the Schwartz space of tempered distributions, consisting of continuous linear maps $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, where to say u is continuous is to say that there exist N and C such that

$$|\langle f, u\rangle| \le C p_N(f), \quad \forall\, f \in \mathcal{S}(\mathbb{R}^n), \tag{18}$$

with p_N as in (15). In view of (14), it is natural to define $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle f, \mathcal{F}u\rangle = \langle \mathcal{F}f, u\rangle, \quad f \in \mathcal{S}(\mathbb{R}^n), \quad u \in \mathcal{S}'(\mathbb{R}^n), \tag{19}$$

and similarly for $\mathcal{F}^*$. Fourier inversion on $\mathcal{S}'(\mathbb{R}^n)$ simply amounts to

$$\langle f, \mathcal{F}^*\mathcal{F}u\rangle = \langle \mathcal{F}^* f, \mathcal{F}u\rangle = \langle \mathcal{F}\mathcal{F}^* f, u\rangle = \langle f, u\rangle. \tag{20}$$

We have, for example,

$$\mathcal{F}\delta(\xi) = (2\pi)^{-n/2}, \quad \mathcal{F}1(x) = (2\pi)^{n/2}\delta. \tag{21}$$

Let us also make note of the behavior of $\mathcal{F}$ and $\mathcal{F}^*$ on $L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Since $(f, g)_{L^2} = \int f\overline{g}\, dx$, (14) gives $(f, \mathcal{F}^* g)_{L^2} = (\mathcal{F}f, g)_{L^2}$ for $f, g \in \mathcal{S}(\mathbb{R}^n)$; hence

$$(\mathcal{F}f, \mathcal{F}g)_{L^2} = (f, \mathcal{F}^*\mathcal{F}g)_{L^2} = (f, g)_{L^2}, \tag{22}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$, the last identity by Fourier inversion on $\mathcal{S}(\mathbb{R}^n)$. This implies $\mathcal{F}$ extends uniquely to $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, as an isometry; ditto for $\mathcal{F}^*$, and Fourier inversion extends, so $\mathcal{F}$ is unitary, with inverse $\mathcal{F}^*$, on $L^2(\mathbb{R}^n)$.

Recalling how $\mathcal{F}$ intertwines ∂_j and multiplication by $-i\xi_j$, we get an extra dividend from (16). Namely,

$$u(t, x) = (2\pi)^{-n/2} \int e^{-t|\xi|^2} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \tag{23}$$

solves the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \quad u(0, x) = f(x), \tag{24}$$

so the solution is given by

$$u(t, x) = \int p(t, x - y) f(y)\, dy, \tag{25}$$

with $p(t, x)$ as in (17). We can write (25) as $e^{t\Delta} f$. In particular, we have the fundamental solution

$$e^{t\Delta}\delta(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \tag{26}$$

for $t > 0$. In fact, Fourier multiplication by $e^{-t|\xi|^2}$ is well behaved on $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, and $L^2(\mathbb{R}^n)$, not just for real $t > 0$, but also for complex t such that $\operatorname{Re} t \geq 0$, and we can pass by analytic continuation from (26) to the formula

$$e^{it\Delta}\delta(x) = (4\pi it)^{-n/2}e^{-|x|^2/4it}, \quad t \in \mathbb{R}, \tag{27}$$

for the fundamental solution to the Schrödinger equation on $\mathbb{R} \times \mathbb{R}^n$. (Analytic continuation picks the correct root for $(4\pi it)^{-n/2}$ if n is odd.)

The distributional theory of Fourier series is also quite useful. Given $f \in L^1(\mathbb{T}^n)$ ($\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$), we set

$$\hat{f}(k) = (2\pi)^{-n}\int\limits_{\mathbb{T}^n} f(x)e^{-ik\cdot x}\,dx, \quad k \in \mathbb{Z}^n, \tag{28}$$

and the Fourier inversion formula reads

$$f(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)e^{ik\cdot x}. \tag{29}$$

For $f \in L^2$, this follows from the fact that $\{e^{ik\cdot x} : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n, (2\pi)^{-n}dx)$; thus $\mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2$ is an isomorphism. We also have isomorphisms

$$\mathcal{F} : C^\infty(\mathbb{T}^n) \longrightarrow s(\mathbb{Z}^n), \quad \mathcal{F} : \mathcal{D}'(\mathbb{T}^n) \longrightarrow s'(\mathbb{Z}^n), \tag{30}$$

where $s(\mathbb{Z}^n)$ consists of rapidly decreasing functions $\mathbb{Z}^n \to \mathbb{C}$, and $s'(\mathbb{Z}^n)$ consists of polynomially bounded functions $\mathbb{Z}^n \to \mathbb{C}$. The Fourier inversion formula on $\mathcal{D}'(\mathbb{T}^n)$ is

$$u \in \mathcal{D}'(\mathbb{T}^n) \Rightarrow u = \sum_k \hat{u}(k)e^{ik\cdot x}, \quad \hat{u}(k) = (2\pi)^{-n}\langle e^{-ik\cdot x}, u\rangle. \tag{31}$$

For example, with $n = 1$,

$$\delta = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{ikx}, \tag{32}$$

the sum converging in $\mathcal{D}'(\mathbb{T}^1)$.

We illustrate some concepts discussed above with the following problem. Describe the fundamental solution $S(t, x) = e^{-it\Delta}\delta(x)$ for $x \in \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$. For general $t \in \mathbb{R}$, this is a real mess, but we can get interesting explicit formulas when t is a rational multiple of π. We have the Fourier series representation

$$S\Big(2\pi\frac{m}{n},x\Big)=\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}e^{2\pi ik^2m/n}e^{ikx}. \tag{33}$$

Setting $k=nj+\ell$, we obtain a double sum,

$$S\Big(2\pi\frac{m}{n},x\Big)=\sum_{\ell=0}^{n-1}e^{2\pi i\ell^2m/n}e^{i\ell x}\sum_{j=-\infty}^{\infty}e^{injx}. \tag{34}$$

Now, via (32),

$$\sum_{j=-\infty}^{\infty}e^{injx}=\frac{2\pi}{n}\sum_{j=0}^{n-1}\delta_{2\pi j/n}, \tag{35}$$

where δ_p is defined by $\langle f,\delta_p\rangle=f(p)$. Hence

$$S\Big(2\pi\frac{m}{n},x\Big)=\frac{1}{n}\sum_{j=0}^{n-1}G(m,n,j)\delta_{2\pi j/n}, \tag{36}$$

where

$$G(m,n,j)=\sum_{\ell=0}^{n-1}e^{2\pi i\ell^2m/n}e^{2\pi i\ell j/n}. \tag{37}$$

These coefficients are known as Gauss sums, and they have number theoretical significance.

The formula (36)–(37) is intrinsically interesting. When $t=2\pi m/n$, $e^{-it\Delta}\delta\in\mathcal{D}'(\mathbb{T}^1)$ is a finite linear combination of delta functions, supported on $\{2\pi j/n : 0\le j\le n-1\}$, a set that becomes denser in $\mathbb{T}^1$ as the denominator n increases. The explicit calculation (37) of the coefficients (36) is also of interest. Matters become even much more interesting when the following is taken into account.

Namely, there is another way to compute $e^{-it\Delta}\delta$ on $\mathbb{T}^1$; use (27) (with t replaced by $-t$) and periodize it. We get

$$S(t,x)=\sum_{\nu=-\infty}^{\infty}e^{-it\Delta}\delta(x-2\pi\nu), \tag{38}$$

where, on the line,

$$e^{-it\Delta}\delta(x)=\frac{1+i}{\sqrt{2}}\frac{1}{\sqrt{4\pi t}}e^{-ix^2/4t}\quad (t>0), \tag{39}$$

so

$$
\begin{aligned}
S\Big(2\pi\frac{m}{n},x\Big) &= \frac{1+i}{4\pi}\Big(\frac{n}{m}\Big)^{1/2}e^{-ix^2n/8\pi m} \\
&\quad\times\sum_{\nu=-\infty}^{\infty}e^{-\pi i\nu^2 n/2m}e^{i\nu nx/2m}.
\end{aligned}
\tag{40}
$$

We can set $\nu = 2mj + \ell$ and convert this into a double sum over $\ell \in \{0,\dots,2m-1\}$, $j \in \mathbb{Z}$. Again using (35), we get

$$
S\Big(2\pi\frac{m}{n},x\Big) = \frac{1}{n}\sum_{j=0}^{n-1}\widetilde{G}(m,n,j)\delta_{2\pi j/n},
\tag{41}
$$

parallel to (36), as it must be. However, and this is quite significant, the calculation leads to an expression $\widetilde{G}(m,n,j)$ with an appearance different from (37). In its place, we get

$$
\begin{aligned}
\widetilde{G}(m,n,j) &= \frac{1+i}{2}\Big(\frac{n}{m}\Big)^{1/2}e^{-\pi ij^2/2mn} \\
&\quad\times\sum_{\ell=0}^{2m-1}e^{-\pi i\ell^2 n/2m}e^{\pi ij\ell/m}.
\end{aligned}
\tag{42}
$$

It follows that these coefficients of $\delta_{2\pi j/n}$ must be equal:

$$
G(m,n,j) = \widetilde{G}(m,n,j).
\tag{43}
$$

In particular, when $j=0$, we get

$$
\sum_{\ell=0}^{n-1}e^{2\pi i\ell^2 m/n} = \frac{1+i}{2}\Big(\frac{n}{m}\Big)^{1/2}\sum_{\ell=0}^{2m-1}e^{-\pi i\ell^2 n/2m}.
\tag{44}
$$

If we further specialize to $m=1$, we get

$$
\sum_{\ell=0}^{n-1}e^{2\pi i\ell^2/n} = \frac{1+i}{2}n^{1/2}(1+i^{-n}).
\tag{45}
$$

The identity (45) is the classical Gauss sum evaluation, produced by Gauss and used in one of his proofs of the quadratic reciprocity theorem. The results (44) and (43) are generalizations of this, to Gauss sum identities established by Dirichlet (by different means, to be sure).

This discussion has intended to illustrate the usefulness of the distribution theory approach to problems in Fourier analysis and PDE, allowing one to obtain substantial conclusions, often with impact in other areas of math. One frequently finds that the arguments involved are fun and easy, and everyone can play!

Actually, there is more to be said about this last assertion, and this bears on the *raison d'etre* for the book under review. Since the classic [6], there have been other treatments of distribution theory made available, some of which have also become classics. One is the three volume set [1]. Another is the treatment in Chapter 6 of Yosida [8]. This chapter was influenced by the notes [2], and for many students and researchers served as an introduction to [3], which made full use of distribution theory as a tool in linear PDE. The book [3] contained a very brief introduction to distribution theory. The book [4] fleshed out this presentation of distribution theory considerably. It has become common (we mention [5] and [7] as examples), though not universal, for a PDE text to include a chapter on distribution theory.

All these references share in common that they are addressed to a reader who has already acquired a significant background in functional analysis (for [8], by getting through Chapters 1–5). Also a background in basic measure theory and Lebesgue integration is assumed. From one point of view, this is reasonable. Distribution theory is essentially written in the language of functional analysis, and the space of distributions on a domain $\Omega \subset \mathbb{R}^n$ contains the set of locally finite measures on Ω. These factors notwithstanding, practitioners of the art have long noted that distribution theory, for the most part, is easier than measure theory, and only rarely does one need to confront heavy duty functional analysis.

With these notions in mind, the authors have set out to produce a text on distribution theory accessible to the undergraduate (or beginning graduate) student at the point in his or her studies where it could serve as an alternative to a beginning course in measure theory. This involves a bit of a balancing act. The authors intend to include material of serious use in PDE, so the presentation cannot be too lightweight.

I would say the authors have done an admirable job of finding the sweet spot for this presentation. They introduce ideas of basic functional analysis as needed, throughout the text. The occasional need for results in measure theory is met by an appendix (labeled Chapter 20). The approach in this appendix involves producing a measure from a positive linear functional (the Daniell approach to integration), which is close in spirit to the main ideas of distribution theory. The authors discuss many fascinating results in distribution theory, with an emphasis on the use of the subject in linear PDE, but with also interesting side trips, involving such topics as Shannon's sampling theorem. The book succeeds as both a basic and a rich account of results in distribution theory.

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