

EERSTE DEELTENTAMEN WISB 212

Analyse in Meer Variabelen

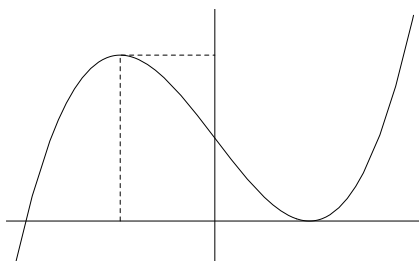
18-04-2006 14-17 uur

- *Zet uw naam en collegekaartnummer op elk blad alsmede het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *Bij dit tentamen mogen boeken, syllabi, aantekeningen en/of rekenmachine **NIET** worden gebruikt.*
- *Het eerste vraagstuk telt voor 60 % van de uitslag en het tweede voor 40 %.*

Exercise 0.1 (Family of cubic curves). Define the monic cubic polynomial function

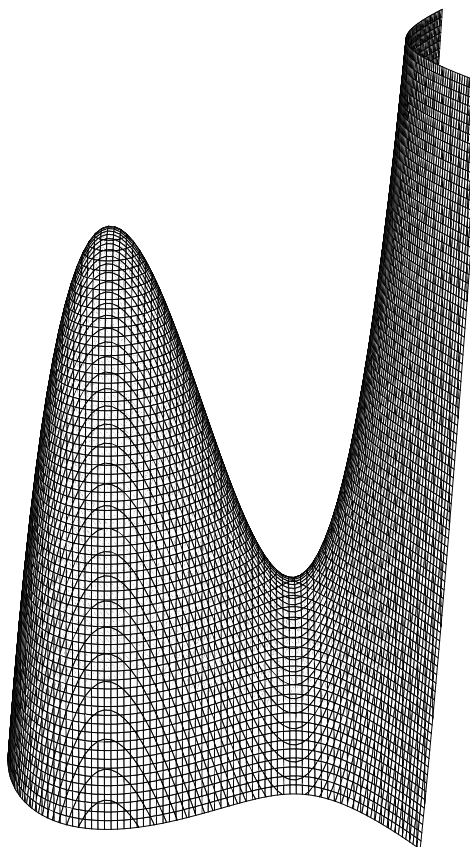
$$p : \mathbf{R} \rightarrow \mathbf{R} \quad \text{by} \quad p(x) = x^3 - 3x + 2.$$

- (i) Prove that the extrema of p are a local maximum of value 4 occurring at -1 and a local minimum 0 at 1. Determine the zeros of p and decompose p into a product of linear factors.



Next introduce the cubic polynomial function

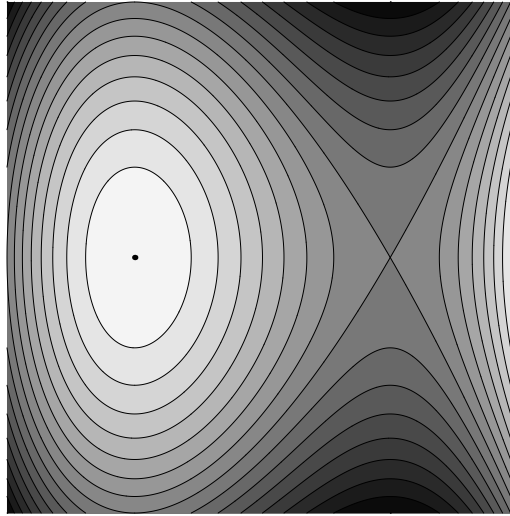
$$g : \mathbf{R}^3 \rightarrow \mathbf{R} \quad \text{by} \quad g(x) = p(x_1) - x_2^2 - x_3 \quad \text{and the set} \quad V = \{x \in \mathbf{R}^3 \mid g(x) = 0\}.$$



- (ii) Show that V is a C^∞ submanifold in \mathbf{R}^3 of dimension 2 by representing it as the graph of a C^∞ function.
- (iii) Verify again the claim about V as in part (ii), but now by considering $Dg(x)$, for all $x \in V$. Further, prove that $(-1, 0, 4)$ and $(1, 0, 0)$ are the only points of V at which the tangent plane of V is given by the linear subspace $\mathbf{R}^2 \times \{0\}$ of \mathbf{R}^3 .

For every $c \in \mathbf{R}$, define the function

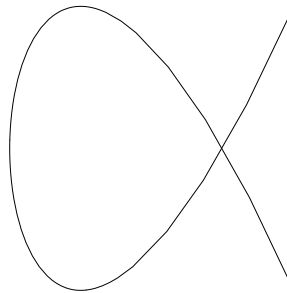
$$g_c : \mathbf{R}^2 \rightarrow \mathbf{R} \quad \text{by} \quad g_c(x_1, x_2) = g(x_1, x_2, c) \quad \text{and the set} \quad V_c = \{x \in \mathbf{R}^2 \mid g_c(x) = 0\}.$$



- (iv) For every $c \in \mathbf{R} \setminus \{0, 4\}$, demonstrate that V_c is a C^∞ submanifold in \mathbf{R}^2 of dimension 1. Prove that V_0 is a C^∞ submanifold in \mathbf{R}^2 of dimension 1 in all of its points with the possible exception of $(1, 0)$. Furthermore, using part (i) show that V_4 is the disjoint union of a point (which?) and a C^∞ submanifold in \mathbf{R}^2 of dimension 1.
- (v) Set $I = [-2, \infty[\subset \mathbf{R}$ and prove by means of part (i) that $V_0 \subset I \times \mathbf{R}$. Next, use this fact to write V_0 as the union of the graphs G_+ and G_- of two distinct functions defined on I that are C^∞ on the interior of I . Furthermore, derive that $(1, 0) \in V_0$ is a point where G_+ and G_- intersect and that $\frac{\pi}{3}$ is the smallest angle between the tangent lines at $(1, 0)$ of G_+ and G_- , respectively.
- (vi) From the previous part it follows that every $x \in V_0$ satisfies $x_1 \geq -2$; in this case, therefore, one may write $x_1 = t^2 - 2$ with $t \in \mathbf{R}$. Deduce $V_0 = \text{im } \phi$, where

$$\phi : \mathbf{R} \rightarrow \mathbf{R}^2 \quad \text{is given by} \quad \phi(t) = (t^2 - 2, t^3 - 3t).$$

Verify that ϕ is an embedding on $\mathbf{R} \setminus \{\pm\sqrt{3}\}$.



Finally, suppose that $p : \mathbf{R} \rightarrow \mathbf{R}$ is an arbitrary monic cubic polynomial with real coefficients and consider $C = \{x \in \mathbf{R}^2 \mid p(x_1) = x_2^2\}$.

- (vii) Show that C possesses a singular point only if p has a root at least of multiplicity two. Describe the geometry of C if p has a root of multiplicity three.

Background. Families of curves in \mathbf{R}^2 of the type studied above occur in *number theory* and in the *theory of differential equations*.

Exercise 0.2 (Primal and dual problem in the sense of optimization theory). Suppose $C \in \text{End}(\mathbf{R}^p)$ to be symmetric and positive definite; that is, $\langle Cy, y \rangle = \langle y, Cy \rangle$ and $\langle y, Cy \rangle \geq 0$ for all $y \in \mathbf{R}^p$, with equality only if $y = 0$. Furthermore, let $n \leq p$ and suppose $A \in \text{Lin}(\mathbf{R}^n, \mathbf{R}^p)$ to be injective.

- (i) Prove that $C \in \text{Aut}(\mathbf{R}^p)$ and that $A^tCA \in \text{End}(\mathbf{R}^n)$ is symmetric and positive definite, and therefore satisfies $A^tCA \in \text{Aut}(\mathbf{R}^n)$. (Recall that $A^t \in \text{Lin}(\mathbf{R}^p, \mathbf{R}^n)$ is defined by $\langle A^ty, x \rangle = \langle y, Ax \rangle$, for all $y \in \mathbf{R}^p$ and $x \in \mathbf{R}^n$.)

Let $0 \neq a \in \mathbf{R}^n$ be fixed and define the quadratic function

$$P : \mathbf{R}^n \rightarrow \mathbf{R} \quad \text{by} \quad P(x) = \frac{1}{2} \langle A^tCAx, x \rangle - \langle a, x \rangle.$$

- (ii) For $x \in \mathbf{R}^n$, show by means of part (i) that $DP(x) = 0$ if and only if x satisfies the linear equation $A^tCAx = a$ and that such an x is unique. Conclude that P attains the value $p := -\frac{1}{2} \langle a, (A^tCA)^{-1}a \rangle$ at its only critical point.

In the sequel it may be used without proof that $\min_{x \in \mathbf{R}^n} P(x) = p$. (This fact can be proved using compactness and consideration of the asymptotic behavior of $P(x)$ as $\|x\| \rightarrow \infty$.)

Now we come to the main issue of the exercise, namely, the study of the quadratic function

$$Q : \mathbf{R}^p \rightarrow \mathbf{R} \quad \text{given by} \quad Q(y) = \frac{1}{2} \langle C^{-1}y, y \rangle, \quad \text{under the constraint} \quad A^ty = a.$$

- (iii) Demonstrate that, for all $y \in V := \{y \in \mathbf{R}^p \mid A^ty = a\}$ and $x \in \mathbf{R}^n$, we have the following identity, in which an *uncoupled* expression occurs at the left-hand side,

$$Q(y) + P(x) = \frac{1}{2} \langle C(C^{-1}y - Ax), C^{-1}y - Ax \rangle.$$

Deduce, for $y \in V$ and $x \in \mathbf{R}^n$, that we have $Q(y) \geq -P(x)$, with equality if and only if $y = CAx$. Using part (ii), show, for all $y \in V$,

$$Q(y) \geq -p = \max_{x \in \mathbf{R}^n} -P(x), \quad \text{and conclude} \quad \min_{y \in V} Q(y) = \max_{x \in \mathbf{R}^n} -P(x).$$

In other words, the constrained minimum of Q equals the unconstrained maximum of $-P$. As an example of a different approach, we now study the preceding problem by introducing the Lagrange function

$$L : \mathbf{R}^p \times \mathbf{R}^n \rightarrow \mathbf{R} \quad \text{with} \quad L(y, x) = Q(y) - \langle x, (A^ty - a) \rangle.$$

- (iv) Using L , determine the points $y \in V$ where the extrema of $Q|_V$ are attained and derive the same results as in part (iii).

Background. The result above is one of the simplest cases of a duality that plays an important role in *optimization theory*. In this manner, the *primal problem* of minimizing Q under constraints is replaced by the *dual problem* of maximizing P .

Solution of Exercise 0.1

- (i) $p'(x) = 3(x^2 - 1) = 0$ implies $x = \pm 1$; with corresponding values $p(-1) = 4$ and $p''(-1) = -6$, hence a local maximum; and $p(1) = 0$ and $p''(1) = 6$, hence a local minimum. Since $\lim_{x \rightarrow \pm\infty} p(x) = \pm\infty$, the extrema are not absolute. In view of $p(1) = p'(1) = 0$, one may write $p(x) = (x - 1)^2(x - a) = x^3 + \dots - a$ (see Application 3.6.A), which implies $a = -2$; hence the factorization is $p(x) = (x - 1)^2(x + 2)$.
- (ii) $g(x) = 0$ implies $x_3 = p(x_1) - x_2^2$. This leads to $V = \{ (x_1, x_2, p(x_1) - x_2^2) \in \mathbf{R}^3 \mid (x_1, x_2) \in \mathbf{R}^2 \}$, displaying V as the graph of a C^∞ function on \mathbf{R}^2 .
- (iii) $Dg(x) = (p'(x_1), -2x_2, -1)$, and this element in $\text{Mat}(1 \times 3, \mathbf{R})$ is of rank 1, for all $x \in \mathbf{R}^3$; therefore g is submersive on all of \mathbf{R}^3 . The assertion about V now follows from the Submersion Theorem 4.5.2. Furthermore, $\text{grad } g(x)$ is perpendicular to $T_x V$, for any $x \in V$ (see Example 5.3.5); hence $T_x V = \mathbf{R}^2 \times \{0\}$ if and only if $p'(x_1) = 0$, $x_2 = 0$ and $g(x) = 0$. But this implies $x_1 = \pm 1$, $x_2 = 0$ and $x_3 = p(\pm 1)$.
- (iv) According to the Submersion Theorem 4.5.2, the set V_c is a C^∞ submanifold in \mathbf{R}^2 of dimension 1 in $x \in V_c$ if $Dg_c(x) = (p'(x_1), -2x_2) \neq (0, 0)$ and $c = p(x_1) - x_2^2$. That is, V_c possibly does not possess the desired properties at x if

$$x_1 = \pm 1, \quad x_2 = 0 \quad \text{and} \quad c \in \{p(\pm 1)\} = \{0, 4\}.$$

If $c = 0$, and $c = 4$, only the point $(1, 0) \in V_0$, and $(-1, 0) \in V_4$, respectively, satisfies all these conditions. Actually, the point $(-1, 0)$ is an isolated point of V_4 . Indeed, on the basis of part (i) one finds for $x \in V_4$ sufficiently close to $(-1, 0)$ that $4 = p(-1) \geq p(x_1) = x_2^2 + 4$. But this implies $x_2 = 0$ and so $x_1 = -1$.

- (v) For $x \in V_0$ one has $0 \leq x_2^2 = p(x_1)$, but then part (i) implies $x_1 \geq -2$. Under the latter assumption, the condition $x_2^2 = p(x_1) = (x_1 - 1)^2(x_1 + 2)$ on x is equivalent to

$$x_2 = \pm(x_1 - 1)\sqrt{x_1 + 2} =: f_\pm(x_1),$$

where $f_\pm : I \rightarrow \mathbf{R}$ is a C^∞ function on the interior of I . Now set $G_\pm = \text{graph } f_\pm$. Since $f_\pm(1) = 0$, one sees $(1, 0) \in \bigcap_\pm G_\pm$, while f_\pm is C^∞ near 1. Furthermore,

$$Df_\pm(x_1) = \pm(\sqrt{x_1 + 2} + (x_1 - 1) \cdots), \quad \text{in particular} \quad \text{graph } Df_\pm(1) = \mathbf{R}(1, \pm\sqrt{3}).$$

Noting that the norms of the two preceding generators of the tangent spaces of G_+ and G_- at $(1, 0)$ are equal to 2 and writing α for the angle between these, one gets

$$\cos \alpha = \frac{\langle (1, \sqrt{3}), (1, -\sqrt{3}) \rangle}{\|(1, \sqrt{3})\| \|(1, -\sqrt{3})\|} = \frac{1 - 3}{2 \cdot 2} = -\frac{1}{2}, \quad \text{that is} \quad \alpha = \frac{2\pi}{3}.$$

It follows that the smallest angle between the tangent lines equals $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

- (vi) Writing $x_1 = t^2 - 2$ for $x \in V_0$, one finds on the basis of part (i)

$$x_2^2 = p(x_1) = (x_1 - 1)^2(x_1 + 2) = (t^2 - 3)^2 t^2 = (t^3 - 3t)^2.$$

This implies $V_0 \subset \text{im } \phi$, whereas the reverse implication is a straightforward calculation. $D\phi(t) = (2t, 3(t^2 - 1))$ is of rank 1, for all $t \in \mathbf{R}$; hence ϕ is an immersion on \mathbf{R} . Further, $\phi(t) = \phi(t')$, for t and $t' \in \mathbf{R}$, leads to $t = \pm t'$, hence $t(t^2 - 3) = 0$; therefore $t = \pm\sqrt{3}$ and $t' = \mp\sqrt{3}$. If $t \neq \pm\sqrt{3}$ and $x = \phi(t)$, then $x_1 - 1 \neq 0$, which implies that $\phi(t) = x \mapsto \frac{x_2}{x_1 - 1} = t$ defines a continuous mapping. This demonstrates that ϕ is an embedding on $\mathbf{R} \setminus \{\pm\sqrt{3}\}$.

- (vii) If $x \in C$ is a singular point of C , then $p(x_1) = x_2^2$ and $(p'(x_1), -2x_2) = (0, 0)$ imply $x_2 = 0$ and $p(x_1) = p'(x_1) = 0$; in other words, p must possess a root of multiplicity at least two. Suppose $p(x_1) = (x_1 - c)^3$, for some $c \in \mathbf{R}$, then the points of C satisfy the equation $(x_1 - c)^3 = x_2^2$, which is an ordinary cusp as in Example 5.3.8.

Solution of Exercise 0.2

- (i) Suppose that $Cy = 0$, then $\langle y, Cy \rangle = 0$, hence $y = 0$. Accordingly, C is injective and thus $C \in \text{Aut}(\mathbf{R}^p)$. Next, $(A^tCA)^t = A^tC^tA^{tt} = A^tCA$, which proves the symmetry. Further, assume $x \in \mathbf{R}^n$ satisfies $A^tCAx = 0$. Then, in view of C being positive definite and A injective,

$$\langle x, A^tCAx \rangle = \langle Ax, CAx \rangle = 0 \implies Ax = 0 \implies x = 0.$$

Finally, apply the first argument to A^tCA .

- (ii) The first assertion on $DP(x)$ follows from Corollary 2.4.3.(ii), while the uniqueness of x is a consequence of $A^tCA \in \text{Aut}(\mathbf{R}^n)$. Furthermore,

$$P((A^tCA)^{-1}x) = \frac{1}{2}\langle a, (A^tCA)^{-1}a \rangle - \langle a, (A^tCA)^{-1}a \rangle.$$

- (iii) For all $y \in V$ and $x \in \mathbf{R}^n$ one obtains, using $A^ty = a$ and the positive definiteness of C ,

$$\begin{aligned} Q(y) + P(x) &= \frac{1}{2}\langle C^{-1}y, y \rangle + \frac{1}{2}\langle A^tCAx, x \rangle - \langle a, x \rangle \\ &= \frac{1}{2}\langle C(C^{-1}y), C^{-1}y \rangle + \frac{1}{2}\langle CAx, Ax \rangle - \langle A^ty, x \rangle \\ &= \frac{1}{2}\langle C(C^{-1}y - Ax), C^{-1}y - Ax \rangle + \frac{1}{2}\langle y, Ax \rangle + \frac{1}{2}\langle CAx, C^{-1}y \rangle - \langle y, Ax \rangle \\ &= \frac{1}{2}\langle C(C^{-1}y - Ax), C^{-1}y - Ax \rangle \geq 0. \end{aligned}$$

Once more on the basis of C being positive definite, one has equality if and only if $C^{-1}y - Ax = 0$, in other words, $y = CAx$. In turn, this implies $Q(y) \geq -P(x)$, for all $y \in V$ and $x \in \mathbf{R}^n$. In particular, this is the case if $x^0 \in \mathbf{R}^n$ is the unique element satisfying $A^tCAx^0 = a$ (see part (ii)); this implies, for all $y \in V$,

$$Q(y) \geq -P(x^0) = \max_{x \in \mathbf{R}^n} -P(x) = -\min_{x \in \mathbf{R}^n} P(x) = -p.$$

Now consider $y^0 = CAx^0 \in \mathbf{R}^p$. Then $A^ty^0 = A^tCAx^0 = a$, that is, $y^0 \in V$; and the preceding arguments imply $Q(y^0) = -P(x^0) = -p$. This proves $\min_{y \in V} Q(y) = -p$.

- (iv) Applying the method of Lagrange multipliers, one obtains that extrema for $Q|_V$ occur at points $y \in V$ satisfying

$$D_y L(y, x) = C^{-1}y - Ax = 0 \implies y = CAx \quad \text{and} \quad a = A^ty = A^tCAx.$$

However, for such y and x ,

$$\begin{aligned} Q(y) &= \frac{1}{2}\langle C^{-1}CAx, CAx \rangle = \frac{1}{2}\langle Ax, CAx \rangle = \frac{1}{2}\langle A^tCAx, x \rangle \\ &= -\frac{1}{2}\langle A^tCAx, x \rangle + \langle a, x \rangle = -P(x). \end{aligned}$$

C^{-1} being positive definite implies that Q attains a minimum on V ; indeed, the graph of the restriction of Q to V is the intersection of an elliptic paraboloid and an affine submanifold (if necessary, use that continuity of the function Q implies that it attains extrema on compact subsets of V). Therefore $\min_{y \in V} Q(y) = -P(x)$ where $x = (A^t C A)^{-1} a \in \mathbf{R}^n$. Finally, use part (ii) to obtain the desired equality.

Background. The method of Lagrange multipliers enables one to obtain the dual quadratic form P , given the primal form Q together with its constraint, by explicitly computing the minimal value of Q .