HERKANSINGSTENTAMEN WISB 212

Analyse in Meer Variabelen

29–08–2006 14–17 uur

- *Zet uw naam en collegekaartnummer op elk blad alsmede het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *Bij dit tentamen mogen boeken, syllabi, aantekeningen en/of rekenmachine* **NIET** *worden gebruikt.*
- *De antwoorden mogen uiteraard in het Nederlands worden gegeven, ook al zijn de vraagstukken in het Engels geformuleerd.*
- *De drie vraagstukken tellen evenzwaar.*

Exercise 0.1 (Diffeomorphism from plane onto hyperbolic domain). We want to parametrize the points belonging to the unbounded open set

$$U = \{\, x \in \mathbf{R}^2 \mid |x_1 x_2| < 1 \,\}$$

by points in all of $\mathbf{R}^2$. Given $x \in U$, note there exists $y \in \mathbf{R}^2$ such that

$$x_1^2 x_2^2 = \frac{y_1^2 y_2^2}{1 + y_1^2 y_2^2}.$$

This suggests to consider

$$\Psi : \mathbf{R}^2 \to \mathbf{R}^2 \qquad \text{given by} \qquad \Psi(y) = f_+(y)\, y \qquad \text{where} \qquad f_\pm(y) = \frac{1}{\sqrt[4]{1 \pm y_1^2 y_2^2}}.$$

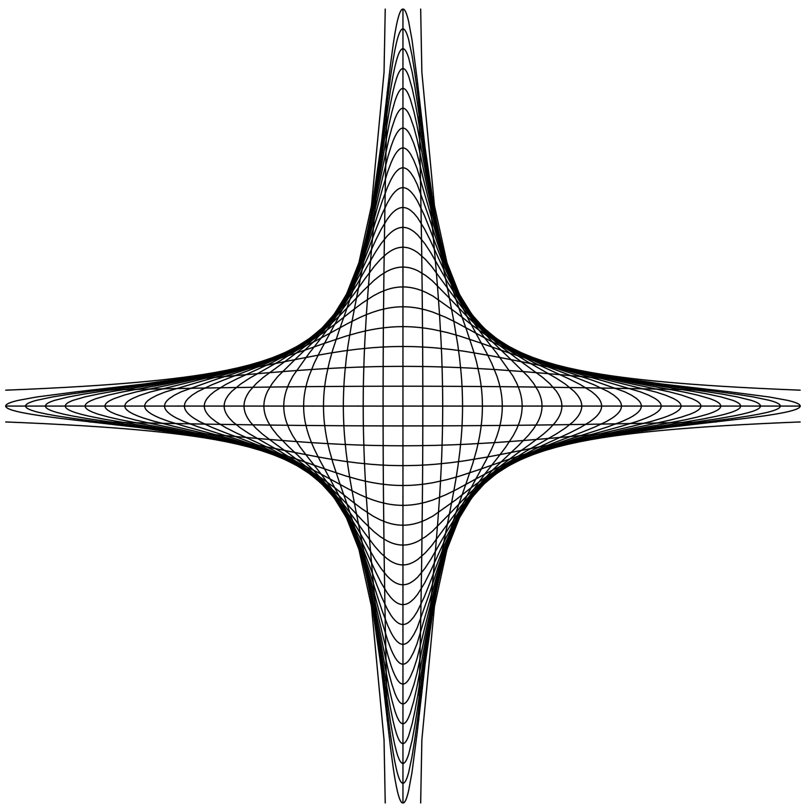
(i) Show that $\Psi : \mathbf{R}^2 \to U$ is a C^∞ diffeomorphism by computing that its inverse $\Phi : U \to \mathbf{R}^2$ satisfies $\Phi(x) = f_-(x)\, x$.

(ii) Prove that $2 y_j\, D_j f_+(y) = -y_1^2 y_2^2\, f_+(y)^5$, for $1 \le j \le 2$. Use this to deduce

$$D\Psi(y) = \frac{f_+(y)^5}{2} \begin{pmatrix} 2 + y_1^2 y_2^2 & -y_1^3 y_2 \\ -y_1 y_2^3 & 2 + y_1^2 y_2^2 \end{pmatrix} \qquad \text{and} \qquad \det D\Psi(y) = f_+(y)^6.$$

(iii) Given $y \in \mathbf{R}^2$, consider the curves $s \mapsto \Psi(s, y_2)$ and $t \mapsto \Psi(y_1, t)$ in U. Demonstrate that the curves are C^∞ submanifolds in U of dimension 1. These two submanifolds obviously intersect at the point $\Psi(y)$; show that it is the only point of intersection.

(iv) Verify that the submanifolds from part (iii) are perpendicular at $\Psi(y)$ if and only if $\Psi(y)$ belongs to the intersection of one of the coordinate axes with U.



Exercise 0.2 (Averaging norms of vectors over a set). For a bounded C^1 submanifold V in $\mathbf{R}^n$ of dimension d, define

$$A(V) = \int_V \|x\|\, d_d x. \qquad \text{Then} \qquad \frac{A(V)}{\mathrm{vol}_d(V)}$$

represents the average norm of a vector belonging to the set V.

(i) Consider $B^2 = \{\, x \in \mathbf{R}^2 \mid \|x\| < 1 \,\}$ and compute $A(B^2)$.

(ii) Set $\square = \{\, x \in \mathbf{R}^2 \mid 0 < x_j < 1,\ 1 \le j \le 2 \,\}$ and show

$$A(\square) = \frac{1}{3}\sqrt{2} + \frac{1}{3}\log(1+\sqrt{2}) = 0.765\,195 \cdots .$$

Hint. Introduce polar coordinates (r, α) in $\square$. Next, one may apply without proof

$$\int \frac{1}{\cos^3\alpha}\, d\alpha = \int \frac{1}{(1-\sin^2\alpha)^2}\, d(\sin\alpha) = \frac{\sin\alpha}{2\cos^2\alpha} + \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

Furthermore, use that $\cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8} = \cos\frac{\pi}{4} = \frac{1}{2}\sqrt{2}$ implies

$$\left\{ \begin{array}{l} \sin \\ \cos \end{array} \right. \frac{\pi}{8} = \frac{1}{2}\sqrt{2 \mp \sqrt{2}} \qquad \text{and} \qquad \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right. \frac{3\pi}{8} = \left\{ \begin{array}{l} \cos \\ \sin \end{array} \right. \left(\frac{\pi}{2} - \frac{3\pi}{8}\right) = \frac{1}{2}\sqrt{2 \pm \sqrt{2}}.$$

(iii) Evaluate $A(V)$ where

$$V = \mathrm{im}(\phi) \qquad \text{with} \qquad \phi : \,]-1,1[\, \to \mathbf{R}^3 \qquad \text{given by} \qquad \phi(t) = (\cos t, \sin t, t).$$

Hint. In the computation of the integral, set $t = \tan\alpha$.

Background. The value of the integral in part (ii) occurs, for instance, in the calculation of the expected distance between two random points on different sides of the unit square.

Exercise 0.3 (Vector field on open set is uniquely determined by its curl, divergence and restriction to the boundary of its normal component). We call an open set $\Omega \subset \mathbf{R}^3$ *admissible* if it satisfies the conditions of the Theorem on Integration of a Total Derivative. Let g be a C^2 function on an open neighborhood of an admissible set Ω and denote by $f : \Omega \to \mathbf{R}^3$ the gradient vector field associated to g. Suppose

$$\mathrm{div}\, f = 0 \quad \text{on } \Omega \qquad \text{and} \qquad \langle f, \nu \rangle = 0 \quad \text{on } \partial\Omega.$$

Here $\nu(y)$ denotes, as usual, the outer normal to $\partial\Omega$ at $y \in \partial\Omega$.

(i) Prove $\mathrm{curl}\, f = 0$ on Ω.

(ii) Using Green's first identity show that $f = 0$ on Ω.

Next, consider the special case of

$$g : \mathbf{R}^3 \setminus \{0\} \to \mathbf{R} \qquad \text{with} \qquad g(x) = -\frac{1}{\|x\|} \qquad \text{and set} \qquad f(x) = \mathrm{grad}\, g(x) = \frac{1}{\|x\|^3}\, x.$$

(iii) Verify $\mathrm{div}\, f = 0$ on $\mathbf{R}^3 \setminus \{0\}$.

(iv) Deduce from the preceding two parts that there exists no admissible open set $\Omega \subset \mathbf{R}^3 \setminus \{0\}$ having the property that $\mathbf{R}y$ is contained in the tangent space of $\partial\Omega$ at y, for all $y \in \partial\Omega$.

(v) Can you give an example of an admissible set open $\Omega \subset \mathbf{R}^3 \setminus \{0\}$ having the property in part (iv) for "more or less half" of the points $y \in \partial\Omega$?

Background. The conditions $\mathrm{div}\, f$ and $\langle f, \nu \rangle = 0$ on the vector field f assert that it is incompressible and that it has no flux through the boundary of Ω. Loosely speaking, these conditions force f to be the vector field of a circulation within Ω, but that is ruled out by the condition that f be irrotational.

Solution of Exercise ??

(i) Given $x \in U$, consider the equation $x = \Psi(y)$ for $y \in \mathbf{R}^2$. If a solution y exists, then $\operatorname{sgn}(x_j) = \operatorname{sgn}(y_j)$, for $1 \le j \le 2$. Obviously, $y = 0$ is the only solution of $0 = \Psi(y)$. So we may assume that either x_1 or $x_2 \neq 0$, say $x_2 \neq 0$. Then $y_2 \neq 0$ and $\frac{x_1}{x_2} = \frac{y_1}{y_2}$, in other words, $x_1 y_2 = x_2 y_1$. Raising the identity $x_j = \Psi_j(y)$ to the fourth power and taking the indices modulo 2, we obtain

$$x_j^4 = \frac{y_j^4}{1 + y_1^2 y_2^2}, \qquad \text{so} \qquad y_j^4 = x_j^4 + (x_j y_j)^2 (x_j y_{j+1})^2 = x_j^4 + (x_j y_j)^2 (x_{j+1} y_j)^2 = x_j^4 + (x_1^2 x_2^2) y_j^4.$$

In other words,

$$(1 - x_1^2 x_2^2) y_j^4 = x_j^4 \qquad \text{and so} \qquad y_j = f_-(x)\, x_j,$$

where $f_-(x)$ is well-defined because $x \in U$. This proves that there exists a unique solution $y \in \mathbf{R}^2$. In other words, the inverse Φ of $\Psi : \mathbf{R}^2 \to U$ is as given and Ψ is a bijection with an inverse of class C^∞.

(ii) We have

$$D_j f_+(y) = D_j (1 + y_1^2 y_2^2)^{-\frac{1}{4}} = -\frac{1}{4}\,(1 + y_1^2 y_2^2)^{-\frac{5}{4}}\, \frac{2 y_1^2 y_2^2}{y_j} = -\frac{y_1^2 y_2^2}{2 y_j}\, f_+(y)^5.$$

Hence the matrix for $D\Psi(y)$ follows from, for $1 \le i, j \le 2$,

$$D_j \Psi_i(y) = \delta_{ij}\, f_+(y) + D_j f_+(y)\, y_i = \frac{f_+(y)^5}{2} \Big(2\delta_{ij}\,(1 + y_1^2 y_2^2) - \frac{y_i y_1^2 y_2^2}{y_j} \Big).$$

This implies

$$\det D\Psi(y) = \frac{f_+(y)^{10}}{4} (4 + 4 y_1^2 y_2^2) = f_+(y)^6.$$

(iii) All assertions are a direct consequence of the fact that Ψ is a C^∞ diffeomorphism.

(iv) The curves intersect orthogonally at $\Psi(y)$ if and only if the cosine of the angle of intersection is equal to zero. Modulo a strictly positive factor, this cosine is given by

$$\langle\, D_1 \Psi(y),\ D_2 \Psi(y)\, \rangle = -\frac{f_+(y)^{10}}{4} (2 + y_1^2 y_2^2) \|y\|^2\, y_1 y_2.$$

Hence it equals zero if and only if $y_1 y_2 = 0$, and this is the case if and only if $\Psi(y)$ belongs to one of the coordinate axes.

Solution of Exercise ??

(i) Using polar coordinate (r, α) in B^2 one finds

$$\int_{B^2} \|x\|\, dx = \int_{-\pi}^{\pi} \int_0^1 r^2\, dr\, d\alpha = 2\pi \frac{1}{3} = \frac{2\pi}{3}.$$

(ii) Introduction of polar coordinates leads to (compare with Exercise 6.15)

$$\int_{\square} \|x\|\, dx = 2 \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\alpha}} r^2\, dr\, d\alpha = \frac{2}{3} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 \alpha}\, d\alpha.$$

Now

$$\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\alpha}\,d\alpha = \frac{1}{2}\sqrt{2}+\frac{1}{2}\log\Big(\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\Big) = \frac{1}{2}\sqrt{2}+\frac{1}{2}\log\Big(\frac{2+\sqrt{2}}{\sqrt{4-2}}\Big) = \frac{1}{2}\sqrt{2}+\frac{1}{2}\log(1+\sqrt{2}).$$

The penultimate equality is obtained by multiplying the numerator and denominator of the argument of the log by $\sqrt{2+\sqrt{2}}$.

(iii) One obtains, for $-1 < t < 1$,

$$\|D\phi(t)\| = \|(-\sin t, \cos t, 1)\| = \sqrt{2} \qquad \text{and} \qquad \|\phi(t)\| = \sqrt{1+t^2} = \sqrt{1+\tan^2\alpha} = \frac{1}{\cos\alpha}.$$

Here $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$, because $\tan\frac{\pi}{4} = 1$. Accordingly $\frac{dt}{d\alpha} = \frac{d\tan\alpha}{d\alpha} = \frac{1}{\cos^2\alpha}$ implies

$$\int_V \|x\|\,d_1x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \|\phi(t)\|\sqrt{2}\,dt = 2\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\alpha}\,d\alpha = 2+\sqrt{2}\log(1+\sqrt{2}).$$

Solution of Exercise ??

(i) For every $x \in \Omega$, the matrix of $Df(x) \in \mathrm{End}(\mathbf{R}^3)$ is given by $(D_jD_ig(x))_{1\le i,j\le 3}$, which is symmetric on account of Theorem 2.7.2. Therefore $Af(x) = 0$, and this leads to $\mathrm{curl}\, f = 0$ on Ω.

(ii) Green's first identity implies

$$\int_\Omega (g\,\Delta g)(x)\,dx + \int_\Omega \|\operatorname{grad} g(x)\|^2\,dx = \int_{\partial\Omega} \Big(g\,\frac{\partial g}{\partial\nu}\Big)(y)\,d_1y.$$

By our assumptions on f we have $\Delta g = \operatorname{div}\operatorname{grad} g = \operatorname{div} f = 0$ on Ω and $\frac{\partial g}{\partial\nu} = \langle \operatorname{grad} g, \nu\rangle = \langle f, \nu\rangle = 0$ on $\partial\Omega$. It follows that $\int_\Omega \|\operatorname{grad} g(x)\|^2\,dx = 0$. Since the integrand is a nonnegative continuous function on Ω, it follows that $f = \operatorname{grad} g = 0$ on Ω.

(iii) See Example 7.8.4 in the case of $n = 3$.

(iv) Note that the vector $f(y)$ is proportional to y, for all $y \in \partial\Omega$. Now argue by contradiction. Indeed, suppose Ω is a set having the properties described in this part. Then the outer normal $\nu(y)$ is perpendicular to y, and so to $f(y)$, for $y \in \partial\Omega$; but this means $\langle f, \nu\rangle = 0$ on $\partial\Omega$. Part (iii) then implies that the conclusion of part (ii) holds; in other words, $f = 0$ on Ω. This is a contradiction because f is nowhere zero on Ω.