EERSTE DEELTENTAMEN WISB 212

Analyse in Meer Variabelen

17–04–2007 14–17 uur

- *Zet uw naam en collegekaartnummer op elk blad alsmede het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *Bij dit tentamen mogen boeken, syllabi, aantekeningen en/of rekenmachine* **NIET** *worden gebruikt.*
- *De vier vraagstukken tellen ieder voor 25% van het totaalcijfer.*
- *Het tentamen telt* **VIER** *bladzijden.*

Exercise 0.1 (Laplacian of composition of norm and linear mapping). For x and $y \in \mathbf{R}^n$, recall that $\langle x,\, y\rangle = x^t\, y$ where x^t denotes the transpose of the column vector $x \in \mathbf{R}^n$; and furthermore, that $\|x\| = \sqrt{\langle x,\, x\rangle}$. Fix $A \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}^p)$ and recall $\ker A = \{\, x \in \mathbf{R}^n \mid Ax = 0\,\}$. Now define

$$f : \mathbf{R}^n \backslash \ker A \to \mathbf{R} \qquad \text{by} \qquad f = \|\cdot\| \circ A, \qquad \text{i.e.} \qquad f(x) = \|Ax\|; \qquad \text{and set} \qquad f^2(x) = f(x)^2.$$

(i) Give an argument without computations that f is a positive C^∞ function.

(ii) By application of the chain rule to f^2 show, for $x \in \mathbf{R}^n \setminus \ker A$ and $h \in \mathbf{R}^n$,

$$Df(x)h = \frac{\langle Ax,\, Ah\rangle}{f(x)}.$$

Deduce that

$$Df(x) \in \mathrm{Lin}(\mathbf{R}^n, \mathbf{R}) \qquad \text{is given by} \qquad Df(x) = \frac{1}{f(x)} x^t\, A^t\, A.$$

Denote by $(e_1, \ldots, e_n)$ the standard basis vectors in $\mathbf{R}^n$.

(iii) For $1 \le j \le n$, derive from part (ii) that

$$D_j f(x) = \frac{\langle Ax,\, Ae_j\rangle}{f(x)} \qquad \text{and deduce} \qquad D_j^2 f(x) = \frac{\|Ae_j\|^2}{f(x)} - \frac{\langle Ax,\, Ae_j\rangle^2}{f^3(x)}.$$

As usual, write $\Delta = \sum_{1\le j\le n} D_j^2$ for the Laplace operator acting in $\mathbf{R}^n$ and $\|A\|^2_{\mathrm{Eucl}} = \sum_{1\le j\le n} \|Ae_j\|^2$.

(iv) Now demonstrate

$$\Delta(\|\cdot\| \circ A)(x) = \frac{\|A\|^2_{\mathrm{Eucl}}\, \|Ax\|^2 - \|A^t\, Ax\|^2}{\|Ax\|^3}.$$

(v) Which form takes the preceding identity if A equals the identity mapping in $\mathbf{R}^n$?

Exercise 0.2 (Application of Implicit Function Theorem). Suppose that $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a C^∞ function and that there exists a C^∞ function $g : \mathbf{R} \to \mathbf{R}$ satisfying

$$g(0) \neq 0 \qquad \text{and} \qquad f(x; 0) = x\, g(x) \qquad (x \in \mathbf{R}).$$

Consider the equation $f(x; y) = t$, where x and $t \in \mathbf{R}$, while $y \in \mathbf{R}^n$.

(i) Prove the existence of an open neighborhood V of 0 in $\mathbf{R}^n \times \mathbf{R}$ and of a unique C^∞ function $\psi : V \to \mathbf{R}$ such that, for all $(y, t) \in V$

$$\psi(0) = 0 \qquad \text{and} \qquad f(\psi(y, t);\, y) = t.$$

(ii) Establish the following formulae, where D_1 and D_2 denote differentiation with respect to the variables in $\mathbf{R}^n$ and $\mathbf{R}$, respectively:

$$D_1\psi(0) = -\frac{1}{g(0)} D_1 f(0; 0) \qquad \text{and} \qquad D_2\psi(0) = \frac{1}{g(0)}.$$

Exercise 0.3 (Quintic diffeomorphism). Recall that $\mathbf{R}_+ = \{\, x \in \mathbf{R} \mid x > 0 \,\}$ and define

$$\Phi : \mathbf{R}^2_+ \to \mathbf{R}^2_+ \qquad \text{by} \qquad \Phi(x) = \frac{1}{(x_1x_2)^2}(x_1^5, x_2^5).$$

(i) Prove that Φ is a C^∞ mapping and that $\det D\Phi(x) = 5$, for all $x \in \mathbf{R}^2_+$.

(ii) Verify that Φ is a C^∞ diffeomorphism and that its inverse is given by

$$\Psi : \mathbf{R}^2_+ \to \mathbf{R}^2_+ \qquad \text{with} \qquad \Psi(y) = (y_1y_2)^{\frac{2}{5}}(y_1^{\frac{1}{5}}, y_2^{\frac{1}{5}}).$$

Compute $\det D\Psi(y)$, for all $y \in \mathbf{R}^2_+$.

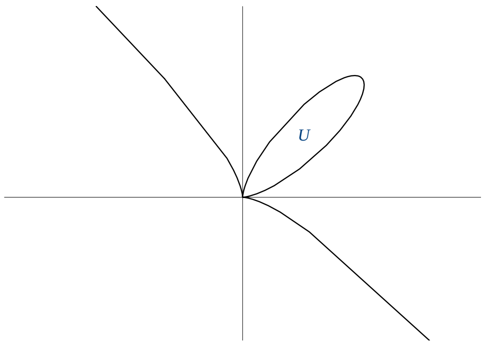
Let $a > 0$ and define

$$g : \mathbf{R}^2 \to \mathbf{R} \qquad \text{by} \qquad g(x) = x_1^5 + x_2^5 - 5a(x_1x_2)^2.$$

Now consider the bounded open sets

$$U = \{\, x \in \mathbf{R}^2_+ \mid g(x) < 0 \,\} \qquad \text{and} \qquad V = \{\, y \in \mathbf{R}^2_+ \mid y_1 + y_2 < 5a \,\}.$$

Then U has a curved boundary, while V is an isosceles rectangular triangle.



(iii) Show that $g \circ \Psi(y) = (y_1y_2)^2(y_1 + y_2 - 5a)$, for all $y \in \mathbf{R}^2_+$. Deduce that the restriction $\Psi|_V : V \to U$ is a diffeomorphism.

Background. By means of parts (ii) and (iii) one immediately computes the area of U to be $\frac{5a^2}{2}$.

Exercise 0.4 (Quintic analog of Descartes' folium). Let $g : \mathbf{R}^2 \to \mathbf{R}$ be the function from Exercise 0.3 and denote by F the zero-set of g (see the curve in the illustration above).

(i) Prove that F is a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1 at every point of $F \setminus \{0\}$.

(ii) By means of intersection with lines through 0 obtain the following parametrization of a part of F:

$$\phi : \mathbf{R} \setminus \{-1\} \to \mathbf{R}^2 \qquad \text{satisfying} \qquad \phi(t) = \frac{5at^2}{1+t^5}\begin{pmatrix} 1 \\ t \end{pmatrix}.$$

(iii) Compute that

$$\phi'(t) = \frac{5at}{(1+t^5)^2}\begin{pmatrix} 2 - 3t^5 \\ t(3 - 2t^5) \end{pmatrix}.$$

Show that ϕ is an immersion except at 0.

(iv) Demonstrate that F is not a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1 at 0.

The remainder is for extra credit and is no part of the regular exam. For $|x_2|$ small, x_2^5 is negligible; hence, after division by the common factor x_1^2 the equation $g(x) = 0$ takes the form $x_1^3 = 5ax_2^2$, which is the equation of an ordinary cusp. This suggests that F has a cusp at 0.

(v) Prove that F actually possesses two cusps at 0. This can be done with simple calculations; if necessary, however, one may use without proof

$$\begin{aligned}
\phi''(t) &= \frac{10a}{(1+t^5)^3}\begin{pmatrix} 6t^{10} - 18t^5 + 1 \\ t(3t^{10} - 19t^5 + 3) \end{pmatrix}, \\
\phi'''(t) &= -\frac{30a}{(1+t^5)^4}\begin{pmatrix} 5t^4(2t^{10} - 16t^5 + 7) \\ 4t^5(t^{10} - 17t^5 + 13) - 1 \end{pmatrix}.
\end{aligned}$$

Solution of Exercise 0.1

(i) The function $\sqrt{\cdot} : \,]\,0,\infty\,[\; \to \mathbf{R}$ is of class C^∞. Hence, f is the composition of C^∞ functions, therefore the assertion follows from the chain rule.

(ii) $f^2(x) = \langle x,\, x\rangle$ implies $Df^2(x)h = 2\langle Ax,\, Ah\rangle$ according to Corollary 2.4.3.(ii). Hence the desired formula follows from $2f(x)\,Df(x)h = Df^2(x)h = 2\langle Ax,\, Ah\rangle$ on account of the chain rule. Furthermore

$$Df(x)h = \frac{\langle Ax,\, Ah\rangle}{f(x)} = \frac{1}{f(x)}\,(Ax)^t\,Ah = \frac{1}{f(x)}x^t\,A^t\,A\,h.$$

(iii) We have

$$D_j f(x) = Df(x)e_j = \frac{\langle Ax,\, Ae_j\rangle}{f(x)}.$$

Application of Corollary 2.4.3.(iii) and (ii) as well as part (ii) implies

$$D_j^2 f(x) = D(D_j f)(x)e_j = \frac{\|Ae_j\|^2}{f(x)} - \frac{\langle Ax,\, Ae_j\rangle^2}{f^3(x)}.$$

(iv) Summation of the preceding identity for j running from 1 to n and $\langle Ax,\, Ae_j\rangle = \langle A^t\,Ax,\, e_j\rangle$ gives

$$\Delta f(x) = \sum_{1\le j\le n} D_j^2 f(x) = \frac{1}{f(x)}\sum_{1\le j\le n}\|Ae_j\|^2 - \frac{1}{f^3(x)}\sum_{1\le j\le n}\langle A^t\,Ax,\, e_j\rangle^2.$$

Furthermore, note that, for all $y\in\mathbf{R}^n$,

$$\sum_{1\le j\le n}\langle y,\, e_j\rangle^2 = \Big\|\sum_{1\le j\le n}\langle y,\, e_j\rangle e_j\Big\|^2 = \|y\|^2.$$

(v) In this case we obtain $\Delta(\|\cdot\|)(x) = \frac{n-1}{\|x\|}$, for $x\in\mathbf{R}^n\setminus\{0\}$ (compare with Exercise 2.40.(iii)).

Solution of Exercise 0.2

(i) Define $F : \mathbf{R}\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ by $F(x;y,t) = f(x;y) - t$. Then F is a C^∞ function satisfying

$$F(0;0,0) = f(0;0) = 0 \qquad\text{and}\qquad D_1F(0;0,0) = \frac{d}{dx}\Big|_{x=0}(x\,g(x)) = g(0)\neq 0.$$

The desired conclusion now follows from the Implicit Function Theorem 3.5.1.

(ii) Furthermore on account of the aforementioned theorem we obtain

$$D\psi(y,t) = -D_xF(\psi(y,t);y,t)^{-1}\circ D_{(y,t)}F(\psi(y,t);y,t).$$

In particular, this is valid for $(\psi(y,t);y,t) = (0;0,0)$. We have

$$D_{(y,t)}F(0;0,0) = (D_y f(0;0),-1) \qquad\text{and so}\qquad D\psi(0,0) = -\frac{1}{g(0)}(D_1 f(0;0),-1),$$

and this leads to the desired formulae.

Solution of Exercise 0.3

(i) Φ is a composition of C^∞ mappings. We have

$$D\Phi(x) = \begin{pmatrix} 3\Big(\dfrac{x_1}{x_2}\Big)^2 & -2\Big(\dfrac{x_1}{x_2}\Big)^3 \\ -2\Big(\dfrac{x_2}{x_1}\Big)^3 & 3\Big(\dfrac{x_2}{x_1}\Big)^2 \end{pmatrix} \qquad \text{and so} \qquad \det D\Phi(x) = 9 - 4 = 5.$$

(ii) Given arbitrary $y \in \mathbf{R}^2_+$, consider the equation $\Phi(x) = y$ for $x \in \mathbf{R}^2_+$; then $\frac{x_1^3}{x_2^2} = y_1$ and $\frac{x_2^3}{x_1^2} = y_2$. Multiplication and division of these equalities leads to

$$x_1x_2 = y_1y_2 \qquad \text{and} \qquad \Big(\frac{x_1}{x_2}\Big)^5 = \frac{y_1}{y_2}. \qquad \text{So} \qquad x_1x_2 = y_1y_2 \qquad \text{and} \qquad \frac{x_1}{x_2} = \frac{y_1^{\frac{1}{5}}}{y_2^{\frac{1}{5}}},$$

and multiplication of the equalities now gives $x_1^2 = y_1^{\frac{6}{5}} y_2^{\frac{4}{5}}$. Accordingly, $x_1 = y_1^{\frac{3}{5}} y_2^{\frac{2}{5}} = (y_1y_2)^{\frac{2}{5}} y_1^{\frac{1}{5}}$ because x_1, y_1 and $y_2 \in \mathbf{R}_+$. Similarly, we obtain the desired formula for x_2. It follows that Φ and Ψ are each other's inverses. On $\mathbf{R}^2_+$ the mapping Ψ is of class C^∞, which implies that Φ is a C^∞ diffeomorphism. From part (i) and the multiplicative property of the determinant we obtain $\det D\Psi(y) = \frac{1}{5}$.

(iii) We find

$$g \circ \Psi(y) = (y_1y_2)^2(y_1 + y_2) - 5a(y_1y_2)^{\frac{8}{5}}(y_1y_2)^{\frac{2}{5}} = (y_1y_2)^2(y_1 + y_2 - 5a).$$

This implies $x = \Psi(y) \in U$ if and only if $g(x) = (y_1y_2)^2(y_1 + y_2 - 5a) < 0$ if and only $y_1 + y_2 - 5a < 0$ if and only if $y \in V$.

Solution of Exercise 0.4

(i) We have

$$Dg(x) = 5(x_1(x_1^3 - 2ax_2^2),\ x_2(x_2^3 - 2ax_1^2)).$$

This matrix is of rank 1 unless (a) $x = 0$ or (b) $x_1^3 = 2ax_2^2$ and $x_2^3 = 2ax_1^2$. In case (b) we may assume $x \neq 0$ and we also obtain $x_1^9 = 8a^3x_2^6 = 32a^5x_1^4$, that is, $x_1^5 = (2a)^5$, which holds if and only if $x_1 = 2a$. In turn this implies $x_2 = 2a$, but $g(2a, 2a) = 64a^5 - 80a^5 = -16a^5 < 0$; in other words, $(2a, 2a) \notin F$. It follows that g is submersive at every point of $F \setminus \{0\}$. The desired conclusion follows from the Submersion Theorem 4.5.2.(ii).

(ii) We eliminate x_2 from the equations $g(x) = 0$ and $x_2 = tx_1$, for fixed $t \in \mathbf{R}$. This leads to $(1 + t^5)x_1^5 = 5at^2x_1^4$, with solutions $x_1 = 0$ (as was to be expected) or $x_1 = \frac{5at^2}{1+t^5}$; thus the desired formula for ϕ holds.

(iii) The formula for ϕ' is a consequence of

$$\phi'(t) = \frac{5a}{(1+t^5)^2}\begin{pmatrix} 2t\,(1+t^5) - t^2\,5t^4 \\ 3t^2(1+t^5) - t^3\,5t^4 \end{pmatrix} = \frac{5at}{(1+t^5)^2}\begin{pmatrix} 2 + 2t^5 - 5t^5 \\ t(3 + 3t^5 - 5t^5) \end{pmatrix}.$$

If $t \neq 0$, then the assumption $\phi'(t) = 0$ implies $2 - 3t^5 = 0$ and $3 - 2t^5 = 0$. This gives $9t^5 = 6 = 4t^5$, that is $5t^5 = 0$, and so arrived at a contradiction. Therefore $\phi'(t) \neq 0$ if $t \neq 0$; hence $\phi'(t)$ is of rank 1, which proves that ϕ is everywhere immersive except at 0.

(iv) F has self-intersection at 0 as follows from $\lim_{t\to\pm\infty} \phi(t) = 0 = \phi(0)$. Indeed, $\widetilde{\phi} : \mathbf{R} \setminus \{-1\} \to \mathbf{R}^2$ with $\widetilde{\phi}(u) = \phi(\frac{1}{u})$ also defines a parametrization of F. Now $\phi(t)$ approaches 0 in a vertical direction as $t \downarrow 0$, while $\widetilde{\phi}(u)$ approaches 0 in a horizontal direction as $u \downarrow 0$.

(v) Select $t_0 > 0$ sufficiently small, that is, suppose $2 - 3t_0^5 > 0$ and $3 - 2t_0^5 > 0$. For t running from -1 to t_0, the sign of the first component $t(2 - 3t^5)$ of $\phi'(t)$ changes from negative to positive at $t = 0$, whereas the sign of the second component $t^2(3 - 2t^5)$ remains nonnegative and vanishes for $t = 0$ only. This behavior of ϕ' near $\underline{0}$ is characteristic for a vertical cusp of F at 0. Mutatis mutandis, the same argument applied to $\widetilde{\phi}$ gives the existence of a second, horizontal, cusp of F at 0. Alternatively, it follows that

$$\phi''(0) = 10a\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \phi'''(0) = 30a\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

According to Definition 5.3.9 this implies the existence of an ordinary cusp of F at 0.