EXAM ANALYSE C

January 28, 1995 9.30–12.30 hour

- *Put your name on every sheet, and on the first sheet the name of the staff member in charge of the practice sessions as well as the number of pages you handed in.*
- *Different parts of the problems are independent as much as possible. If you can't do a specific part, continue and do the next parts.*
- *The problems have equal weights*

Exercise 0.1 (Quadrature of parabola). Define $\phi : \mathbf{R} \to \mathbf{R}^2$ by $\phi(t) = (t, t^2)$; then $P = \mathrm{im}(\phi)$ is a parabola. Consider arbitrary t_+ and $t_- \in \mathbf{R}$ with $t_+ > t_-$, and define

$$\delta = \frac{1}{2}(t_+ - t_-), \qquad t_0 = \frac{1}{2}(t_+ + t_-).$$

(i) Suppose $t \in \mathbf{R}$ satisfies $t_+ > t > t_-$. Prove that the area of the triangle in $\mathbf{R}^2$ with vertices $\phi(t_+)$, $\phi(t)$ and $\phi(t_-)$, is given by the following determinant

$$\frac{1}{2}\begin{vmatrix} t_+ - t_- & t_+ - t \\ t_+^2 - t_-^2 & t_+^2 - t^2 \end{vmatrix} \in \mathbf{R}.$$

Determine the unique t such that the corresponding triangle $\Delta(t_+, t_-)$ has maximal area, and show that this area is equal to δ^3.

(ii) Determine the t for which the direction of the tangent space to P in $\phi(t)$ is equal to that of the straight line $l(t_+, t_-)$ through the points $\phi(t_+)$ en $\phi(t_-)$.

(iii) Prove by successive integration that $\mathrm{area}\,(S(t_+, t_-)) = \dfrac{4}{3}\delta^3$, if $S(t_+, t_-)$ is the sector of the parabola P with basis $l(t_+, t_-)$, ie, the bounded subset in $\mathbf{R}^2$ that is bounded by the parabola P and the straight line $l(t_+, t_-)$.
Hint: Verify

$$S(t_+, t_-) = \{\, x \in \mathbf{R}^2 \mid t_- \le x_1 \le t_+,\ x_1^2 \le x_2 \le (t_0 - \delta)^2 + 2t_0(x_1 - t_-) \,\}.$$

Moreover you may use without proof $\frac{1}{3}(t_+^3 - t_-^3) = \frac{2}{3}\delta^3 + 2t_0^2\delta$.

The parts (i) and (iii) imply the *quadrature of the parabola* according to Archimedes: for all t_+ and $t_- \in \mathbf{R}$ with $t_+ > t_-$ we have

$$\mathrm{area}\,(S(t_+, t_-)) = \frac{4}{3}\,\mathrm{area}\,(\Delta(t_+, t_-)).$$

Next we give a direct proof of this resultat. To this end we define $\Psi : \mathbf{R}^2 \to \mathbf{R}^2$ by means of

$$\Psi(y) = \begin{pmatrix} t_0 + \quad \delta\, y_1 \\ t_0^2 + 2t_0\delta\, y_1 + \delta^2 y_2 \end{pmatrix} = \phi(t_0) + \delta \begin{pmatrix} 1 & 0 \\ 2t_0 & \delta \end{pmatrix} y.$$

(iv) Show that Ψ is a C^∞ diffeomorphism. Verify $\Psi \circ \phi(t) = \phi(t_0 + \delta\, t)$, for all $t \in \mathbf{R}$. Conclude that Ψ maps the parabola P into itself; and that the triangle $\Delta(t_+, t_-)$ in part (i) is the image under Ψ of the triangle with vertices $(1, 1)$, $(0, 0)$ and $(-1, 1)$.

(v) Prove $\det D\Psi(y) = \delta^3$, for every $y \in \mathbf{R}^2$. Deduce from part (iv) that the quadrature of the parabola is reduced to a special case.

Exercise 0.2 (Kissoid). Let $0 \leq y < \sqrt{2}$. Then the vertical line in $\mathbf{R}^2$ through the point $(2 - y^2,\, 0)$ has a unique point of intersection, say $\chi(y) \in \mathbf{R}^2$, with the circle segment

$$\{\, x \in \mathbf{R}^2 \mid \|x - (1,\, 0)\| = 1,\ x_2 \geq 0 \,\};$$

and we have $\chi(y) \neq (0,\, 0)$. The straight line through $(0,\, 0)$ and $\chi(y)$ intersects the vertical line through $(y^2,\, 0)$ in a unique point, say $\phi(y) \in \mathbf{R}^2$. (Make a drawing!)

(i) Verify that we have

$$\phi(y) = \Big(y^2,\ \frac{y^3}{\sqrt{2 - y^2}}\Big) \qquad (0 \leq y < \sqrt{2}).$$

(ii) Show that the mapping $\phi : \,]\, 0,\ \sqrt{2}\, [\, \to \mathbf{R}^2$ given by $y \mapsto \phi(y)$, is a C^∞ embedding.

The *kissoid of Diokles* is the set $V \subset \mathbf{R}^2$ defined by

$$V = \Big\{ \Big(y^2,\ \frac{y^3}{\sqrt{2 - y^2}}\Big) \in \mathbf{R}^2 \ \Big|\ -\sqrt{2} < y < \sqrt{2} \Big\}.$$

(iii) Prove that $V \setminus \{(0,0)\}$ is a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1.

(iv) Show

$$V = g^{-1}(\{0\}) \qquad \text{where} \qquad g(x) = x_1^3 + (x_1 - 2)x_2^2.$$

(v) Verify that $g : \mathbf{R}^2 \to \mathbf{R}$ is surjective, and moreover that, for every $c \in \mathbf{R} \setminus \{0\}$, the level set $g^{-1}(\{c\})$ is a C^∞ submanifold in $\mathbf{R}^2$ of dimension 1.

(vi) Using a Taylor expansion prove that V has an ordinary cusp in the point $(0,0)$, and deduce from this that V is not a C^1 submanifold in $\mathbf{R}^2$ of dimension 1 in $(0,0)$.

Exercise 0.3. Let $\Omega \subset \mathbf{R}^3$ be the bounded open subset bounded by:

the unit sphere	$\{\, x \in \mathbf{R}^3 \mid \|x\| = 1 \,\}$;
the cylinder	$\{\, x \in \mathbf{R}^3 \mid x_1^2 +_2^2 = 1 \,\}$;
the two horizontal planes	$\{\, x \in \mathbf{R}^3 \mid x_3 = h_i \,\} \quad (1 \leq i \leq 2)$,

where $-1 < h_1 < h_2 < 1$. Accordingly the boundary $\partial\Omega$ is the union of a set S_1 on the sphere, a set S_2 on the cylinder, and two plane pieces. It is our goal to show in two different fashions that the sets S_1 and S_2 have equal area.

(i) Prove this by application of Gauss' divergence Theorem to the vector field $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by:

$$f(x) = \Big(\frac{x_1}{x_1^2 + x_2^2},\ \frac{x_2}{x_1^2 + x_2^2},\ 0\Big).$$

(ii) Prove this also by direct computation of both areas.