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On Euler numbers, Hilbert sums, Lobachevskiĭ integrals, and their asymptotics

Dedicated to Tom Koornwinder on the occasion of his 60^{th} birthday

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1. INTRODUCTION.

The numbers mentioned in the title (see the formulae (5), (6) and (7) below for definitions) depend on an $n \in \mathbf{N}$ and can be associated with a central B-spline, specifically, the n-fold convolution of the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}]$ with itself. Therefore, their asymptotic behavior for $n \to \infty$ is an immediate consequence of the central limit theorem. Furthermore, define

$$(1) \qquad I(n,m) = \int_0^\infty \frac{\sin^n \lambda}{\lambda^m}\, d\lambda \qquad (n, m \in \mathbf{N},\, n - m \geq 0).$$

The asymptotics of the $I(n, n-k)$, for $n \to \infty$ and $k \in \{0\} \cup \mathbf{N}$, is determined and the $I(n,m)$, for all values of n and m, are explicitly computed by means of (rudimentary) distribution theory. For $n - m$ odd, these formulae seem to occur neither in the standard tables of integrals or integral transforms, nor in computer algebra packages like Mathematica (for large values of n and m).

Convolution. Denote by χ the characteristic function of the interval $[-\frac{1}{2}, \frac{1}{2}] \subset \mathbf{R}$, and by $\chi_n = \chi * \cdots * \chi$ the n-fold convolution product ($n \in \mathbf{N}$). Then χ_n is a spline function, viz. a C^{n-2}-function on $\mathbf{R}$ that is piecewise polynomial of degree $n-1$, for $n \geq 2$, having $[-\frac{n}{2}, \frac{n}{2}]$ as its support. Indeed, let $\chi_n^{(i)}$ ($0 \leq i \leq n$) be the i-th derivative of χ_n. We have the equality $\chi^{(1)} = \delta_{-\frac{1}{2}} - \delta_{\frac{1}{2}}$ of distributions on $\mathbf{R}$, where δ_x denotes the Dirac measure at $x \in \mathbf{R}$; whence

(2) $$\chi_n^{(n)} = \chi^{(1)} * \cdots * \chi^{(1)} = \sum_{0\le j\le n} (-1)^j \binom{n}{j} \delta_{-\frac{n}{2}+j}.$$

By $(n-i)$-fold integration, requiring the continuity of the $\chi_n^{(i)}$ for $0 \le i \le n-2$ and the vanishing of $\chi_n^{(n-1)}$ on $(-\infty, -\frac{n}{2})$, we find

(3) $$\chi_n^{(i)}(x) = \frac{1}{(n-1-i)!} \sum_{0\le j\le [x+\frac{n}{2}]} (-1)^j \binom{n}{j} \left(x+\frac{n}{2}-j\right)^{n-1-i} \qquad (0 \le i \le n-1).$$

Furthermore, observe that

(4) $$\chi_n^{(i)} = \chi_{n-i} * \chi_i^{(i)} = \chi_{n-i} * \sum_{0\le j\le i} (-1)^j \binom{i}{j} \delta_{-\frac{i}{2}+j}.$$

In particular, we find the Euler numbers $A(n-1,k)$ (cf. [1, OO pp. 419-421])

(5) $$(n-1)!\,\chi_n\left(\frac{n}{2}-k\right) = A(n-1,k) := \sum_{0\le j\le k} (-1)^j \binom{n}{j} (k-j)^{n-1}$$
$$(k \in \{0\} \cup \mathbf{N}).$$

χ_n is an even function; hence $\chi_n^{(2i+1)}(0) = 0$, for $0 \le 2i \le n-3$. Therefore (cf. [7, Lemma 3.4.7])

$$\sum_{0\le j\le [\frac{n}{2}]} (-1)^j \binom{n}{j} \left(\frac{n}{2}-j\right)^{n-2-2i} = 0 \qquad (3 \le n,\ 0 \le 2i \le n-3).$$

And, taking (3) with $i=2$ and $x=0$, we have the following sums, which Hilbert introduced in his work on invariant theory (cf. [4, §9]):

(6) $$H(n) := \sum_{0\le j\le [\frac{n}{2}]} (-1)^j \binom{n}{j} \left(\frac{n}{2}-j\right)^{n-3} = (n-3)!\,\chi_n^{(2)}(0) \qquad (3 \le n).$$

Notice that (4), with $i=2$, implies $H(n) = A(n-3, \frac{n}{2}-2) - 2A(n-3, \frac{n}{2}-1) + A(n-3, \frac{n}{2})$, for n even.

Fourier inversion. The Fourier transform of χ_n is $\lambda \mapsto \left((\sin\frac{1}{2}\lambda)/\frac{1}{2}\lambda\right)^n$. Using Fourier inversion we get the integrals of Lobachevskiĭ (cf. [5, p. 170])

(7) $$\int_0^\infty \frac{\sin^n \lambda}{\lambda^{n-2i}} \cos(2x\lambda)\,d\lambda = (-1)^i \frac{\pi}{2^{2i+1}} \chi_n^{(2i)}(x) \qquad (2 \le n,\ 2 \le n-2i,\ x \in \mathbf{R}).$$

In particular, with $n=2$, $i=0$, $x=0$, it follows that

(8) $$\frac{1}{\pi} \int_0^\infty \frac{1-\cos(2x\lambda)}{x^2}\,dx = \frac{2}{\pi} \int_0^\infty \frac{\sin^2(x\lambda)}{x^2}\,dx = \lambda.$$

Furthermore, we obtain from (7)

$$\frac{1}{\pi} \int_0^\infty \frac{\sin^n \lambda}{\lambda^{n-2i}}\, \frac{\cos(2x\lambda)-1}{x^2}\,d\lambda = \frac{(-1)^i}{2^{2i+1}} \frac{1}{x^2} \left(\chi_n^{(2i)}(x) - \chi_n^{(2i)}(0)\right).$$

Integrate this identity with respect to x from 0 to ∞, interchange the order of integration and use (8) and the evenness of $\chi_n^{(2i)}$. Then it follows that, in the notation of (1),

$$
(9) \qquad I(n, n-1-2i) = \frac{(-1)^{i+1}}{2^{2i+1}} \int_{-\infty}^{0} \frac{1}{x^2} \left(\chi_n^{(2i)}(x) - \chi_n^{(2i)}(0) \right) dx
$$
$$
(2 \leq n,\ 2 \leq n-1-2i).
$$

As to the restriction on i, notice that $\chi_n^{(2i)}$ is piecewise polynomial of degree $n-1-2i$. Suppose that $2i = n-2$. Then we see from (3) that $\chi_n^{(2i)} - \chi_n^{(2i)}(0)$ is piecewise linear and nonvanishing on $[-1, 0]$. Therefore the RHS, as is the LHS, in (9) is divergent in this case.

Asymptotics. The asymptotics of these sums and integrals is a direct consequence of the local form of the central limit theorem (cf. [6, Thm. VIII.2.1]). In our situation, this comes down to the following. Because $\int_{-\infty}^{\infty} \chi(x)\, dx = 1$, $\int_{-\infty}^{\infty} x\chi(x)\, dx = 0$ and $\int_{-\infty}^{\infty} x^2\chi(x)\, dx = \frac{1}{12}$, we have

$$
\lim_{n\to\infty} \left(\frac{n}{12}\right)^{\frac{1}{2}} \chi_n\left(x\sqrt{\tfrac{n}{12}}\right) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} \qquad (x \in \mathbf{R}).
$$

Even stronger, if we write $H_p(x)$ for the p-th Hermite polynomial $(-1)^p e^{x^2} E^{(p)}(x)$ where $E(x) = e^{-x^2}$, it follows that

$$
(10) \qquad \lim_{n\to\infty} \left(\frac{n}{6}\right)^{i+\frac{1}{2}} \chi_n^{(2i)}\left(x\sqrt{\tfrac{n}{6}}\right) = \frac{1}{\sqrt{\pi}} H_{2i}(x) e^{-x^2} \qquad (i \in \{0\} \cup \mathbf{N},\ x \in \mathbf{R}).
$$

In particular, for the Euler numbers we get, if $x \in \mathbf{R}$ and $\frac{n}{2} - x\sqrt{n} \in \mathbf{N}$,

$$
\lim_{n\to\infty} \frac{\sqrt{n}}{(n-1)!} A\left(n-1, \frac{n}{2} - x\sqrt{n}\right) = \sqrt{\frac{6}{\pi}} e^{-6x^2}.
$$

Moreover, from (10) with $x = 0$ we get Springer's result for the Hilbert sums (cf. [7, Lemma 3.4.7])

$$
\lim_{n\to\infty} \left(\frac{n}{12}\right)^{i+\frac{1}{2}} \frac{1}{(n-1-2i)!} \sum_{0 \leq j \leq [\frac{n}{2}]} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{n-1-2i} = (-1)^i \frac{1}{\sqrt{2\pi}} (2i-1)!!,
$$

with the notation $(2i-1)!! = (2i-1)(2i-3)\cdots 3 \cdot 1$; and furthermore (cf. [7, Lemma 3.4.11])

$$
(11) \qquad \lim_{n\to\infty} \left(\frac{n}{3}\right)^{i+\frac{1}{2}} I(n, n-2i) = \sqrt{\frac{\pi}{2}} (2i-1)!!.
$$

Actually, we have the following generalization of (11):

$$
(12) \qquad \lim_{n\to\infty} \left(\frac{n}{6}\right)^{\frac{k+1}{2}} I(n, n-k) = \frac{1}{2} \Gamma\left(\frac{k+1}{2}\right) \qquad (k \in \{0\} \cup \mathbf{N}).
$$

Proof of (12). Obviously the case of $k = 2i$ is covered by (11). Now suppose $k = 2i + 1$. From (9) we obtain

$$I(n, n-1-2i) = \frac{(-1)^{i+1}}{2^{2i+1}} \left(\frac{n}{6}\right)^{-\frac{1}{2}} \int_0^\infty \frac{1}{x^2} \left(\chi_n^{(2i)}(x\sqrt{\tfrac{n}{6}}) - \chi_n^{(2i)}(0)\right) dx$$
$$(0 \le 2i \le n-3).$$

But this equality, taken in conjunction with (10), implies

$$(13) \quad \lim_{n\to\infty} \left(\frac{n}{6}\right)^{i+1} I(n, n-1-2i) = \frac{(-1)^{i+1}}{2^{2i+1}} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{x^2} (H_{2i}(x)e^{-x^2} - H_{2i}(0))\, dx.$$

Using induction over i, integration by parts, the recurrences $H_{2i}^{(1)} = 4iH_{2i-1}$ and $H_{2i+1}(x) = 2xH_{2i}(x) - 4iH_{2i-1}(x)$, and the orthogonality relations for the Hermite polynomials one easily verifies the following formulae:

$$\int_0^\infty \frac{1}{x^2} (H_{2i}(x)e^{-x^2} - H_{2i}(0))\, dx = (-1)^{i+1} 4^i i! \sqrt{\pi} \qquad (i \in \{0\} \cup \mathbf{N}),$$
$$\int_0^\infty \frac{1}{x} H_{2i-1}(x) e^{-x^2}\, dx = (-1)^{i-1} 4^{i-1} (i-1)! \sqrt{\pi} \qquad (i \in \mathbf{N}).$$

Therefore the RHS in (13) becomes $\frac{1}{2}\Gamma(i+1)$; that is, (12) is also valid for $k = 2i + 1$.

Evaluation of the integrals $I(n,m)$. These have the following values (see [2] and [3] for a completely different derivation).

(i) Case of $n - m$ even:

$$\text{(i.a)} \quad m = 1, \quad I(2k+1, 1) = \frac{\pi}{2^{k+1}} \frac{(2k-1)!!}{k!} \qquad (k \in \{0\} \cup \mathbf{N}),$$
$$\text{(i.b)} \quad m = 2, \quad I(2k, 2) = \frac{\pi}{2^k} \frac{(2k-3)!!}{(k-1)!} \qquad (k \in \mathbf{N}),$$
$$\text{(i.c)} \quad m > 2, \quad I(n,m) = \frac{\pi(-1)^{\frac{n-m}{2}}}{2^{n-m+1}(m-1)!} \sum_{0 \le j \le \frac{n-1}{2}} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{m-1}.$$

(ii) Case of $n - m$ odd:

$$\text{(ii.a)} \quad m = 1, \quad I(2k, 1) = \infty \qquad (k \in \mathbf{N}),$$
$$\text{(ii.b)} \quad m > 1,\ n \text{ even}, \quad I(n,m) = \frac{(-1)^{\frac{n-m-1}{2}}}{2^{n-m}(m-1)!} \sum_{0 \le j \le \frac{n-4}{2}} (-1)^j \binom{n}{j} \left(\frac{n}{2} - j\right)^{m-1} \log\left(\frac{n}{2} - j\right),$$
$$\text{(ii.c)} \quad m > 1,\ n \text{ odd}, \quad I(n,m) = \frac{(-1)^{\frac{n-m-1}{2}}}{2^{n-1}(m-1)!} \sum_{0 \le j \le \frac{n-3}{2}} (-1)^j \binom{n}{j} (n-2j)^{m-1} \log(n-2j).$$

Proof of (i). For (i.a), notice that χ_{2k+1} is a polynomial of degree $2k$ on $[-\frac{1}{2}, \frac{1}{2}]$. Hence in (7) we are allowed to replace n by $2k+1$ and to choose $i = k$ and $x = 0$. We simplify the answer using the binomial identities. Now (i.b). Similarly we can replace n by $2k$, and choose $i = k - 1$ and $x = 0$. Finally (i.c) is immediate from (7).

Proof of (ii). Case (ii.a) has been discussed already. In view of (9) we compute

$$\int_{-\infty}^{0} \frac{1}{x^2}\left(\chi_n^{(n-m-1)}(x) - \chi_n^{(n-m-1)}(0)\right) dx$$
$$= -\frac{2}{n}\chi_n^{(n-m-1)}(0) + \int_{-\frac{n}{2}}^{0} \frac{1}{x^2}\left(\chi_n^{(n-m-1)}(x) - \chi_n^{(n-m-1)}(0)\right) dx.$$

Integrate by parts twice and use that $\chi_n^{(n-m)}(0) = \chi_n^{(n-m-1)}(-\frac{n}{2}) = \chi_n^{(n-m)}(-\frac{n}{2}) = 0$. The integral above takes the form $-\int_{-\frac{n}{2}}^{0} \chi_n^{(n-m+1)}(x)\log(-x)\,dx$. Next, denote the p-fold antiderivative of log by $\log^{[p]}$; then there exist constants $c(p) \in \mathbf{R}$ such that $\log^{[p]}(-x) = \frac{1}{p!}x^p \log(-x) + c(p)\,x^p$. In particular, $\log^{[p]}(0) = 0$, for $p \in \mathbf{N}$. Since $\chi_n^{(i)}(-\frac{n}{2}) = 0$, for $0 \le i \le n-2$, and the resulting integrals are convergent, we can continue integrating by parts, viz. $m-2$ many times; and we obtain $(-1)^{m-1}\int_{-\frac{n}{2}}^{0} \chi_n^{(n-1)}(x)\log^{[m-2]}(-x)\,dx$. Since $-\frac{n}{2} + [\frac{n-1}{2}] = -1$ or $-\frac{1}{2}$, we get in view of (2) after one more integration by parts

$$(-1)^m \sum_{0\le j\le[\frac{n-1}{2}]} (-1)^j \binom{n}{j} \log^{[m-1]}\left(-\left(-\frac{n}{2}+j\right)\right)$$
$$= \frac{-1}{(m-1)!}\sum_{0\le j\le[\frac{n-1}{2}]} (-1)^j \binom{n}{j}\left(\frac{n}{2}-j\right)^{m-1}\log\left(\frac{n}{2}-j\right) - c(m-1)\sum_{0\le j\le[\frac{n-1}{2}]} (-1)^j \binom{n}{j}\left(\frac{n}{2}-j\right)^{m-1}.$$

But the last sum is equal to $(m-1)!\chi_n^{(n-m)}(0) = 0$. The computation of the $I(n,m)$ now can be completed by combining the results above.

Remarks. The formulae in the section above also can be obtained by means of complex analysis. The author is grateful to J.J. Duistermaat and T.A. Springer for discussions concerning this manuscript.

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