MRI SPRING SCHOOL 2004
Lie Groups in Analysis, Geometry and Mechanics

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Literature

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  * Recommended to readers interested in applications in algebra and algebraic geometry.
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  * Nicely written. A modern treatment of H. Weyl's classic: *The Classical Groups, Their Invariants and Representations*. Princeton University Press 1939. May be read at different levels of difficulty.
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  * Excellent text for a first introduction as it restricts itself to matrix groups.
- Simon, B.: *Representations of finite and compact groups*. Graduate Studies in Math., Volume 10. Amer. Math. Soc.1996
  * The theory from an analyst's perspective.
- Varadarajan, V.S.: *Lie Groups, Lie Algebras, and their Representations*. 2nd printing. Springer-Verlag 1984
  * The classical reference. Many exercises with results from the literature.

GROUP ACTIONS

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(1)	Th	June	3	[LG]: Sections 1.11, 2.1 - 2.5
(2)	Fr	June	4	[LG]: Sections 2.6 - 2.8
(3)	Mo	June	7	[LG]: Sections 3.0 - 3.4

(1)	Quotients, Proper group actions, Bochner's Linearization Theorem, Slices, Associated fiber bundles, Smooth functions on the orbit space
(2)	Orbit types and local action types, The stratification by orbit types, Principal and regular orbits
(3)	Introduction, Centralizers, The adjoint action, Connectedness of stabilizers, The group of rotations and its covering group

For smooth actions of a compact Lie group on a smooth manifold, or slightly more generally for proper Lie group actions, very detailed descriptions can be given of the orbits, the action in a suitable invariant neighborhood of an orbit, the decomposition of the manifold into orbit types, the space of invariant smooth functions, and various other aspects of the action.

The local action, in a suitable invariant neighborhood of any given point, is isomorphic to the action on associated vector bundles, which is described entirely in terms of the Lie group itself, the stabilizer subgroup of the point, and a linear representation of this stabilizer subgroup. It is a theorem of Schwarz that the algebra of smooth functions which are invariant under such a linear representation is generated by finitely many homogeneous polynomials. This leads to a local description of the orbit space as a semi-algebraic variety with a conic singularity. These features of actions will be used in the course *Symmetry in Mechanics*.

Particular examples are the actions by conjugation of a compact Lie group on itself and on its Lie algebra. The theory then leads to the structure theory of compact Lie groups and their Lie algebras, including the description of the maximal tori, roots and root spaces, the Weyl group and Weyl's integration theorem. See also the course *Structure Theory of Lie Groups and Lie Algebras*. Another example arises when the stabilizer group is trivial. When the orbit space is smooth we have a principal fiber bundle, which appears in the course *Analysis on Principal Fiber Bundles*.

There is a close relation between proper smooth actions and algebraic actions of reductive complex algebraic groups on complex affine varieties, but there will probably not be enough time to work this out in detail.

0.1 Group Actions: First Lecture

0.1.1 Group actions

Definition 0.1.2. An *action* of a group G on a set M is a homomorphism A from G to the group of bijective mappings: $M \to M$. Writing

$$A(g)(x) = A(g, x) = g \cdot x \qquad (g \in G, x \in M), \tag{1}$$

we can also describe the action of G on M as a mapping $A : G \times M \to M$ such that

$$A(gh, x) = A\big(g, A(h, x)\big) \qquad (g, h \in G, x \in M). \tag{2}$$

If G is a Lie group and M a C^k manifold, with $1 \leq k \leq \omega$, then a C^k *action* of G on M is an action A that is C^k as a mapping: $G \times M \to M$. For each $x \in M$, the *orbit through* x for the action A is defined as the set

$$A(G)(x) = G \cdot x = \{\, A(g)(x) \mid g \in G \,\} \subset M. \tag{3}$$

❍

The relation $y \in G \cdot x$ is an equivalence relation in M, which partitions M into orbits. If R is an equivalence relation in a topological space M, we denote by M/R the set of equivalence classes and by $\pi : M \to M/R$ the canonical projection which assigns to each $x \in M$ its equivalence class $\{\, y \in M \mid (x, y) \in R \,\}$. Defining a subset V of M/R to be open if and only if $\pi^{-1}(V)$ is open in M, we get a topology on M/R for which π is continuous. Actually this is the strongest topology on M/R with this property, and it is called the *quotient topology* on M/R. The quotient topology of M/R has the Hausdorff property if and only if R is a closed subset of $M \times M$.

The collection of orbits is called the *set-theoretic quotient* $G\backslash M$ of M under the action A of G on M. The surjective mapping

$$\pi : x \mapsto G \cdot x : M \to G\backslash M$$

is called the *canonical projection.* Even for analytic actions of Lie groups on analytic manifolds it may happen that the quotient $G\backslash M$ cannot be provided with the structure of a Hausdorff topological space making the canonical projection continuous. In certain algebraic situations one avoids this by taking as a quotient the space of closed orbits. However, in this section we will restrict ourselves to actions which are so nice that the set-theoretic quotient space can be provided with the structure of a smooth manifold, making the canonical projection into a smooth fibration, see Theorem 0.1.6 below.

Exercise 0.1.3. The action $(t, (x_1, x_2)) \mapsto (x_1 + t x_2, x_2)$ of $(\mathbf{R}, +)$ on $\mathbf{R}^2$ is an example with all orbits being closed. For each $x_1,\ x_1' \in \mathbf{R}$, any two invariant neighborhoods of $(x_1, 0)$ and $(x_1', 0)$ intersect each other, making the quotient topology a non-Hausdorff topology.

Definition 0.1.4. The action is said to be *transitive* if M is an orbit, that is, if there is an $x \in M$ such that for each $y \in M$ we have $y = g \cdot x$, for some $g \in G$. In this case M is also called a *homogeneous space.*

A subset N of M is said to be *invariant under the action* A or G*-invariant* if $g \cdot y \in N$, whenever $g \in G$, $y \in N$. That is, $A(G \times N) \subset N$; then the restriction of A to $G \times N$ defines an action of G on N, called the *restriction to* N *of the action* of G on M.

For each $x \in M$,

$$G_x = \{ g \in G \mid g \cdot x = x \}$$

is clearly a subgroup of G, called the *isotropy group* of the action at the point x, or the *stabilizer* of x under the action. The mapping

$$A_x : g \mapsto A(g)(x) : G \to G \cdot x \tag{4}$$

is surjective, by the definition of $G \cdot x$. On the other hand, $A(h)(x) = A(g)(x)$ if and only if $g^{-1}h \in G_x$, that is, $h \in gG_x$. This shows that A_x is injective if and only if $G_x = \{1\}$; in this case the action is said to be *free at* x. The action is said to be *free* if it is free at x, for every $x \in M$, that is, if each orbit is in a bijective correspondence with G by means of the mappings A_x, with $x \in M$. ❍

Let H be any subgroup of G. Then

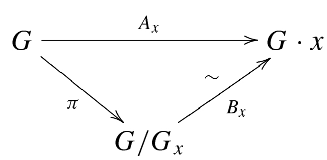
$$h \mapsto \mathrm{L}(h) : x \mapsto hx, \qquad \text{and} \qquad h \mapsto \mathrm{R}(h)^{-1} : x \mapsto xh^{-1}, \tag{5}$$

define a free action of H on G, called the *action from the left*, and *the right*, respectively, of H on G. The orbits are the Hx (that is, the right cosets equal the right translates of H), and the xH, respectively, with $x \in G$ (that is, the left cosets equal the left translates of H in G). The corresponding quotient spaces are denoted by, respectively,

$$H \backslash G = \{ Hx \mid x \in G \} \qquad \text{and} \qquad G/H = \{ xH \mid x \in G \}.$$

The remark following (4) shows that the fibers of $A_x : G \to G \cdot x$ are just the cosets gG_x, for $g \in G$; and there is a unique bijective mapping

$$B_x : G/G_x \xrightarrow{\sim} G \cdot x, \tag{6}$$



such that $A_x = B_x \circ \pi$, if π denotes the canonical projection: $G \to G/G_x$.

Exercise 0.1.5. As a further exercise with the definitions, we note that, for any subgroup H of G,

$$g \mapsto (xH \mapsto gxH) \qquad (g, x \in G),$$

defines a transitive action of G on G/H, with isotropy group at $1H$ equal to H. (The isotropy group at $xH \in G/H$ is equal to xHx^{-1}.) This shows that every subgroup H of G is equal to the isotropy group, at some point, of some action of G.

Finally, we recall that the product

$$(xH) \cdot (yH) = (xy)H \qquad (x, y \in G),$$

in G/H is well-defined if and only if H is a *normal subgroup* of G, that is,

$$xHx^{-1} = H \qquad (x \in G).$$

This makes $G/H = H \backslash G$ into a group. The canonical projection $G \to G/H$ is a group homomorphism with kernel equal to H. Because the kernel of every group homomorphism is a normal subgroup, this shows that the normal subgroups are precisely the kernels of group homomorphisms from G to some other group G'. In Corollary 0.1.10 we shall see that if G is a Lie group, then the closed normal subgroups are precisely the kernels of homomorphisms of Lie groups $G \to G'$, with G' some other Lie group.

Until now we have only discussed some set-theoretical generalities about group actions. We now turn to their topological and smoothness properties. Recall that a continuous mapping Φ from a topological space U to a topological space V is said to be *proper* if $\Phi^{-1}(K)$ is compact in U for every compact subset K of V. If U and V are Hausdorff, this implies that $\Phi(C)$ is closed in V for every closed subset C of U. A continuous action of a topological group G on a topological space M is said to be a *proper action* if

$$(g, x) \mapsto (g \cdot x, x) \quad \text{is a proper mapping } : G \times M \to M \times M.$$

For a proper continuous action, the quotient topology on the orbit space has the Hausdorff property.

Theorem 0.1.6. *Let G be a Lie group, M a C^k manifold, for $k \geq 1$, and A a C^k action of G on M that is* **proper and free**. *Then the orbit space $G\backslash M$ has a unique structure of a C^k manifold, of dimension equal to $\dim M - \dim G$, with the following properties. If $\pi : M \to G\backslash M$ is the canonical projection $x \mapsto G \cdot x$, then for every point in $G\backslash M$ there is an open neighborhood S in $G\backslash M$ and a C^k diffeomorphism*

$$\tau : x \mapsto (\chi(x), s(x)) : \pi^{-1}(S) \to G \times S, \qquad \begin{array}{ccccc} M & \hookleftarrow & \pi^{-1}(S) & \xrightarrow{\tau} & G \times S \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \\ G\backslash M & \hookleftarrow & S & & \end{array}$$

such that, for $x \in \pi^{-1}(S)$, $g \in G$, we have $s(x) = \pi(x) = G \cdot x$, so

$$\begin{array}{ccc} x & \xmapsto{\tau} & (\chi(x), G \cdot x) \\ {\scriptstyle\pi}\downarrow & \swarrow & \\ G \cdot x & & \end{array} \quad \text{and} \quad \tau(g \cdot x) = (g\chi(x), s(x)), \quad \text{i.e.} \quad \begin{array}{ccc} g \cdot x & \xmapsto{\tau} & (g\chi(x), G \cdot x) \\ {\scriptstyle\pi}\downarrow & \swarrow & \\ G \cdot x & & \end{array}.$$

The topology of $G\backslash M$ is equal to the quotient topology.

The τ in Theorem 0.1.6 are local trivializations making $\pi : M \to G\backslash M$ into a C^k fiber bundle; they also translate the action of G on M into the left action of G on the first component in $G \times S$. If $\sigma : \pi^{-1}(T) \to G \times T$ is another such trivialization then we have

$$\tau \circ \sigma^{-1} : (g, x) \mapsto (g\phi(x), x) : G \times (S \cap T) \to G \times (S \cap T), \tag{7}$$

for a C^k function $\phi : S \cap T \to G$. A fiber bundle M with a Lie group G as fiber and retrivializations $\tau \circ \sigma^{-1}$ as in (7), is called a *principal fiber bundle with structure group G*. One verifies easily that the left G-actions on the first factor in the trivializations $G \times S$ induce an action of G on the principal fiber bundle that is proper and free, with the fibers as the orbits. In this way having a proper and free action of G on M is equivalent to saying that M is a principal fiber bundle with structure group G.

Below, we shall drop the hypothesis that the action is free and investigate general proper actions, with applications to the structure of compact Lie groups. Later, in the study of noncompact Lie groups, we shall also have to deal with Lie group actions which are neither proper nor free.

In the following we fix some notation concerning Lie groups.

Definition 0.1.7. Let G be a Lie group. In order to detect a possible noncommutativity of G on the infinitesimal level at 1 (that is, in terms of finitely many derivatives at 1), we have to turn to second-order derivatives at 1. This can be done as follows. Write, for each $x \in G$,

$$\mathbf{Ad}\, x : y \mapsto xyx^{-1}, \tag{8}$$

for the *conjugation* by x in the group G. Noncommutativity of G means that $\mathbf{Ad}\, x$ is not equal to the identity: $G \to G$, for each $x \in G$. Because $(\mathbf{Ad}\, x)(1) = 1$, the tangent mapping of $\mathbf{Ad}\, x$ at 1 is a linear mapping

$$\mathrm{Ad}\, x := \mathrm{T}_1(\mathbf{Ad}\, x) : \mathfrak{g} \to \mathfrak{g},$$

called the *adjoint mapping* of x, or the *infinitesimal conjugation* by x in $\mathfrak{g}$. Because

$$\mathbf{Ad}(ab) = (\mathbf{Ad}\, a) \circ (\mathbf{Ad}\, b) \qquad (a, b \in G),$$

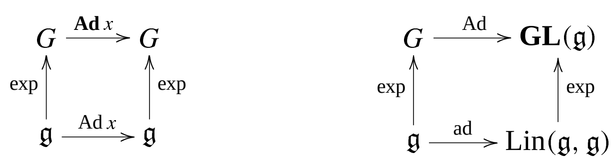
an application of the chain rule for tangent mappings shows that

$$\mathrm{Ad}(ab) = (\mathrm{Ad}\, a) \circ (\mathrm{Ad}\, b) \qquad (a, b \in G),$$

as well. That is, the mapping

$$\mathrm{Ad} : x \mapsto \mathrm{Ad}\, x : G \to \mathbf{GL}(\mathfrak{g})$$

is a homomorphism of groups; it is called the *adjoint representation* of G in $\mathfrak{g} = \mathrm{T}_1\, G$.



The next step is to observe that $\mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g})$ is a C^1 mapping. So again we can take the tangent mapping at 1, and we obtain a linear mapping

$$\mathrm{ad} := \mathrm{T}_1\, \mathrm{Ad} : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{g}),$$

if we identify the tangent space at I of the open subset $\mathbf{GL}(\mathfrak{g})$ of $\mathrm{Lin}(\mathfrak{g}, \mathfrak{g})$ with $\mathrm{Lin}(\mathfrak{g}, \mathfrak{g})$, as is usual. For each $X, Y \in \mathfrak{g}$,

$$[\, X, Y\,] := (\mathrm{ad}\, X)(Y) \in \mathfrak{g} \tag{9}$$

is called the *Lie bracket* of X and Y. The condition that ad is a linear mapping from $\mathfrak{g}$ into $\mathrm{Lin}(\mathfrak{g}, \mathfrak{g})$ just means that

$$(X, Y) \mapsto [\, X, Y\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is a bilinear mapping; hence it can be viewed as a product structure turning $\mathfrak{g}$ into an algebra over $\mathbf{R}$.

The tangent space $\mathfrak{g} = \mathrm{T}_1\, G$, provided with the Lie bracket (9) as the product structure, is called the *Lie algebra* of the Lie group G. ❍

Exercise 0.1.8. In the example $G = \mathbf{GL}(V)$ of the general linear group, we get

$$\mathfrak{g} = \mathrm{T}_{\mathrm{I}}\, \mathbf{GL}(V) = \mathrm{Lin}(V, V),$$

$$\mathrm{Ad}\, x : Y \mapsto x \circ Y \circ x^{-1} \qquad (x \in \mathbf{GL}(V),\ Y \in \mathrm{Lin}(V, V)). \tag{10}$$

So in this case $\mathbf{Ad}\, x$ is just the restriction of $\mathrm{Ad}\, x$ to the open subset $\mathbf{GL}(V)$ of $\mathrm{Lin}(V, V)$. Anyhow, differentiating the right hand side in (10) with respect to x, at $x = \mathrm{I}$ and in the direction of $X \in \mathrm{Lin}(V, V)$, we get

$$[\, X, Y\,] = (\mathrm{ad}\, X)(Y) = X \circ Y - Y \circ X,$$

the *commutator* of X and Y in $\mathrm{Lin}(V, V)$.

Example 0.1.9. For any Lie group G, the action $(x, X) \mapsto (\mathrm{Ad}\, x)(X) : G \times \mathfrak{g} \to \mathfrak{g}$ of G on $\mathfrak{g}$ is called the *adjoint action*. Lie group actions for which the transformations are linear transformations in a vector space, are called *representations*. This is the reason why the adjoint action is also called the adjoint representation. Let us take a look at the simplest examples. For $G = \mathbf{SO}(3)$ or $\mathbf{SU}(2)$, the orbits of the adjoint action are either points or two-dimensional spheres, the stabilizer groups G_x then are either the whole group G or a circle subgroup $T = S^1$ of G. It follows that $G \to G/T$ exhibits $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ as a principal circle bundle over the two-dimensional sphere S^2, both being nontrivial, that is, not equivalent to a Cartesian product. It is interesting to identify these bundles in the classification of all principal circle bundles over S^2, see Steenrod [1951]. For $G = \mathbf{SU}(2)$, we get the three-dimensional sphere S^3 as a principal circle bundle over S^2, which is the same as the action of the circle $\{ z \in \mathbf{C} \mid |z| = 1 \}$ as a multiplicative subgroup of $\mathbf{C} \setminus \{0\}$ on the unit sphere in $\mathbf{C}^2 \simeq \mathbf{R}^4$ by means of the complex multiplication by the scalars z. In differential topology this is known as the *Hopf fibration*; it occurs implicitly at many places in the classical literature. For $G = \mathbf{SL}(2, \mathbf{R})$, the adjoint orbits are diffeomorphic to a point, the punctured plane (a one-sheeted hyperboloid or a punctured half-cone) or the plane (a sheet of the two-sheeted hyperboloid). In this case the stabilizer groups are either G itself or $\mathbf{R} \times (\mathbf{Z}/2\mathbf{Z})$, or the circle. The circle fibration of $\mathbf{SL}(2, \mathbf{R})$ over the plane is trivial, but the fibration of $\mathbf{SL}(2, \mathbf{R})$ over the punctured plane is not: $\mathbf{SL}(2, \mathbf{R})$ is connected and the fibers are not connected. ☆

For a Lie group G, the mapping $(g, x) \mapsto (gx, x)$ is bijective: $G \times G \to G \times G$. Since the inverse $(y, x) \mapsto (yx^{-1}, x)$ is continuous, it follows that the left action of G on G is proper and free. Similarly the right action of G on G is proper and free. If H is a closed subgroup of G then it follows immediately that the left and the right action of H on G are proper and free as well. Because a closed subgroup of a Lie group is a Lie subgroup we therefore obtain the following:

Corollary 0.1.10. *For any closed subgroup H of a Lie group G, there is a unique structure of an analytic manifold on $H \backslash G$, and G/H, making $G \to H \backslash G$, and $G \to G/H$, respectively, into an analytic principal fiber bundle with structure group H. The right, and left action, of G induces an analytic transitive action of G on $H \backslash G$, and G/H, respectively, with $G_{1H} = H$. Finally, if H is also a normal subgroup, then this analytic structure makes G/H into a Lie group and the canonical projection into a homomorphism of Lie groups.*

For a general C^k action of a Lie group G on a C^k manifold M, we can apply this to $H = G_x$, which is a Lie subgroup of G. Turning to the right action of G_x on G, we then see that the mapping B_x of (6) is a C^k immersion from the analytic manifold G/G_x into M, mapping G/G_x bijectively onto $G \cdot x$. This exhibits the orbit $G \cdot x$ as an immersed C^k submanifold of M, of dimension equal to $\dim G - \dim G_x$. Note also that the Lie algebra of G_x is equal to

$$\mathrm{T}_1(G_x) = \mathfrak{g}_x = \ker\, \mathrm{T}_1\, A_x, \qquad \text{whereas} \qquad \mathrm{T}_x(G \cdot x) = \mathrm{im}\, \mathrm{T}_1\, A_x \xleftarrow{\sim} \mathfrak{g}/\mathfrak{g}_x.$$

Especially, a C^k homogeneous space is C^k diffeomorphic to an analytic manifold of the form G/H where H is a closed Lie subgroup of a Lie group G; this analytic structure is the unique one for which the action is analytic.

0.1.11 Proper group actions

It is the main purpose of this section to show that for proper C^k actions of Lie groups on manifolds, a quite detailed description of the orbit structure can be given; in particular, the orbit space is a locally finite union of C^k manifolds, pieced together in a nice way.

We now return to the general C^k action of G on M. Then G_x, the isotropy group at x, is a closed, and therefore a Lie subgroup of G. The mapping $A_x : G \to M$ induces a bijective mapping $B_x : G/G_x \to G \cdot x$; furthermore, B_x intertwines (cf. Definition 0.1.16 below) the left action of G on G/G_x with the action of G on $G \cdot x$, that is, $A(g)|_{G \cdot x} = B_x \circ \mathrm{L}_{G/G_x}(g) \circ B_x^{-1}$, for every $g \in G$.

$$\begin{array}{ccc} G/G_x & \xrightarrow{\mathrm{L}_{G/G_x}(g)} & G/G_x \\ {\scriptstyle B_x}\big\downarrow & & \big\downarrow{\scriptstyle B_x} \\ G \cdot x & \xrightarrow{A(g)} & G \cdot x \end{array}$$

In particular, each transitive action can be identified with the left action of G on G/H for some closed Lie subgroup H of G; this fact reduces the theory of transitive actions to the structure theory of Lie groups.

Write $\alpha_x = \mathrm{T}_1 A_x : \mathfrak{g} \to \mathrm{T}_x M$ for the *infinitesimal action* at x, then the Lie algebra of G_x is equal to $\mathfrak{g}_x = \ker \alpha_x$. We conclude this introduction with the following general description, which however is local both in M and in G:

Lemma 0.1.12. *Let A be a C^k action ($k \geq 1$) of the Lie group G on the manifold M. For $x_0 \in M$, let S be a C^k submanifold of M through x_0 such that*

$$\mathrm{T}_{x_0} M = \alpha_{x_0}(\mathfrak{g}) \oplus \mathrm{T}_{x_0} S, \tag{11}$$

and let C be a C^k submanifold of G through 1 such that

$$\mathfrak{g} = \mathfrak{g}_{x_0} \oplus \mathrm{T}_1 C.$$

Then there is an open neighborhood C_0, and S_0, of 1, and x_0, in C, and S, respectively, such that $A_0 = A|_{C_0 \times S_0}$ is a C^k diffeomorphism from $C_0 \times S_0$ onto an open neighborhood M_0 of x_0 in M.

Remark 0.1.13. The set $\alpha_{x_0}(\mathfrak{g})$ is equal to the tangent space at x_0 of the orbit $G \cdot x_0$ through x_0, the latter viewed as an immersed submanifold of M. Condition (11) says that S intersects $G \cdot x_0$ transversally and has complementary dimension.

The mapping $\pi_2 \circ A_0^{-1} : M_0 \to S_0$, where π_2 is the projection $C_0 \times S_0 \to S_0$ onto the second factor, is a (trivial) C^k fibration, whose fibers are submanifolds of orbits. To be precise, $c \mapsto c \cdot s$ is a C^k diffeomorphism from C_0 onto the fiber over $s \in S_0$. In particular, all (local) orbits of neighborhoods of x_0, as in Lemma 0.1.12 intersect S_0 near x_0 transversally. In the next section it will be shown that, if the action is proper at x_0, then S_0 can be chosen such that the orbits near x_0 intersect S_0 in orbits for the action of G_{x_0}, the isotropy group at x_0. However, in general such a nice description of the intersections of the nearby orbits with S_0 is not possible.

The nearby local orbits intersect S_0 in isolated points if and only if $\dim \mathfrak{g}_x = \dim \mathfrak{g}_{x_0}$ for all x near x_0. Note that $\mathfrak{g}_{x_0} = \{ X \in \mathfrak{g} \mid \alpha_{x_0}(X) = 0 \}$ implies that $\dim \mathfrak{g}_x \leq \dim \mathfrak{g}_{x_0}$ for all x near x_0. In the special case that $\mathfrak{g}_{x_0} = 0$, that is, if the action is infinitesimally (locally) free at x_0, then C_0 is an open neighborhood of 1 in G. In the local identification of M with $C_0 \times S_0$, the local action of $g \in G$ then consists of left multiplication by G only on the first factor. Theorem 0.1.6 is a global version of this, but it needs the much stronger assumption of a globally free and proper action.

It is also clear that Lemma 0.1.12 remains true for locally defined actions of a local Lie group; this in turn is given by its infinitesimal action, which is a finite-dimensional Lie algebra $\mathfrak{g}$ of vector fields on M. If $G = (\mathbf{R}, +)$, and $\dim \mathfrak{g} = 1$, respectively, the action is the flow of a vector field X on M. The condition that $\mathfrak{g}_{x_0} = 0$ means that $X(x_0) \neq 0$, and Lemma 0.1.12 is the "flow box theorem" stating that in suitable local coordinates the flow after time t is equal to the translation $(x_1, x_2, \dots, x_n) \mapsto (x_1 + t, x_2, \dots, x_n)$ over t in the first variable. In this situation the manifold S is also called a *local Poincaré section* for the vector field X; condition (11) then just expresses that

$$\dim S = \dim M - 1 \quad \text{and} \quad X(x_0) \notin \mathrm{T}_{x_0} S.$$

✰

0.1.14 Bochner's Linearization Theorem

Let M be a real-analytic manifold and K a compact topological group, acting continuously on M by means of C^k transformations ($1 \leq k \leq \omega$). It may be proved that this implies that the action A is a continuous homomorphism from K to the topological group $\mathrm{Diff}^k(M)$ of C^k diffeomorphisms of M. The following theorem says that, near a fixed point, the action of K can be identified with the linear action of a closed Lie subgroup of the orthogonal group, acting on a ball centered at the origin in a Euclidean space.

Theorem 0.1.15 (Bochner's Linearization Theorem). *Let A be a continuous homomorphism from a compact group K to $\mathrm{Diff}^k(M)$, with $k \geq 1$ and let $x_0 \in M$, $A(k)(x_0) = x_0$, for all $k \in K$. Then there exists a K-invariant open neighborhood U of x_0 in M and a C^k diffeomorphism χ from U onto an open neighborhood V of 0 in $\mathrm{T}_{x_0} M$, such that*

$$\chi(x_0) = 0, \quad \mathrm{T}_{x_0} \chi = \mathrm{I} : \mathrm{T}_{x_0} M \to \mathrm{T}_{x_0} M$$

and

$$\chi(A(k)(x)) = \mathrm{T}_{x_0} A(k) \chi(x) \quad (k \in K,\ x \in U), \qquad \textit{i.e.} \qquad \begin{array}{ccccc} U & \xrightarrow{A(k)} & U & \hookrightarrow & M \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} & & \\ V & \xrightarrow[T_{x_0} A(k)]{} & V & \hookrightarrow & \mathrm{T}_{x_0} M \end{array} \tag{12}$$

If g is an arbitrary inner product on $\mathrm{T}_{x_0} M$, then

$$\bar{g} = \int_K \mathrm{T}_{x_0} A(k)^* g \, dk$$

is an inner product on $\mathrm{T}_{x_0} M$ that is invariant under the tangent action of K on $\mathrm{T}_{x_0} M$. In other words,

$$K' = \{ \mathrm{T}_{x_0} A(k) \mid k \in K \}$$

is a compact, and hence closed Lie subgroup of the orthogonal group of the Euclidean space $E = (\mathrm{T}_{x_0} M, \bar{g})$. Let B be an open ball around 0 in E that is contained in V, with V as in the Theorem above. Then B is K'-invariant, so, in view of (12), $\chi^{-1}(B)$ is a K-invariant open neighborhood of x_0 in U on which K acts as described in the sentence preceding the theorem.

Definition 0.1.16. If A, and B, are actions of a group G on a space X, and Y, respectively, then one says that a mapping $\Phi : X \to Y$ *intertwines* A with B, or is G-equivariant: $X \to Y$, if

$$\Phi \circ A(g) = B(g) \circ \Phi \quad (g \in G), \qquad \text{that is,} \qquad \begin{array}{ccc} X & \xrightarrow{A(g)} & X \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ Y & \xrightarrow{B(g)} & Y \end{array} \tag{13}$$

This means that $A(g)$, for each $g \in G$, maps each fiber of Φ onto a fiber of Φ; one also sometimes says that the action A *covers* the action B with respect to the mapping Φ.

If G is a Lie group and A, and B, is a C^k action of G on the manifold X, and Y, respectively, then Φ is an *equivalence of* C^k *actions* if Φ is a C^k diffeomorphism: $X \to Y$, intertwining A with B; the actions A and B are said to be C^k *equivalent* if there exists an equivalence of C^k actions between A and B. ❍

In this terminology, Theorem 0.1.15 says that the action of K, restricted to a suitable K-invariant open neighborhood U in M of the fixed point x_0, is equivalent to the linear tangent action of K on $\mathrm{T}_{x_0} M$, restricted to an open neighborhood of 0 in $\mathrm{T}_{x_0} M$. Indeed, (12) is just (13) with G, g, Φ, $B(g)$ replaced by K, k, χ, $\mathrm{T}_{x_0} A(k)$, respectively.

0.1.17 Slices

Definition 0.1.18. Let $A : G \times M \to M$ be a C^k action ($k \geq 1$) of a Lie group G on a manifold M. A C^k *slice at* $x_0 \in M$ *for the action* A is a C^k submanifold S of M through x_0 such that, in the notation of Lemma 0.1.12,

(i) $\mathrm{T}_{x_0} M = \alpha_{x_0}(\mathfrak{g}) \oplus \mathrm{T}_{x_0} S$; and $\mathrm{T}_x M = \alpha_x(\mathfrak{g}) + \mathrm{T}_x S$, $(x \in S)$;

(ii) S is G_{x_0}-invariant;

(iii) if $x \in S$, $g \in G$, and $A(g)(x) \in S$, then $g \in G_{x_0}$.

❍

It follows that the identity mapping: $S \to M$ induces a bijective mapping, even a homeomorphism: $G_{x_0} \cdot x \mapsto G \cdot x$, from the space $G_{x_0}\backslash S$ of G_{x_0}-orbits in S onto an open neighborhood of $G \cdot x_0$ in the space $G\backslash M$ of G-orbits in M. Note that the action of G_{x_0} on S has x_0 as a fixed point, by definition.

Definition 0.1.19. The action A is said to be *proper at* x_0 if for every sequence x_j in M, and g_j, in G such that $\lim_{j\to\infty} x_j = x_0$, and $\lim_{j\to\infty} g_j \cdot x_j = x_0$, respectively, there is a subsequence $j = j(k)$ such that $g_{j(k)}$ converges in G as $k \to \infty$. ❍

If G is not compact, one can find a sequence of compact subsets K_j of G such that g_j has no convergent subsequence whenever $g_j \notin K_j$, for all j. Using this one obtains that the action is proper at x_0 if and only if there exists a neighborhood U of x_0 in M such that $\{\, g \in G \mid A(g)(U) \cap U \neq \emptyset \,\}$ has a compact closure in G. Note that properness of the action at x_0 implies that G_{x_0} is a compact subgroup of G.

Application of Bochner's Linearization Theorem 0.1.15 now may be used to prove the following:

Theorem 0.1.20 (Slice Theorem). *Let A be a C^k action ($k \geq 1$) of the Lie group G on the manifold M, and suppose that the action is proper at $x_0 \in M$. Then there exists a C^k slice S at x_0 for the action A.*

0.1.21 Associated Fiber Bundles

The Tube Theorem 0.1.22 below asserts that, in a suitable G-invariant neighborhood of any orbit $G \cdot x$, the action is equivalent to a standard one that is constructed in terms of the Lie group G, the stabilizer group G_x (a closed Lie subgroup of G), and the tangent representation of G_x on $\mathrm{T}_x M / \mathrm{T}_x(G \cdot x)$. This construction can be described in the framework of *associated fiber bundles*; we shall start by defining this useful general concept, which will play a role in the course *Representation Theory and Applications in Classical Quantum Mechanics*.

Let X and Y be C^k manifolds and let H be a Lie group acting in a C^k fashion both on X and Y. The action of $h \in H$ on X will be denoted by $x \mapsto x \cdot h^{-1}$, and the one on Y by $y \mapsto h \cdot y$. Furthermore assume that the action of H on X is proper and free, so that the orbit space X/H is a C^k manifold of dimension equal to $\dim X - \dim H$, and $X \to X/H$ is a principal fiber bundle with structure group H, cf. Theorem 0.1.6 and the remarks thereafter.

Under these conditions, the action of H on $X \times Y$, defined by

$$(h, (x, y)) \mapsto (x \cdot h^{-1}, h \cdot y) \quad \big(h \in H, (x, y) \in X \times Y\big), \tag{14}$$

is proper and free as well; for this it suffices to look at what happens with the first component. The quotient manifold is a C^k manifold, which will be denoted by

$$X \times_H Y = \big\{ \{ (x \cdot h^{-1}, h \cdot y) \mid h \in H \} \bigm| (x, y) \in X \times Y \big\},$$

and $X \times Y \to X \times_H Y$ is another principal fiber bundle with structure group H. The projection $X \times Y \to X$ onto the first factor induces a mapping $X \times_H Y \to X/H$, the unique one which makes the diagram

$$\begin{array}{ccccc} & & X \times Y & & \\ & \swarrow & & \searrow \scriptstyle{\text{principal f.b. with structure group } H} & \\ X & & & & X \times_H Y \\ & \scriptstyle{\text{principal f.b. with structure group } H} \searrow & & \swarrow \scriptstyle{\text{associated f.b. with fiber } Y} & \\ & & X/H & & \end{array}$$

commutative. The claim is that $X \times_H Y \to X/H$ is a C^k fiber bundle over X/H with fiber equal to Y; this will be called the *fiber bundle over X/H with fiber Y, associated to the principal fiber bundle $X \to X/H$ with structure group H and using the action of H on Y.*

Now let G be another Lie group, with an action $(g, x) \mapsto g \cdot x$ on X that **commutes** with the action $(h, x) \mapsto x \cdot h^{-1}$ of H on X, that is, $x \mapsto g \cdot x$ commutes with $x \mapsto x \cdot h^{-1}$ for every $g \in G$ and $h \in H$, or

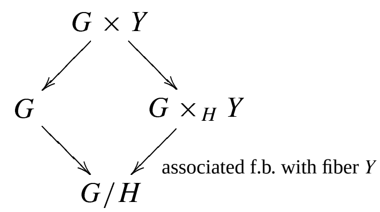
$$g \cdot (x \cdot h^{-1}) = (g \cdot x) \cdot h^{-1} \quad (g \in G,\ x \in X,\ h \in H). \tag{15}$$

The equivalent formulation of this condition is that $((g, h), x) \mapsto (g \cdot (x \cdot h^{-1}))$ is an action of $G \times H$ on X. In the case of two commuting actions, the custom to write one action as a left multiplication and the other as a right multiplication, makes the commutativity of the action look like an associative law in (15). Compare this with the actions of left and right multiplication of a group on itself.

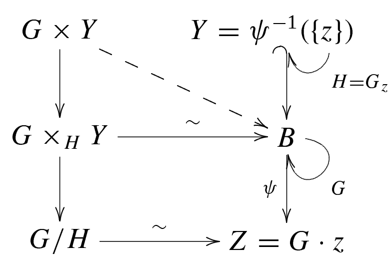
From (15) it follows that the action A_X of G on X maps H-orbits onto H-orbits, so it defines a unique action $A_{X/H}$ of G on X/H covered by A_X with respect to the projection $X \to X/H$. Using the local coordinate charts for X/H from Theorem 0.1.6, one verifies immediately that $A_{X/H}$ is a C^k action on X/H.

The action $(g, (x, y)) \mapsto (g \cdot x, y)$ of G on $X \times Y$ also commutes with the action (14) of H on $X \times Y$, so it covers a unique action of G on $X \times_H Y$ with respect to the projection $X \times Y \to X \times_H Y$. This action is C^k and in fact all the arrows in the diagram above are C^k fibrations and intertwine the respective actions of G on X, $X \times_H Y$ and X/H.

An interesting special case occurs if H is a closed (hence Lie) subgroup of G, $X = G$, and if we let G and H act on G by means of left and right multiplications, respectively. In this case G acts transitively on the base space G/H of the fibration $G \times_H Y \to G/H$ (with fiber Y).



Conversely, let $\psi : B \to Z$ be a C^k fibration, intertwining a C^k action of G on B with a **transitive** C^k action of G on Z; this is called a *homogeneous G-bundle*. For some $z \in Z$, write $Y = \psi^{-1}(\{z\})$, the fiber in B over z, which is a closed C^k submanifold of B. Also, $H = G_z$, the stabilizer of z in G, is a closed subgroup of G, which acts on Y. A straightforward verification shows that the mapping $(g, y) \mapsto g \cdot y : G \times Y \to B$ induces a C^k diffeomorphism: $G \times_H Y \to B$. In the diagram



all arrows are C^k fibrations intertwining the respective G-actions, and the horizontal ones are C^k diffeomorphisms. This shows that each homogeneous G-bundle is equivalent to one of the form $G \times_H Y \to G/H$, for a suitable closed Lie subgroup H of G and C^k action of H on a manifold Y, the fiber of the original bundle.

The action of G on $Z \overset{\sim}{\to} G/H$ is proper if and only if $G_z = H$ is a compact subgroup of G; for the "if" part use that the canonical projection $G \to G/H$ is a proper mapping if H is compact. In this case the action of G on the homogeneous G-bundle $G \times_H Y$ is also proper.

Another interesting special case occurs when $Y = E$ is a finite-dimensional vector space on which H acts by linear transformations (or a "linear representation of H"). Then each fiber of $X \times_H E \to X/H$ has a unique structure of a vector space for which $e \mapsto \{ (x \cdot h^{-1}, h \cdot e) \mid h \in H \}$ is a linear mapping from E to the fiber over xH, for each $x \in X$. This makes $X \times_H E$ into a C^k vector bundle over X/H, called the *associated vector bundle over X/H with fiber E, defined by the given representation of H in E*.

We now come to the standard model for proper actions in G-invariant neighborhoods, as announced in the beginning of this section. Its proof is based on the Slice Theorem 0.1.20.

Theorem 0.1.22 (Tube Theorem). *Let A be a C^k action of a Lie group G on a manifold M, proper at $x_0 \in M$. Then there exists a G-invariant open neighborhood U of x_0 in M such that the G-action in U is C^k equivalent to the action of G on $G \times_{G_{x_0}} B$. Here B is an open G_{x_0}-invariant neighborhood of 0 in $\mathrm{T}_{x_0} M / \alpha_{x_0}(\mathfrak{g})$, on which G_{x_0} acts linearly, via the tangent action $k \mapsto \mathrm{T}_{x_0} A(k)$ modulo $\alpha_{x_0}(\mathfrak{g})$.*

Essentially the Tube Theorem says that the action of G near x_0 can be completely described in terms of G_{x_0} and the linear action of G_{x_0} on $\mathrm{T}_{x_0} S$.

Observe that the equivalence $U \to G \times_{G_{x_0}} B$, followed by the projection $G \times_{G_{x_0}} B \to G/G_{x_0}$, defines a G-equivariant C^k fibration $U \to G/G_{x_0}$, for which the orbit $G \cdot x_0$ is a global section. Also notice that the properness of the G-action on $G \times_{G_{x_0}} B$ implies that the action of G on the G-invariant open neighborhood U of x_0 in M is proper. Apparently "proper at x_0" is equivalent to "proper on a G-invariant neighborhood".

0.1.23 Smooth Functions on the Orbit Space

Let G be a Lie group acting properly on the manifold M. This latter condition is equivalent to: the action is proper at x for each $x \in M$, and the topology of the orbit space $G \backslash M$ is Hausdorff.

Indeed, the properness of the action implies that the orbit relation $\{ (x, g \cdot x) \in M \times M \mid x \in M,\ g \in G \}$ is closed in $M \times M$, which in turn is equivalent to the Hausdorff property for $G \backslash M$, see Lemma 1.11.3. Now conversely suppose that $x_j \to x$ and $g_j \cdot x_j \to y$ in M as $j \to \infty$, for sequences x_j, and g_j, in M, and G, respectively. The closedness of the orbit relation implies that $y = g \cdot x$, for some $g \in G$; and then $(g^{-1} g_j) \cdot x_j = g^{-1} \cdot (g_j \cdot x_j) \to g^{-1} \cdot (g \cdot x) = x$, as $j \to \infty$, because of the continuity of the action. Now the properness of the action at x implies that a subsequence of the $g^{-1} g_j$ converges and then the corresponding subsequence of the g_j converges as well.

An example of an action which is proper at all points without being proper, is obtained by taking the flow of the vector field $(x, y) \mapsto (\frac{x^2+y^2}{x^2+1}, 0)$ on $M = \mathbf{R}^2 \setminus \{(0, 0)\}$. Note that every invariant neighborhood of $(-1, 0)$ intersects every invariant neighborhood of $(1, 0)$, whereas $(-1, 0)$ and $(1, 0)$ do not belong to the same orbit; and therefore the quotient space has no Hausdorff topology.

Because the canonical projection $\pi : M \to G \backslash M$ is continuous and maps open subsets of M onto open subsets of $G \backslash M$, the orbit space $G \backslash M$ is locally compact. Also it is locally pathwise connected, even locally contractible. In order to describe the connected components of $G \backslash M$, we note that, for each $g \in G$, the transformation $A(g)$ maps any connected component C of M diffeomorphically onto a connected component C' of M. Furthermore, $A(g)(C) = C$ if $g \in G^\circ$; so we get a natural action of the discrete group G/G° on the discrete space $\pi_0(M)$ of connected components of M. For each connected component $\underline{C}$ of $G \backslash M$, the set $\pi^{-1}(\underline{C})$ is equal to the union of the sets C in a G/G°-orbit in $\pi_0(M)$, and $\underline{C} = \pi(C)$, for any such C. The group $G_{(C)} = \{ g \in G \mid A(g)(C) = C \} = \{ g \in G \mid A(g)(C) \cap C \neq \emptyset \}$ is open and closed in G and acts on C; and $\underline{C}$ can be identified with $G_{(C)} \backslash C$.

Assuming from now on that M is paracompact, we have that each connected component C is equal to the union of a countable collection of compact subsets C_i. Hence $\underline{C} = \pi(C) = \bigcup_i \pi(C_i)$, and $\pi(C_i)$ is compact because of the continuity of π; and this shows that $G \backslash M$ **is paracompact**. Here we have used the theorem that a Hausdorff, locally compact space is paracompact if and only if it is the disconnected union of spaces, each of which is a union of countably many compact subsets, cf. Bourbaki [1951], §9, No.10, Th.5. The "if-part" of this criterion has been used before in Theorem 1.9.1 to prove that every group is paracompact.

Although in general $G \backslash M$ is not a smooth manifold (in the sequel we shall see in more detail how close we can get), it is natural to call a function f on an open subset V of $G \backslash M$ to be *of class* C^k if and only if $\pi^* f = f \circ \pi$ is a function of class C^k on M. These functions $\pi^* f$ are precisely the C^k functions ϕ on M that are constant on the G-orbits; or $\phi(A(g)(x)) = \phi(x)$, for all $x \in M$, $g \in G$, or

$A(g)^*\phi = \phi$, for each $g \in G$. These are the *G-invariant* C^k *functions on M*. In turn this means that $A^*\phi = \pi_2^*\phi$, where A is the action map: $G \times M \to M$ and $\pi_2 : G \times M \to M$ the projection onto the second factor. The space of G-invariant C^k functions on M is denoted by $\mathrm{C}^k(M)^G$, and the gist of the remarks above is that π^* is an isomorphism from $\mathrm{C}^k(G\backslash M)$ onto $\mathrm{C}^k(M)^G$, more or less by definition.

Replacing M by $U = A(G \times S)$, where S is a slice for the G-action at x_0 as in the proof of Theorem 2.4.1, we get that the C^k function ϕ on U is G-invariant if and only if $A^*\phi|_{(G\times S)} = \pi_2^*\psi$, for a C^k function ψ on S which is G_{x_0}-invariant. In this way not only $G\backslash U$ is homeomorphic to $G_{x_0}\backslash S$, but also the space of C^k functions on $V = G\backslash U$ gets canonically identified with the space of C^k functions on $G_{x_0}\backslash S$.

Moreover the G_{x_0}-action on S is C^k equivalent to the restriction to an open neighborhood B of 0 in $E = \mathrm{T}_{x_0} M/\alpha_{x_0}(\mathfrak{g})$ of the tangent action of G_{x_0} on $\mathrm{T}_{x_0} M$ modulo $\alpha_{x_0}(\mathfrak{g})$; and the latter is by orthogonal linear transformations with respect to some invariant inner product in E. In particular, any function of the distance to the origin is G_{x_0}-invariant, and we can find such a function χ of class C^∞ such that $\chi \geq 0$, $\chi(0) > 0$ and the support of χ is contained in any given neighborhood of 0 in B. Transporting this to $V \subset G\backslash M$ and extending the resulting function by 0 to $G\backslash M$, we see that for every $\underline{x}_0 \in G\backslash M$ and every neighborhood V of $\underline{x}_0$ in $G\backslash M$ there exists a function f of class $\mathrm{C}^{\min(k,\infty)}$ on $G\backslash M$, such that $f \geq 0$, $f(\underline{x}_0) > 0$ and the support of f is contained in V. (Notice that the previous argument doesn't apply in the real-analytic category.) In combination with the paracompactness of $G\backslash M$, we have proved the following

Lemma 0.1.24. *For every open covering $\mathcal{V}$ of $G\backslash M$ there is a partition of unity $\{f_j\}$ on $G\backslash M$, of class $\mathrm{C}^{\min(k,\infty)}$ and subordinate to $\mathcal{V}$. That is, each f_j is a $\mathrm{C}^{\min(k,\infty)}$ function on $G\backslash M$, $f_j \geq 0$, and the support* $\mathrm{supp}(f_j)$ *of f_j is contained in some $V_j \in \mathcal{V}$. Moreover, the* $\mathrm{supp}(f_j)$ *form a locally finite family of compact subsets of $G\backslash M$, and $\sum f_j = 1$ on $G\backslash M$.*

Such partitions of unity can be used for piecing together G-invariant structures that are defined in G-invariant neighborhoods in M, to global ones in M. The structures should belong to a category where one can take arbitrary convex linear combinations. As an application we give the

Proposition 0.1.25. *Let G be a Lie group acting properly and in a C^k fashion on the paracompact manifold M, with $1 \leq k \leq \infty$. Then M has a G-invariant Riemannian structure g of class C^{k-1}.*

Conversely, one may prove that if g is a Riemannian structure of class C^{k-1} on a paracompact manifold M with finitely many connected components, then the group I of isometries of the corresponding metric space is equal to the group of automorphisms of (M, g), and is a finite-dimensional Lie group with countably many components. Its action on M is proper and of class C^k, and its Lie group topology coincides with the C^k topology on $I \subset \mathrm{Diff}^k(M)$ and also with the topology of pointwise convergence. Here $k > 2$; if $k = 1$, and $k = 2$, we have to add the condition that g is Hölder continuous, and that the first-order derivatives of g are Hölder continuous, respectively. Any closed subgroup G of I is then also a Lie group acting properly and in a C^k fashion on M.

On the other hand, if g is the G-invariant structure of the Proposition, and Φ_j is a sequence in G such that $A(\Phi_j)$ converges pointwise in M to a mapping $\Phi : M \to M$, then the properness of the action implies that a subsequence of the Φ_j converges in G, to some $\Psi \in G$. Because this implies that $A(\Phi_j)$ converges pointwise to $A(\Psi)$, we get that $\Phi = A(\Psi)$. In other words, $A(G)$ is a closed subgroup of the isometry group I for the Riemannian structure g. As usual we assume that the action is effective

($\ker A = \{1\}$), by passing to the Lie group $G/\ker A$, if necessary. In this sense we have that, for $3 \leq k \leq \infty$ and for a paracompact manifold M with finitely many components, that **proper effective** C^k **actions of Lie groups** and **closed subgroups of isometries for** C^{k-1} **Riemannian structures** can be regarded as the same topics.

0.2 Group Actions: Second Lecture

0.2.1 Orbit Types and Local Action Types

We keep our standing assumption that A is a proper C^k action of a Lie group G on a manifold M.

Definition 0.2.2. We say that $x, y \in M$, and $G \cdot x, G \cdot y \in G\backslash M$, *are of the same type*, with notation $x \sim y$, and $G \cdot x \sim G \cdot y$, respectively, if there exists a G-equivariant bijection from $G \cdot x$ to $G \cdot y$. We say that x *dominates* y, and that $G \cdot x$ *dominates* $G \cdot y$, with notation: $y \lesssim x$, and $G \cdot y \lesssim G \cdot x$, respectively, if there is a G-equivariant mapping from $G \cdot x$ to $G \cdot y$.

Clearly, $\sim$ is an equivalence relation in M, and $G\backslash M$; we denote the equivalence classes, sometimes simply called the *orbit types in* M, and $G\backslash M$, respectively, by

$$M_x^{\sim} = \{\, y \in M \mid y \sim x \,\}, \quad G\backslash M_{G\cdot x}^{\sim} = \{\, G \cdot y \in G\backslash M \mid G \cdot y \sim G \cdot x \,\}.$$

On the other hand, $\lesssim$ is a pre-order in M, and $G\backslash M$, respectively; and we write

$$M_x^{\lesssim} = \{\, y \in M \mid y \lesssim x \,\}, \quad G\backslash M_{G\cdot x}^{\lesssim} = \{\, G \cdot y \in G\backslash M \mid G \cdot y \lesssim G \cdot x \,\}.$$

❍

We start with some direct observations about these notions.

Lemma 0.2.3. (i) *We have $x \sim y$ if and only if G_x is conjugate to G_y within G, that is, $G_y = g^{-1}G_x g := \{\, g^{-1}hg \in G \mid h \in G_x \,\}$, for some $g \in G$.*

(ii) *Also $y \lesssim x$ if and only G_x is conjugate, within G, to a subgroup of G_y. That is, $g^{-1}G_x g \subset G_y$, for some $g \in G$.*

(iii) *If Φ is a G-equivariant mapping: $G \cdot x \to G \cdot y$, then $G_x \subset G_{\Phi(x)}$ and $\Phi = B_{\Phi(x)} \circ \pi \circ B_x^{-1}$.*

$$\begin{array}{ccc} G/G_x & \xrightarrow{\pi} & G/G_{\Phi(x)} \\ {\scriptstyle B_x}\big\downarrow{\scriptstyle \sim} & & {\scriptstyle \sim}\big\downarrow{\scriptstyle B_{\Phi(x)}} \\ G \cdot x & \xrightarrow{\Phi} & G \cdot y \end{array}$$

Here π is the real-analytic, G-equivariant fibration $\pi : gG_x \mapsto gG_{\Phi(x)} : G/G_x \to G/G_{\Phi(x)}$, with fiber $G_{\Phi(x)}/G_x$ induced by $A_x : g \mapsto g \cdot x : G \to G \cdot x$. In particular, Φ automatically is a C^k fibration with fiber diffeomorphic to $G_{\Phi(x)}/G_x$.

(iv) *Finally, $x \sim y$ if and only if $x \lesssim y$ and $y \lesssim x$.*

Definition 0.2.4. For any subgroup H of G, we define the *set of fixed points* for H in M as

$$M^H = \{\, y \in M \mid h \cdot y = y, \text{ for all } h \in H \,\}.$$

❍

For any continuous action, M^H is a closed subset of M. Note that if H is compact, acting in a C^k fashion on M, for $k \geq 1$, then Bochner's Linearization Theorem 0.1.15 implies that locally M^H is a C^k submanifold of M. That is, each connected component of M^H is a closed submanifold of M, but different connected components of M^H can have different dimensions.

On the basis of the Tube Theorem 0.1.22 one may prove:

Lemma 0.2.5. (i) *For any $x \in M$, we have $M_x^{\lesssim} = G \cdot (M^{G_x})$. Moreover, $M_x^{\lesssim}$, and $G\backslash M_x^{\lesssim}$, is a closed subset of M, and $G\backslash M$, respectively.*

(ii) *If S is a C^k slice at $x \in M$ for the G-action on M, then*

$$M_x^{\sim} \cap G \cdot S = G \cdot (S^{G_x}) = M_x^{\lesssim} \cap G \cdot S.$$

(Note that $G \cdot S$ is a G-invariant open neighborhood of x in M.) Restricting S to a suitable G-invariant open neighborhood of x in M, we get that $M_x^{\sim} \cap G \cdot S$ is a closed C^k submanifold of $G \cdot S$, which is G-equivariantly C^k diffeomorphic to $G/G_x \times (S^{G_x})$. Here S^{G_x} is a closed C^k submanifold of S of dimension equal to $\dim(\mathrm{T}_x\, M/\alpha_x(\mathfrak{g}))^{G_x}$, that is

$$M_x^{\sim} \cap G \cdot S \overset{\sim}{\longrightarrow} G/G_x \times (S^{G_x}).$$

Definition 0.2.6. The elements $x,\ y \in M$, and $G \cdot x,\ G \cdot y \in G\backslash M$, respectively, are said to be *of the same local type*, with the notation: $x \approx y$ and $G \cdot x \approx G \cdot y$, if there is a G-equivariant C^k diffeomorphism Φ from an open G-invariant neighborhood U of x in M onto an open G-invariant neighborhood V of y in M. ❍

Clearly this defines an equivalence relation $\approx$ in M, and $G\backslash M$, respectively; it is finer than $\sim$, that is, each $\sim$-equivalence class is partitioned into $\approx$-equivalence classes. These equivalence classes sometimes will be called *local action types* in M, and $G\backslash M$, respectively, and denoted by

$$M_x^{\approx} = \{\, y \in M \mid y \approx x \,\}, \quad G\backslash M_{G\cdot x}^{\approx} = \{\, G \cdot y \in G\backslash M \mid G \cdot y \approx G \cdot x \,\}.$$

Note that the Tube Theorem 0.1.22 shows that $x \approx y$ if and only $x \sim y$ and the actions of G_x, and G_y, on $\mathrm{T}_x\, M/\alpha_x(\mathfrak{g})$, and $\mathrm{T}_y\, M/\alpha_y(\mathfrak{g})$, respectively, are equivalent via a linear intertwining isomorphism, that is, as representations.

Definition 0.2.7. For any subgroup H of G, the *normalizer* of H in G is defined as $\mathrm{N}(H) = \mathrm{N}_G(H) = \{\, g \in G \mid gHg^{-1} = H \,\}$. It is the largest subgroup of G containing H as a normal subgroup. It is closed in G if H is closed in G. ❍

Theorem 0.2.8. (i) *Each local action type is an open and closed subset of the corresponding orbit type.*

(ii) *The set $M_x^{\approx} \cap M^{G_x}$ is open in M^{G_x}, and a locally closed C^k submanifold of M (that is, all its connected components have the same dimension), and it is also $\mathrm{N}(G_x)$-invariant.*

(iii) *The canonical projection:* $M \to G\backslash M$ *maps* $M_x^{\approx} \cap M^{G_x}$ *onto* $G\backslash M_{G.x}^{\approx}$*, its fibers in* $M_x^{\approx} \cap M^{G_x}$ *are the orbits for the* $\mathrm{N}(G_x)$*-action on* $M_x^{\approx} \cap M^{G_x}$*. The action of the group* $\mathrm{N}(G_x)/G_x$ *on* $M_x^{\approx} \cap M^{G_x}$ *is proper and free; hence there is a unique structure of a* C^k *manifold on* $G\backslash M_{G.x}^{\approx}$ *for which* $\pi : M_x^{\approx} \cap M^{G_x} \to G\backslash M_{G.x}^{\approx}$ *is the corresponding principal fibration with structure group* $\mathrm{N}(G_x)/G_x$*.*

(iv) $M_x^{\approx}$ *is a locally closed* G*-invariant* C^k *submanifold of* M*, and the* G*-action induces a* G*-equivariant* C^k *diffeomorphism from the associated fiber bundle* $G/G_x \times_{\mathrm{N}(G_x)/G_x} (M_x^{\approx} \cap M^{G_x})$ *onto* $M_x^{\approx}$*. Further* $\pi : M_x^{\approx} \to G\backslash M_{G.x}^{\approx}$ *is a* C^k *fiber bundle with fiber* G/G_x*. In other words,*

$$\begin{array}{ccccc} M_x^{\approx} \cap M^{G_x} & & G/G_x \times_{\mathrm{N}(G_x)/G_x} (M_x^{\approx} \cap M^{G_x}) & \xrightarrow{\sim} & M_x^{\approx} \\ & \searrow^{\text{p.f.b. with structure group } \mathrm{N}(G_x)/G_x} & \swarrow_{\text{associated f.b. with fiber } G/G_x} & & \downarrow{\scriptstyle \text{f.b. with fiber } G/G_x} \\ & G\backslash M_{G.x}^{\approx} & = \!=\!=\!=\!=\!=\!=\!= & & G\backslash M_{G.x}^{\approx} \end{array}$$

(v) *We have*

$$\dim M_x^{\approx} = \dim G - \dim G_x + \dim(\mathrm{T}_x M/\alpha_x(\mathfrak{g}))^{G_x} \tag{16}$$
$$= \dim G - \dim \mathrm{N}(G_x) + \dim(M_x^{\approx} \cap M^{G_x}); \tag{17}$$
$$\dim G\backslash M_{G.x}^{\approx} = \dim(\mathrm{T}_x M/\alpha_x(\mathfrak{g}))^{G_x} \tag{18}$$
$$= \dim(M_x^{\approx} \cap M^{G_x}) - \dim G_x + \dim \mathrm{N}(G_x). \tag{19}$$

Remark 0.2.9. $G/G_x \times_{\mathrm{N}(G_x)/G_x} (M_x^{\approx} \cap M^{G_x}) \cong M_x^{\approx}$ is a C^k fiber bundle over $G/\mathrm{N}(G_x)$ with fiber $M_x^{\approx} \cap M^{G_x}$, the G-action on it covering the G-action on $G/\mathrm{N}(G_x)$ by left multiplications. However, because the action of $\mathrm{N}(G_x)/G_x$ on $M_x^{\approx} \cap M^{G_x}$ is also proper and free, $M_x^{\approx}$ is also a C^k fiber bundle over $(\mathrm{N}(G_x)/G_x)\backslash(M_x^{\approx} \cap M^{G_x}) \cong G\backslash M_{G\cdot x}^{\approx}$, with fiber G/G_x; of course, this is just the fibration of $M_x^{\approx}$ into its G-orbits. Here the G-action covers the trivial action of G on $G\backslash M_{G\cdot x}^{\approx}$. The common fibers of the intersections of the two fibrations are the $\mathrm{N}(G_y)/G_y$-orbits in $M_x^{\approx} \cap M^{G_y} = M_y^{\approx} \cap M^{G_y}$, for $y \in M_x^{\approx}$.

$$\begin{array}{ccccc} G/G_x \times (M_x^{\approx} \cap M^{G_x}) & \xrightarrow{\text{p.f.b}} & G/G_x \times_{\mathrm{N}(G_x)/G_x} (M_x^{\approx} \cap M^{G_x}) \cong M_x^{\approx} & \xrightarrow{\text{f.b.}} & (\mathrm{N}(G_x)/G_x)\backslash(M_x^{\approx} \cap M^{G_x}) \\ \downarrow & & \downarrow{\scriptstyle \text{f.b. with fiber } M_x^{\approx} \cap M^{G_x}} & & \downarrow{\scriptstyle \cong} \\ G/G_x & \longrightarrow & G/\mathrm{N}(G_x) & & G\backslash M_{G\cdot x}^{\approx} \end{array}$$

Different local orbit types in a given orbit type can have different dimensions. This is one of the reasons why we preferred to formulate Theorem 0.2.8.(ii)–(v) for local action types rather than for orbit types. ✰

0.2.10 The Stratification by Orbit Types

Theorem 0.2.8.(iv) shows that the local action types partition M into locally closed C^k submanifolds, each of which is C^k fibered by G-orbits. In this section we shall study the local properties of this partitioning.

Consider the action of a compact Lie group H acting on a Euclidean space E by means of orthogonal linear transformations. To start with, $E_0^{\sim} = E^H = \{ v \in E \mid h \cdot v = v, \text{ for all } h \in H \}$ is a linear subspace of E, which is H-invariant. Hence F, the orthogonal complement of E^H, is H-invariant as well. The linear isomorphism $+ : (e, f) \mapsto e + f : E^H \times F \to E$ is H-equivariant if we let act H on $E^H \times F$ by $(h, (e, f)) \mapsto (e, h \cdot f)$, for $h \in H$, $(e, f) \in E^H \times F$. Note that $F_0^{\sim} = F^H = \{0\}$. In order to study the H-action on $F \setminus \{0\}$, we observe that the unit sphere $\Sigma = \{ f \in F \bigm| ||f|| = 1 \}$ in F is H-invariant and that in turn the real-analytic diffeomorphism $p : (r, f) \mapsto r\, f : \mathbf{R}_{>0} \times \Sigma \to F \setminus \{0\}$ is H-equivariant if we let act H on $\mathbf{R}_{>0} \times \Sigma$ by $(h, (r, f)) \mapsto (r, h \cdot f)$, for $h \in H$, $(r, f) \in \mathbf{R}_{>0} \times \Sigma$. It follows that the orbit types for the action on the H-invariant open subset $E \setminus E_0^{\sim}$ are the sets of the form $E^H + p(\mathbf{R}_{>0} \times T)$, where T runs over the orbit types for the H-action on Σ.

Next let H be a compact subgroup of a Lie group G and let π denote the projection: $G \times E \to G \times_H E$. Each G-orbit in $G \times_H E$ meets $\pi(\{1\} \times E)$; and $\pi(1, e) = g \cdot \pi(1, e) = \pi(g, e)$ if and only if $(1, e) = (gh^{-1}, h \cdot e)$, that is $g = h$ and $h \cdot e = e$, for some $h \in H$. In other words, $G_{\pi(1,e)} = H_e$; or the orbit types in $G \times_H E$ are the sets of the form $\pi(G \times \mathcal{T})$, where $\mathcal{T}$ runs over the orbit types for the action of H in E. In view of the previous paragraph, the orbit types in $G \times_H E$ therefore are the sets of the form $\pi\bigl(G \times (E^H + p(\mathbf{R}_{>0} \times T))\bigr)$, where T runs over the orbit types for the action of H on Σ.

Using the Tube Theorem 0.1.22 and Theorem 0.2.8.(i) we obtain the following proposition by induction on the dimension of the manifold on which the Lie group acts properly.

Proposition 0.2.11. *For a proper* C^1 *action of a Lie group G on a manifold M, there are locally only finitely many distinct orbit types with locally only finitely many connected components. In particular, one also has locally only finitely many distinct local action types. If M is compact, then there are only finitely many distinct connected components of orbit types, and the same is true if M is a finite-dimensional vector space on which G acts by linear transformations.*

Any connected component of an orbit type for the action of G on $G \times_H E$ that is not the orbit type of $\pi(G \times \{0\})$, has dimension equal to $\dim G - \dim H + \dim E^H + 1 + \dim T'$, where T' is a connected component of an orbit type T for the action of H on Σ. Because the dimension of the orbit type of $\pi(G \times \{0\})$ is equal to $\dim G - \dim H + \dim E^H$, we get, using again the Tube Theorem 0.1.22 and Theorem 0.2.8.(i), the following:

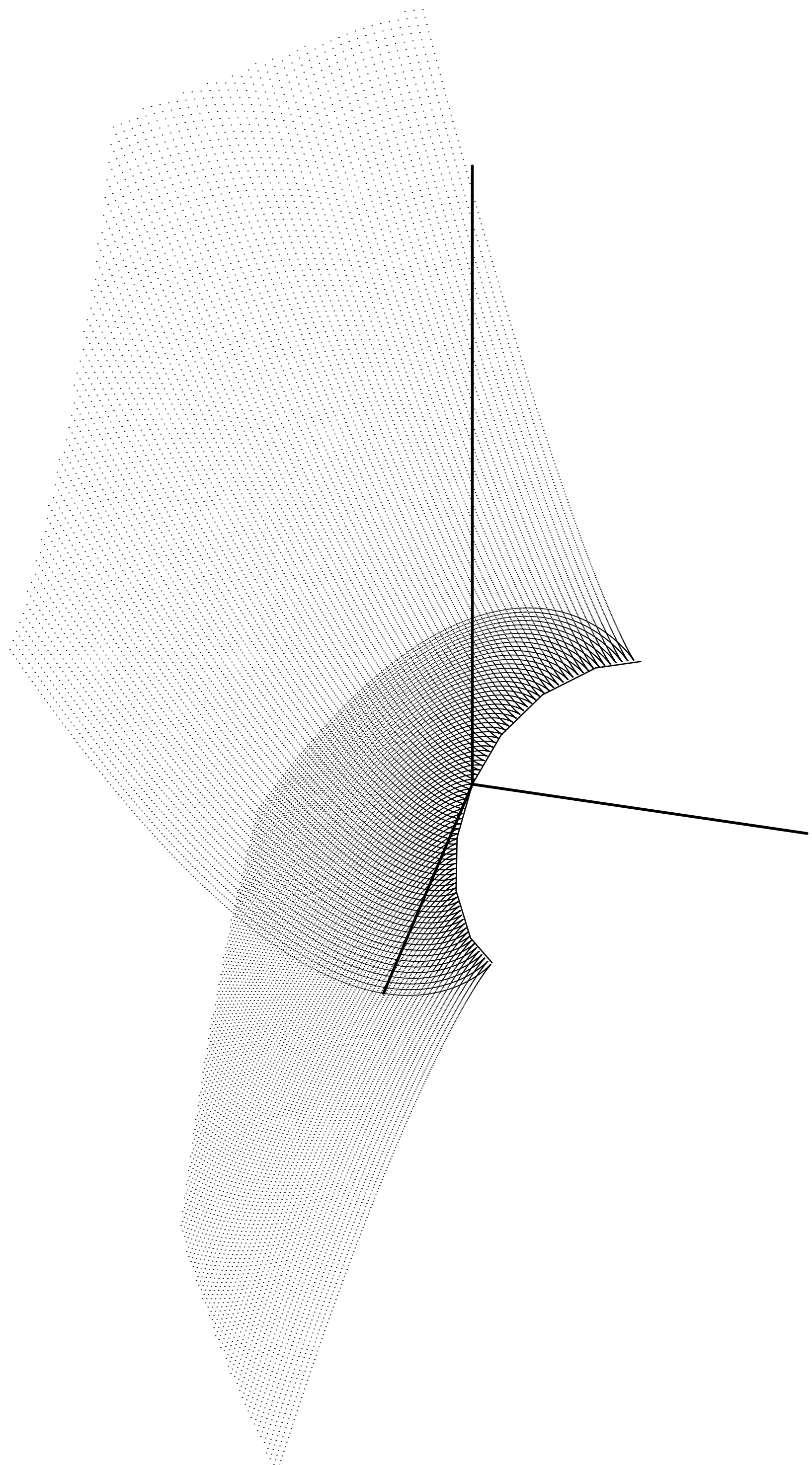
Proposition 0.2.12. *For each $x \in M$ there is a G-invariant open neighborhood U of x in M such that, for each $y \in U \setminus M_x^{\approx}$,*

$$\dim M_y^{\approx} = \dim M_x^{\approx} + 1 + \dim T,$$

where T is some local action type for the tangent action of G_x on the unit sphere in the orthogonal complement of $\alpha_x(\mathfrak{g}) + (\mathrm{T}_x M)^{G_x}$ in $\mathrm{T}_x M$, with respect to a given G_x-invariant inner product on $\mathrm{T}_x M$. In the same vein,

$$\dim G \backslash M_y^{\approx} = \dim G \backslash M_x^{\approx} + 1 + \dim G_x \backslash T.$$

In particular, $\dim M_y^{\approx} > \dim M_x^{\approx}$ *and* $\dim G \backslash M_y^{\approx} > \dim G \backslash M_x^{\approx}$, *for all $y \in U \setminus M_x^{\approx}$. Note also that $x \lesssim y$, hence* $\dim G.y \geq \dim G.x$, *for all y sufficiently close to x. In particular, the estimates above show that "smaller orbits cannot compensate for being small by massive appearance, not even in the orbit space".*



Definition 0.2.13. A C^k *stratification* of the manifold M is a locally finite partition of M into locally closed connected C^k submanifolds M_i $(i \in I)$ of M, called the *strata* of the stratification, such that

the following is satisfied. For each $i \in I$ the closure M_i^c of M_i in M is equal to $M_i \cup \bigcup_{j \in I_i} M_j$, where $I_i \subset I \setminus \{i\}$, and $\dim M_j < \dim M_i$, for each $j \in I_i$.

The stratification is said to be a *Whitney stratification* if the following conditions (a) and (b) are met:

(a) For each $i \in I$, $j \in I_i$ and each sequence x_n in M_i such that $\lim_{n\to\infty} x_n = x \in M_j$ and $\lim_{n\to\infty} \mathrm{T}_{x_n} M_i = L$ in the Grassmann bundle of $\mathrm{T}\, M$, we have $\mathrm{T}_x M_j \subset L$.

(b) If x_n is a sequence as in (a) and y_n is a sequence in the limit stratum M_j, such that $\lim y_n = x$ and $y_n \neq x_n$, for all n, then each limit of the one-dimensional subspaces $\mathbf{R} \cdot \lambda(x_n, y_n)$ of $\mathrm{T}_{x_n} M$, for $n \to \infty$, is contained in L. Here λ is a diffeomorphism from an open neighborhood of the diagonal in $M \times M$ to an open neighborhood of the zero section of $\mathrm{T}\, M$. Clearly the set of limit lines does not depend on the choice of λ.

❍

Theorem 0.2.14. *The connected components of the orbit types in M form a Whitney stratification of M.*

Remark 0.2.15. We feel that the stratification by orbit types has even more special properties than general Whitney stratifications.

The orbit space $G\backslash M$ is stratified by the connected components of the orbit types (cf. Theorem 2.6.7. We define the dimension of a connected component of $G\backslash M$ as the maximum of the dimensions of its strata). But in order to say that this is a Whitney stratification, we have to embed $G\backslash M$ at least locally in a smooth manifold. For convenience, we assume in the following discussion that $k = \infty$.

Locally the space of smooth (C^∞) functions on $G\backslash M$, identified with $\mathrm{C}^\infty(M)^G$, can be identified with $\mathrm{C}^\infty(B)^K$, where $K = G_x$ and B is a K-invariant open neighborhood of 0 in $E = \mathrm{T}_x M/\alpha_x(\mathfrak{g})$. The action of K on E is the one induced by the tangent action $k \mapsto \mathrm{T}_x A(k)$ of $K = G_x$ on $\mathrm{T}_x M$.

The theorem of Schwarz [1975], applied to this representation of the compact group K on E, states that $\mathrm{C}^\infty(E)^K$ contains finitely many polynomials $p_1, \ldots, p_k$, which may be chosen homogeneous of degree $m_1, \ldots, m_k > 0$, such that every $f \in \mathrm{C}^\infty(E)^K$ can be written as $f = \phi \circ p$, for some $\phi \in \mathrm{C}^\infty(\mathbf{R}^k)$. Here p denotes the mapping $x \mapsto (p_1(x), \ldots, p_k(x)) : E \to \mathbf{R}^k$.

The conic structure of multiplication by $r > 0$ in E, which maps K-orbits to K-orbits, is intertwined by p with the action of $r > 0$ in $\mathbf{R}^k$ given by: $(y_1, \ldots, y_k) \mapsto (r^{m_1} y_1, \ldots, r^{m_k} y_k)$. (The invariance of $p(E)$ under such an action is said to be a quasihomogeneous structure on $p(E)$.) In view of the compactness of the unit sphere in E, this leads to the properness of the mapping $p : E \to \mathbf{R}^k$; and it follows that $p(E)$ is a closed subset of $\mathbf{R}^k$.

The mapping p induces a homeomorphism: $K\backslash E \to p(E)$. Let X be a local action type of the K-action on E. Using the local description of the action in the Tube theorem 2.4.1 and of the orbit types preceding Proposition 0.2.11, one sees that, for each $x \in X$, there exist K-invariant C^∞ functions $f_1, \ldots, f_r$ near x such that $df_1, \ldots, df_r$ are linearly independent and $r = \dim K\backslash X$. Combined with the theorem of Schwarz, this implies that $\operatorname{rank} \mathrm{T}_x(p|_X) = \dim K\backslash X$, for all $x \in X$. In turn this implies that $p(X)$ is a smooth submanifold of $\mathbf{R}^k$, diffeomorphic to $K\backslash X$ under the homeomorphism: $K\backslash E \to p(E)$. Because of the properness of p, the $p(X)$ form a stratification of $p(E) \subset \mathbf{R}^k$.

Using that the action of K on E is polynomial (cf. Corollary 14.6.2), one can prove that the strata X are semialgebraic. According to the Tarski-Seidenberg theorem (cf. Hörmander [1983], Appendix

A.2), which says that the image of semi-algebraic sets under polynomial mappings is semi-algebraic, one gets that both $p(E)$ and the $p(X)$'s are semi-algebraic. Łojasiewicz [1965], p.150–153 has proved that every locally-finite partition of a semi-analytic set into semi-analytic sets can be refined into a Whitney stratification. So, in particular, the stratification of $p(E)$ by the $p(X)$'s can be refined to a Whitney stratification of $p(E)$. We have not investigated the question whether the $p(X)$'s themselves automatically form a Whitney stratification. ✩

0.2.16 Principal and Regular Orbits

A part of the information about the orbit type stratification can be encoded in its *directed graph* $\mathcal{T}$. The vertices of $\mathcal{T}$ are the connected components of the orbit types in $G\backslash M$ and we draw an arrow from $A \in \mathcal{T}$ to $B \in \mathcal{T}$, notation: $A \to B$, if $B \subset A^{\mathrm{c}}$; in words, if elements of A converge to elements of B. Clearly the connected components in the graph correspond to the connected components of the orbit space $G\backslash M$.

If $A \to B$, then $A \gtrsim B$ (Lemma 0.2.5.(ii)). If also $A \neq B$, then according to Proposition 0.2.12 we have $\dim A > \dim B$ and $\dim \pi^{-1}(A) > \dim \pi^{-1}(B)$, where π denotes the projection: $M \to G\backslash M$. The latter shows that $A \to B$ defines a partial ordering in $\mathcal{T}$ (that is, if $A \to B$ and $B \to C$ then $A \to C$, and $A = B$ if and only if $A \to B$ and $B \to A$), in which any chain at most has $1 + \dim G\backslash M$ many elements. In pictures, one usually only draws the arrows between immediate successors. Note that the **minimal** elements in the graph $\mathcal{T}$ are precisely the connected components of orbit types in $G\backslash M$ and their preimages in M that are **closed** in $G\backslash M$, and M, respectively.

Example 0.2.17. The graph

$$\underset{B}{\bullet} \longleftarrow \underset{A}{\bullet} \longrightarrow \underset{C}{\bullet}$$

is the directed graph of the action of $\mathbf{SO}(2)$ on the two-dimensional sphere in $\mathbf{R}^3$: A is the orbit type of the circular orbits, and B and C are the fixed points, the North and the South Pole, respectively. (For the action of $\mathbf{SO}(3)$ on $\mathbf{R}^3$, the vertices B and C would get identified since the vertical axis is a single orbit type of fixed points.) The action of $\mathbf{SO}(3)$ on itself by conjugation has the same directed graph: A is the collection of conjugacy classes of rotations through an angle between 0 and π, whereas B is the conjugacy class of the rotation through π and C is the identity. Although of quite a different nature, the action of $\mathbf{SU}(2)$ on itself by conjugation also has this directed graph, with A the collection of two-dimensional conjugacy classes and B, C corresponding to the elements $\pm$ I in $\mathbf{SU}(2)$. ✩

Definition 0.2.18. For a proper C^k action ($k \geq 1$) of the Lie group G on the manifold M, the orbit $G \cdot x$, with $x \in M$, is said to be a *principal orbit* if its local action type $M_x^{\approx}$ is open in M, that is, if it belongs to a maximal element of the directed graph $\mathcal{T}$. We write $M^{\mathrm{princ}} = \{\, x \in M \mid G \cdot x \text{ is a principal orbit } \}$, and $G\backslash M^{\mathrm{princ}} \subset G\backslash M$ for the set of principal orbits. ❍

Clearly M^{princ} is an open G-invariant subset of M, projecting onto the open subset $G\backslash M^{\mathrm{princ}}$ of the orbit space $G\backslash M$. Moreover, M^{princ} is also dense in M as the complement of the union of the, locally finitely many, orbit types of positive codimension in M. The last statement in Theorem 0.2.8.(iv) shows that the open and closed subsets $M_x^{\approx}$ of M^{princ} are open G-invariant subsets of M fibered by G-orbits. These are maximal in the sense that no $x \in M \setminus M^{\mathrm{princ}}$ has a G-invariant open neighborhood that is fibered by G-orbits. Indeed, each nearby orbit intersects a slice at x in some point y, and then $G_y \subset G_x$.

If $\dim G\cdot y = \dim G\cdot x$, then $\dim G_y = \dim G_x$; or G_x/G_y is finite and $G\cdot y \cong G/G_y \to G/G_x \cong G\cdot x$ is a fibration with finite fiber G_x/G_y. This contradicts local triviality of the orbit structure near x, unless $G_y = G_x$, for all $y \in S$ near x. In terms of the description preceding Proposition 0.2.11 we have $x \in M^{\mathrm{princ}}$ if and only if the space F occurring there is equal to $\{0\}$.

It is the main purpose of this section to show that, in each connected component of $G\backslash M$, the set $G\backslash M^{\mathrm{princ}}$ is connected, cf. the Principal Orbit Theorem 0.2.22 below. In other words, **each connected component of the graph $\mathcal{T}$ has a unique maximal element**. If M is connected, this means that there is **only one principal orbit type**.

In the sequel, it will be convenient to use a somewhat coarser partition of $G\backslash M$ than the one into orbit types.

Definition 0.2.19. Two elements x , $y \in M$ are said to be of the same *infinitesimal type*, notation: $x \underset{\mathrm{inf}}{\sim} y$, if there exists $g \in G$ such that $\mathrm{Ad}\, g^{-1}(\mathfrak{g}_y) = \mathfrak{g}_x$, (or $g^{-1}(G_y)^\circ g = G_x^\circ$). One says that y *dominates* x *infinitesimally*, notation: $x \underset{\mathrm{inf}}{\lesssim} y$, if there exists $g \in G$ such that $\mathrm{Ad}\, g^{-1}(\mathfrak{g}_y) \subset \mathfrak{g}_x$, (or $g^{-1}(G_y)^\circ g \subset G_x^\circ$). ❍

Because $\mathfrak{g}_x$ is the Lie algebra of G_x and $\mathrm{Ad}\, g^{-1}(\mathfrak{g}_y)$ is the Lie algebra of $g^{-1}(G_y)g$, we have $x \sim y \Rightarrow x \underset{\mathrm{inf}}{\sim} y$, and $x \lesssim y \Rightarrow x \underset{\mathrm{inf}}{\lesssim} y$; so the infinitesimal version is a coarsening of the partition, and ordering, respectively by orbit types.

Furthermore, if S is a slice at x and $y \in S$, then $G_y \subset G_x$ and hence $\mathfrak{g}_y \subset \mathfrak{g}_x$. So all nearby orbits are dominating infinitesimally. Also $y \underset{\mathrm{inf}}{\sim} x$ if and only if $\mathfrak{g}_y = \mathfrak{g}_x$. The intersection with S of the infinitesimal type of x therefore is equal to the common set of zeroes in S of the vector fields $\alpha_x(X)$, for $X \in \mathfrak{g}_x$. In a Bochner linearization these vector fields are all linear and this set of zeroes is a linear subspace. As in Theorem 0.2.8.(iv), one obtains that the set $M_x^{\underset{\mathrm{inf}}{\sim}}$ of elements of the same infinitesimal type as x locally is a closed C^k submanifold of M. As in Proposition 0.2.12, one gets that the dimension of nearby different infinitesimal types is strictly larger, both in M and in the orbit space $G\backslash M$. As in Theorem 0.2.14, the infinitesimal types form a Whitney stratification of M.

Definition 0.2.20. The set $G\cdot x$ is said to be a *regular orbit* if the dimension of the orbits $G\cdot y$ is constant (not strictly larger), for all y near x. That is, if x is in the interior of its infinitesimal type; or, if $G\cdot x$ belongs to a maximal element of the directed graph $\mathcal{T}^{\mathrm{inf}}$ defined by the infinitesimal type. Elements on regular orbits will be called *regular points* for the action of G on M; and we shall denote the set of these points by M^{reg}, and the set of regular orbits by $G\backslash M^{\mathrm{reg}} \subset G\backslash M$. ❍

Because $\dim G\cdot y = \dim \mathfrak{g} - \dim \mathfrak{g}_y$ and $\{\, y \in M \mid \dim \mathfrak{g}_y > r \,\}$ is a closed subset of M for every $r \in \mathbf{Z}_{>0}$, we get that $G\cdot x$ is a regular orbit if and only if $\dim \mathfrak{g}_x \leq \dim \mathfrak{g}_y$, or $\dim \mathfrak{g}_x = \dim \mathfrak{g}_y$, for all y near x. Almost by definition M^{reg} is a dense open subset of M. It is G-invariant and projects onto the dense open subset $G\backslash M^{\mathrm{reg}}$ of $G\backslash M$. Clearly $M^{\mathrm{princ}} \subset M^{\mathrm{reg}}$, and both M^{princ} and M^{reg} are unions of local action types.

We now consider the orbit type strata of codimension 1 in M.

Lemma 0.2.21. *Suppose that the Lie group G acts properly and in a C^k fashion on the manifold M, for $k \geq 1$. Let $x \in M$ and $\dim M_x^{\approx} = \dim M - 1$. Then $M_x^{\approx} \subset M^{\mathrm{reg}}$ and $\dim_{\mathbf{R}} F = 1$, where F is the subspace appearing in the description preceding Proposition 0.2.11, while G_x acts on F as the group $\boldsymbol{O}(1) = \{1, -1\}$. The orbits near $G \cdot x$ are fibered over $G \cdot x$, with fibers consisting of two elements; if G is connected then these are two-fold coverings.*

Note that near x, the hypersurface $M_x^{\approx}$ disconnects M into two half spaces, but that G_x interchanges the two half spaces. It also follows that near $G \cdot x$ the orbit space $G\backslash M$ can be viewed as a C^k manifold with boundary (a "half space"), where $G \cdot x$ is a point on the codimension-one boundary. This is even so in terms of the Remark 0.2.15: near $G \cdot x$ the orbit space can be identified with $(\mathrm{T}_x M/\alpha_x(\mathfrak{g}))^{G_x} \times \big(\{1, -1\}\backslash F\big)$, so we can take $p_1, \ldots, p_{k-1}$ to be a basis of the linear forms on the first factor, and $p_k = f^2$, if f denotes the f-coordinate. So p maps $E = (\mathrm{T}_x M/\alpha_x(\mathfrak{g}))^{G_x} \times F$ onto the half space $\{ (y_1, \ldots, y_k) \in \mathbf{R}^k \mid y_k \geq 0 \}$. Thereby $S \cap M_x^{\approx}$ gets mapped to a neighborhood of 0 in the boundary $\{ (y_1, \ldots, y_k) \in \mathbf{R}^k \mid y_k = 0 \}$, and $S \cap M^{\mathrm{princ}}$ to a neighborhood of 0 in the interior $\{ (y_1, \ldots, y_k) \in \mathbf{R}^k \mid y_k > 0 \}$.

Theorem 0.2.22 (Principal Orbit Theorem). *Suppose that the Lie group G acts properly and in a C^1 fashion on the manifold M. Then $M \setminus M^{\mathrm{reg}}$ is equal to the union of local orbit types (strata for the stratification described in Section 0.2.10) of codimension ≥ 2 in M. For every connected component M° of M, the subset $M^{\mathrm{reg}} \cap M^\circ$ is connected, open and dense in M°. Each connected component $(G\backslash M)^\circ$ of $G\backslash M$ contains only one principal orbit type, which is a connected, open and dense subset of it.*

The smallest nonvoid, open and closed G-invariant subsets are the preimages of the connected components of $G\backslash M$ or, equivalently, the sets $G \cdot M^\circ$ where M° is a connected component of M. By restricting the discussion to those subsets, we may assume that $G\backslash M$ is connected; and according to Theorem 0.2.22 this implies that there is only one principal orbit type.

Corollary 0.2.23. *Suppose that G acts properly and in a C^k ($k \geq 1$) fashion on M, and assume that $G\backslash M$ is connected. Let $x \in M^{\mathrm{princ}}$ and write C for the connected component of x in $M^{\mathrm{princ}} \cap M^{G_x}$. Then*

(i) $M^{\mathrm{princ}} = M_x^{\approx} = M_x^{\sim}$.

(ii) *The set H, the union of the connected components of M^{G_x} that meet M^{princ}, is a closed C^k submanifold of M, which contains $\mathrm{N}(G_x) \cdot C = M^{\mathrm{princ}} \cap M^{G_x}$ as an open subset.*

(iii) $G_{(C)} = \{ g \in G \mid A(g)(C) = C \}$ *induces a G-equivariant C^k diffeomorphism*

$$G/G_x \times_{G_{(C)}/G_x} C \xrightarrow{\sim} M_x^{\approx} = M^{\mathrm{princ}}.$$

(iv) *If C^{c} denotes the closure of C, then $C^{\mathrm{c}} \subset H$ and $G \cdot C^{\mathrm{c}} = M$.*

The purpose of Theorem 0.2.8 was to obtain reductions to proper and free actions of $\mathrm{N}(G_x)/G_x$ on $M_x^{\approx} \cap M^{G_x}$; if $x \in M^{\mathrm{princ}}$, then $G \cdot (M_x^{\approx} \cap M^{G_x})$ is open in M. The aim of Corollary 0.2.23 is to combine this with connectivity statements. If the action is free at some point $x \in M$, we get that $G_x = \{1\}$, $\mathrm{N}(G_x) = G$ and $M^{G_x} = M$; and the reductions are not very substantial in that case.

For the action by conjugation of a compact Lie group on itself, one has another extreme case, where $\mathrm{N}(G_x)/G_x$, the *Weyl group*, is finite. In this situation, $G_{(C)}/G_x$ is even much smaller and often trivial.

For general proper G-actions, it might be interesting to investigate the relationship between the $\mathrm{N}(G_x)/G_x$-orbit types in H and the intersections with H of the G-orbit types in M. And also, whether the restriction to C^{c} of the G-orbit types leads to Whitney stratification of C^{c}, and what can be said about the fibers of the projection $\pi : C^{\mathrm{c}} \to G\backslash M$.

For the action by conjugation of a compact Lie group on itself, we shall return to these questions in the next lecture. There we will also use the following variation on Theorem 0.2.8 and Corollary 0.2.23. For any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we write

$$M^{\mathfrak{h}} = \{\, y \in M \mid \alpha_y(X) = 0, \text{ for all } X \in \mathfrak{h}\,\},$$

the set of zeroes of the infinitesimal actions of the $X \in \mathfrak{h}$, that is, the fixed point set of the connected Lie subgroup H of G with Lie algebra equal to $\mathfrak{h}$. In addition,

$$\mathrm{N}_G(\mathfrak{h}) = \{\, g \in G \mid \mathrm{Ad}\, g(\mathfrak{h}) = \mathfrak{h}\,\},$$

the normalizer of $\mathfrak{h}$ in G, i.e., the normalizer of H in G. Clearly $\mathrm{N}_G(\mathfrak{h})$ is a closed Lie subgroup of G. Furthermore, if H is contained in a compact subgroup K of G, then the Bochner Linearization Theorem 0.1.15 shows that $M^{\mathfrak{h}}$ is a locally closed subset of M, locally equal to a C^k submanifold, for which the dimensions of different connected components may be different.

Proposition 0.2.24. *Suppose that G acts properly and in a C^k ($k \geq 1$) fashion on M, and assume that $G\backslash M$ is connected. Let $x \in M^{\mathrm{reg}}$ and write C' for the connected component of x in $M^{\mathrm{reg}} \cap M^{\mathfrak{g}_x}$, and H' for the union of the connected components of $M^{\mathfrak{g}_x}$ that meet M^{reg}. Then*

(i) *We have $M^{\mathrm{reg}} = M_x^{\widetilde{\mathrm{inf}}}$.*

(ii) *The set H' is a closed C^k submanifold of M and is $\mathrm{N}_G(\mathfrak{g}_x)$-invariant. Further $M^{\mathrm{reg}} \cap M^{\mathfrak{g}_x}$ is open and dense in H', and equal to the set of regular points for the $\mathrm{N}_G(\mathfrak{g}_x)/(G_x)^\circ$-action on H'. The G-action induces a G-equivariant C^k diffeomorphism from the associated fiber bundle*

$$G/(G_x)^\circ \times_{\mathrm{N}_G(\mathfrak{g}_x)/(G_x)^\circ} (M^{\mathrm{reg}} \cap M^{\mathfrak{g}_x}) \xrightarrow{\sim} M^{\mathrm{reg}}.$$

Also $(\mathrm{N}_G(\mathfrak{g}_x)/(G_x)^\circ)\backslash M^{\mathrm{reg}} \cap M^{\mathfrak{g}_x} \cong G\backslash M^{\mathrm{reg}}$.

(iii) *$\mathrm{N}_G(\mathfrak{g}_x) \cdot C' = M^{\mathrm{reg}} \cap M^{\mathfrak{g}_x}$; and $G_{(C')} := \{\, g \in G \mid A(g)(C') = C'\,\}$ is an open subgroup of $\mathrm{N}_G(\mathfrak{g}_x)$, and the action $A : G \times C' \to M$ induces a G-equivariant C^k diffeomorphism*

$$G/(G_x)^\circ \times_{G_{(C')}/(G_x)^\circ} C' \xrightarrow{\sim} M^{\mathrm{reg}}.$$

(iv) *If $(C')^{\mathrm{c}}$ denotes the closure of C', then $(C')^{\mathrm{c}} \subset H'$ and $G \cdot (C')^{\mathrm{c}} = M$.*

Corollary 0.2.25. *Assume that $G\backslash M$ is connected and also, for $x \in M^{\mathrm{princ}}$, that G_x is connected. Then the submanifold H in Corollary 0.2.23 is equal to the submanifold H' in Proposition 0.2.24, and further $\mathrm{N}_G(\mathfrak{g}_x) = \mathrm{N}_G(G_x)$, $M^{\mathfrak{g}_x} = M^{G_x}$, $M^{\mathrm{princ}} \cap M^{\mathfrak{g}_x}$ is dense in H, $C \subset C' \subset H$, $G_{(C)} = G_{(C')} \subset \mathrm{N}(G_x)$.*

0.3 Group Actions: Third Lecture: Application to Compact Lie Groups

0.3.1 Introduction

Throughout this chapter, G will be a compact Lie group. It acts on itself by means of the conjugation $\mathbf{Ad}\, g(x) = gxg^{-1}$, for $(g, x) \in G \times G$, as defined in (8). Because G is compact, this action is proper. Applying the general principles from the preceding lectures, we shall obtain a detailed description of the structure of G. This includes the basic theorems about connected, compact Lie groups, viz.:

(i) The **Maximal Torus Theorem** 0.4.10, stating that every element of G is conjugate to an element of a maximal torus T. Also, all maximal tori are conjugate to each other and equal to the centralizer of an element of principal orbit type.

(ii) The **Weyl Covering Theorem** 0.4.11 and the **Weyl Integration Theorem** 0.5.4, concerned with the mapping Γ from $G/T \times T$ onto G, defined by the action of conjugation.

In several respects, the action of conjugation of a compact Lie group on itself enjoys special properties when compared to general actions of compact Lie groups. Sometimes therefore one could give direct, easy proofs, without having to refer to the general theory. Still, even in these cases, we deem it instructive to treat the structure theory of compact Lie groups in the framework of general proper group actions.

Another disadvantage of proceeding from the more general theory to this special situation, is that the same principle may appear several times under slightly different guises, which may make orientation more difficult. For this reason it might be helpful to start by browsing through this lecture, in order to see how the final results, say for connected, compact Lie groups, look like.

0.3.2 Centralizers

For the action by conjugation of a compact Lie group G on itself, the general theory of proper actions from the preceding lectures immediately leads to a number of interesting conclusions.

(a) (See Sections 0.1.11 and 0.1.23.) The orbit space $(\mathbf{Ad}\, G)\backslash G$, the space of conjugacy classes in G, is a compact, Hausdorff topological space. Functions on it are identified with the functions on G that are constant on the conjugacy classes, that is

$$f(gxg^{-1}) = f(x) \quad (x, g \in G).$$

These are also called the *class functions* on G. The space of C^k functions ($1 \leq k \leq \omega$) on the orbit space is defined as the space of C^k class functions on G.

(b) In Sections 0.1.14 – 0.1.21, the basic tool is the concept of a *slice* at any point x, a submanifold S through x whose tangent space is complementary to the tangent space of the orbit. It is also invariant under the action of the stabilizer group G_x of x, and according to the Tube Theorem 0.1.22 the action of G near x can be completely described in terms of G_x and the linear action of G_x on $\mathrm{T}_x\, S$.

For the action by conjugation of G on itself, the stabilizer or isotropy group of $x \in G$ is equal to the *centralizer*

$$G_x = \mathrm{Z}_G(x) := \{\, g \in G \mid gx = xg \,\}$$

of x in G. Then G_x is a closed, hence Lie subgroup of G, with Lie algebra equal to

$$\mathrm{T}_1(G_x) = \mathfrak{g}_x := \mathfrak{z}_{\mathfrak{g}}(x) := \ker(\mathrm{Ad}\, x - \mathrm{I}),$$

the *centralizer of x in $\mathfrak{g}$*. Indeed, $Y \in \mathfrak{g}$ belongs to the Lie algebra of G_x if and only if

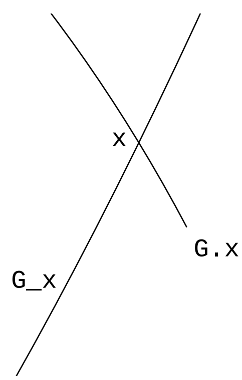
$$\exp tY = x(\exp tY)x^{-1} = \exp \mathrm{Ad} x(tY) = \exp t\, \mathrm{Ad}\, x(Y), \quad \text{for all } t \in \mathbf{R};$$

and this is equivalent to $\mathrm{Ad}\, x(Y) = Y$.

However, because G acts on itself, G_x can also be seen as a subset of the manifold on which G acts. As such, it is a closed real-analytic submanifold, which passes through the point x, that is,

$$x \in G_x. \tag{20}$$

The consequences of this are quite remarkable.



Proposition 0.3.3. (i) *Near x, the set G_x is a slice at x for the action by conjugation, and locally it is the only one. The tangent space to the orbit is given by (cf. (5))*

$$\alpha_x(\mathfrak{g}) := \mathrm{T}_x(\mathbf{Ad}\, G(x)) = \mathrm{T}_1\ \mathrm{R}(x)\big(\mathrm{im}(\mathrm{Ad}\, x - \mathrm{I})\big),$$

and the tangent space to the slice by

$$\mathrm{T}_x(G_x) = \mathrm{T}_1\ \mathrm{R}(x)(\mathfrak{g}_x) = \mathrm{T}_1\ \mathrm{R}(x)\big(\ker(\mathrm{Ad}\, x - \mathrm{I})\big).$$

(ii) *The logarithmic chart at x intertwines the action of G_x in G, near x, with the adjoint representation of G_x in $\mathfrak{g}$, near the origin.*

Combining now the Tube Theorem 0.1.22, see the theory of Sections 0.1.14 - 0.1.21, with Proposition 0.3.3, we get the following conclusions. In an open, conjugacy invariant neighborhood of x in G, the action of conjugation is analytically equivalent to the G-action on $G \times_{G_x} U$, where U is an $\mathrm{Ad}\, G_x$-invariant open neighborhood of 0 in $\mathfrak{g}_x$, and the action of G_x on U is the adjoint one. The C^k class functions near x are identified (in the logarithmic chart and via restriction to $\mathfrak{g}_x$) with the $\mathrm{Ad}\, G_x$-invariant functions of class C^k on $\mathfrak{g}_x$ near 0, or with the G_x-class functions of class C^k on G_x, near the identity element of G_x.

In the sequel, we shall not only use centralizers in G and $\mathfrak{g}$, respectively, of single elements of G, but also of subgroups. Furthermore, we shall need the centralizers in G and $\mathfrak{g}$, respectively, of elements in $\mathfrak{g}$. Below we have collected the definitions of all the combinations that occur later on.

For any subset H of any Lie group G,

$$G_H := \mathrm{Z}_G(H) := \bigcap_{h \in H} G_h = \{\, g \in G \mid gh = hg, \text{ for all } h \in H \,\}$$

is called the *centralizer of H in G*. Here H is regarded as a subset of the manifold on which G acts. However, considering H as a subset of the group which acts, $\mathrm{Z}_G(H)$ can also be recognized as the set of fixed points of H, or, in the notation of Definition 0.2.4,

$$G_H = \mathrm{Z}_G(H) = G^H.$$

$\mathrm{Z}_G(H)$ is a closed, hence Lie subgroup of G, with Lie algebra equal to

$$\mathfrak{z}_{\mathfrak{g}}(H) := \bigcap_{h \in H} \mathfrak{g}_h = \{\, X \in \mathfrak{g} \mid \mathrm{Ad}\, h(X) = X, \text{ for all } h \in H \,\} = \mathfrak{g}^{\mathrm{Ad}\, H},$$

called the *centralizer of H in $\mathfrak{g}$*. If $H = G$, then

$$\mathrm{Z}(G) := \mathrm{Z}_G(G) = \{\, g \in G \mid gx = xg, \text{ for all } x \in G \,\}$$

is called the *center* of G. Its Lie algebra is equal to

$$\mathfrak{z}(G) := \mathfrak{z}_{\mathfrak{g}}(G),$$

which is called the *center of G in $\mathfrak{g}$*.

Note that if H is a **connected** Lie subgroup of G with Lie algebra equal to $\mathfrak{h}$, then

$$\mathrm{Z}_G(H) = \mathrm{Z}_G(\mathfrak{h}) := \{\, g \in G \mid \mathrm{Ad}\, g\,(Y) = Y, \text{ for all } Y \in \mathfrak{h} \,\},$$

the *centralizer of $\mathfrak{h}$ in G*. Furthermore,

$$\mathfrak{z}_{\mathfrak{g}}(H) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) := \{\, X \in \mathfrak{g} \mid [\, X, Y\,] = 0, \text{ for all } Y \in \mathfrak{h} \,\},$$

is said to be the *centralizer of $\mathfrak{h}$ in $\mathfrak{g}$*. If G itself is connected, then (see Definition 0.1.7)

$$\mathrm{Z}(G) = \ker \mathrm{Ad}, \quad \text{with } \mathrm{Ad} : G \to \mathbf{GL}(\mathfrak{g}),$$

and the Lie algebra of the center $\mathrm{Z}(G)$ of G is equal to the *center* $\mathfrak{z}$ of $\mathfrak{g}$, given by

$$\mathfrak{z} := \mathfrak{z}(\mathfrak{g}) := \{\, X \in \mathfrak{g} \mid [\, X, Y\,] = 0, \text{ for all } Y \in \mathfrak{g} \,\} = \ker \mathrm{ad}, \qquad \text{with} \qquad \mathrm{ad} : \mathfrak{g} \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{g}).$$

(c) G has a Riemannian structure which is left and right invariant, and therefore also conjugacy invariant. These Riemannian structures on G are real-analytic, and the mapping $\beta \mapsto \beta_1$ is a bijection from the space of these onto the space of $\mathrm{Ad}\, G$-invariant inner products on $\mathfrak{g}$. The latter are obtained by averaging any inner product on $\mathfrak{g}$ over $\mathrm{Ad}\, G$. The proof is immediate, although one could also argue that it coincides with the proof of Proposition 0.1.25, when applied to the left-right action of $G \times G$ on G.

(d) In Section 0.2.1 the central theme is the study of the *orbit types*

$$G_x^{\sim} = \{ \, y \in G \mid G_y \text{ is conjugate to } G_x \text{ in } G \, \}.$$

From the description of the action of conjugation near x after Proposition 0.3.3, it follows that this action is completely determined, up to equivalence, by the group G_x (which determines the adjoint action of G_x on its own Lie algebra $\mathfrak{g}_x$). It follows that **the orbit types $G_x^{\sim}$ are equal to the local action types $G_x^{\approx}$** from Section 0.2.1. In the further description of the local action types, the fixed point set for the action of G_x, and the normalizer $\mathrm{N}(G_x)$ of G_x play an important part. About these we have some more consequences of (20):

Lemma 0.3.4. (i) *For every $x \in G$, the fixed point set $\mathrm{Z}_G(G_x)$ of G_x in G is contained in G_x, so it is equal to the center $\mathrm{Z}(G_x)$ of the group G_x.*

(ii) *The normalizer $\mathrm{N}(G_x)$ of G_x in G has the same dimension as G_x, so $\mathrm{N}(G_x)/G_x$ is a finite group.*

As for the proof of assertion (ii), we note the following. An element $g \in \mathrm{N}(G_x)$ close to 1 maps $x \in G_x$ to an element in G_x close to x, because of the continuity of the action. Because G_x, near x, is a slice at x by Proposition 0.3.3.(i), we conclude that $g \in G_x$. This proves that $\dim \mathrm{N}(G_x) = \dim G_x$. Because $\mathrm{N}(G_x)$, as a closed subgroup of G, is compact, $\mathrm{N}(G_x)/G_x$ is compact and discrete, hence finite.

Theorem 0.2.8 now leads to the following conclusions. To begin with, the orbit type $G_x^{\sim}$ is a locally closed, real-analytic submanifold of G, of codimension equal to the dimension of $G_x/\mathrm{Z}(G_x)$.

Furthermore, **the fixed point set $\mathrm{Z}(G_x)$ of G_x in G is an Abelian group** (a compact Lie subgroup of G_x, hence of G). Now $G_x^{\sim} \cap \mathrm{Z}(G_x)$ is an open subset of $\mathrm{Z}(G_x)$, containing x. The finite group $\mathrm{N}(G_x)/G_x$ acts freely on $G_x^{\sim} \cap \mathrm{Z}(G_x)$, and the quotient is naturally identified with $(\mathbf{Ad}\, G)\backslash G^{\sim}_{(\mathbf{Ad}\, G)\cdot x}$, the orbit type in the orbit space. In this way the latter gets the structure of a real-analytic manifold, of dimension equal to $\dim \mathrm{Z}(G_x)$.

The action of G induces a G-equivariant analytic diffeomorphism

$$G/G_x \times_{\mathrm{N}(G_x)/G_x} (G_x^{\sim} \cap \mathrm{Z}(G_x)) \xrightarrow{\sim} G_x^{\sim}.$$

The partitioning of $G_x^{\sim}$ into conjugacy classes defines a real-analytic fibration $G_x^{\sim} \to (\mathbf{Ad}\, G)\backslash G^{\sim}_{(\mathbf{Ad}\, G)\cdot x}$, with fiber equal to G/G_x. In other words, we have obtained

$$\begin{array}{ccccc}
G_x^{\sim} \cap \mathrm{Z}(G_x) & & G/G_x \times_{\mathrm{N}(G_x)/G_x} (G_x^{\sim} \cap \mathrm{Z}(G_x)) & \xrightarrow{\sim} & G_x^{\sim} \\
\searrow{\scriptstyle \text{p.f.b. with finite struct. gr. } \mathrm{N}(G_x)/G_x} & & \swarrow{\scriptstyle \text{associated f.b. with fiber } G/G_x} & & \downarrow{\scriptstyle \text{f.b. with fiber } G/G_x} \\
 & (\mathbf{Ad}\, G)\backslash G^{\sim}_{(\mathbf{Ad}\, G)\cdot x} & = & & (\mathbf{Ad}\, G)\backslash G^{\sim}_{(\mathbf{Ad}\, G)\cdot x}
\end{array}$$

(e) From Section 0.2.10, combined with the compactness of the manifold G on which G acts, we get that there are only finitely many orbit types, each of these having only finitely many connected components. These constitute a Whitney stratification in G.

If $y \in (G_x^{\sim})^{\mathrm{c}} \setminus G_x^{\sim}$, a "boundary point" of a stratum in $G_x^{\sim}$, then G_x is conjugate (in G) to a subgroup of G_y. Moreover, $\dim \mathrm{Z}(G_y) < \dim \mathrm{Z}(G_x)$, and therefore a fortiori

$$\operatorname{codim} G_y^{\sim} = \dim G_y/\mathrm{Z}(G_y) > \dim G_x/\mathrm{Z}(G_x) = \operatorname{codim} G_x^{\sim}.$$

(f) By definition and by (d), $x \in G$ is of *principal orbit type* if $G_x^{\sim}$ is open in G. The set of these is denoted by G^{princ}. We have $x \in G^{\text{princ}}$ if and only if $\dim \mathrm{Z}(G_x) = \dim G_x$, or $G_x / \mathrm{Z}(G_x)$ is a finite group, or **the adjoint representation of G_x on its Lie algebra $\mathfrak{g}_x$ is trivial**. This implies that $\mathfrak{g}_x$ is Abelian, and that $(G_x)^\circ = (\mathrm{Z}(G_x))^\circ$.

$G \setminus G^{\text{princ}}$ is a closed subset of G. It is equal to the disjoint union of finitely many locally closed, connected, real-analytic submanifolds of G (the connected components of the other orbit types), of codimension ≥ 1 in G.

In the union ${}'G$ of all connected components of G that meet a given conjugacy class, there is precisely one principal orbit type. In other words, the directed graph of orbit types in ${}'G$ is connected, and has a unique maximal element. In particular, there is only one principal orbit type in G°, the identity component of G. This implies that all G_x, for $x \in G^{\text{princ}} \cap G^\circ$, are conjugate to each other.

(g) An element x is said to be a *regular* element of G if all nearby orbits have the same dimension. That is, $\dim \mathfrak{g}_y = \dim \mathfrak{g}_x$, for all y near x in G; this in turn is equivalent to the condition that $\mathfrak{g}_x$ **is Abelian**. Indeed, let S be a slice at x, which is a neighborhood of x in G_x. If $y \in S$, then $g \in G_y$ implies that $\mathbf{Ad}\, g(y) = y \in S$, hence $g \in G_x$ (this is part of the definition of a slice). Or $G_y \subset G_x$; and in turn this implies that $\mathfrak{g}_y \subset \mathfrak{g}_x$. Because all elements in G near x are conjugate to an element of S, the conclusion is that x is regular if and only if $\mathfrak{g}_x \subset \mathfrak{g}_y$, for all $y \in G_x$ sufficiently close to x. If $y = zx$, this means that $\mathrm{Ad}\, z = \mathrm{I}$ on $\mathfrak{g}_x$ for all $z \in G_x$ near 1, which clearly is equivalent to the condition that $\mathfrak{g}_x$ is Abelian.

The set G^{reg} of regular elements in G is open. The complement $G^{\text{sing}} = G \setminus G^{\text{reg}}$, the set of *singular elements* in G, is equal to the disjoint union of finitely many connected, locally closed, real-analytic submanifolds of G of codimension ≥ 2. These are the *infinitesimal orbit types*, the partitioning of G into them is a coarsening of the partitioning into orbit types. They also define a Whitney stratification in G. In particular, $G^{\text{reg}} \cap C$ is connected for each connected component C of G.

The open set G^{reg} contains the open subset G^{princ}. The orbits in $G^{\text{reg}} \setminus G^{\text{princ}}$ are called *exceptional*; they have the same dimension as the nearby principal orbits. That is, if $x \in G^{\text{reg}} \setminus G^{\text{princ}}$, and $y \in G^{\text{princ}}$ belongs to the slice at x, then G_y is a subgroup of G_x of the same dimension; so these groups only differ by number of their connected components, but they have the same identity component. The injection $G_y \to G_x$ induces a finite covering $G/G_y \to G/G_x$, where G/G_y, and G/G_x, can be identified with the orbit through y, and x, respectively.

(h) Let $x \in G^{\text{princ}}$, write C for the connected component of x in $G^{\text{princ}} \cap \mathrm{Z}(G_x)$. Also, write ${}'G^{\text{princ}} = G^{\text{princ}} \cap {}'G$, where ${}'G$ is as in (f). Combining (d) and Corollary 0.2.23, it follows that every element of ${}'G^{\text{princ}} \cap \mathrm{Z}(G_x)$ is conjugate, by means of an element of $\mathrm{N}(G_x)$, to an element of C.

Furthermore, the group $\mathrm{N}(C) := \{ g \in G \mid gCg^{-1} = C \}$ is an open subgroup of $\mathrm{N}(G_x)$ containing G_x. The finite group $\mathrm{N}(C)/G_x$ acts freely on C, the orbit space is in bijective correspondence with the set of conjugacy classes in ${}'G^{\text{princ}}$. The action of G induces a G-equivariant analytic diffeomorphism

$$G/G_x \times_{\mathrm{N}(C)/G_x} C \xrightarrow{\sim} {}'G^{\text{princ}}.$$

Finally, **each element of ${}'G$ is conjugate to an element of C^{c}, the closure of C in G, which is contained in** $\mathrm{Z}(G_x)$

Proposition 0.3.5. *Let G be connected, $x \in G^{\mathrm{reg}}$. Then*

(i) $T := (G_x)^\circ$ *is a* **torus** *in G, that is, a connected, compact, Abelian subgroup of G.*

(ii) *Each element of G is conjugate to an element of T, and actually to an element in the closure of any connected component C of $T \cap G^{\mathrm{princ}}$.*

(iii) $x \in T$.

Remark 0.3.6. In Corollary 0.3.15, we shall actually see that y is of principal orbit type in a connected, compact Lie group G, if and only if G_y is a torus. ☆

Corollary 0.3.7. *For any connected, compact Lie group G with Lie algebra $\mathfrak{g}$, the exponential mapping is surjective: $\mathfrak{g} \to G$.*

Remark 0.3.8. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ which happens to be the Lie algebra of a **closed** subgroup H of G. Then $\exp \mathfrak{h}$ is a compact subset of G, and also a subgroup, because it is equal to H°. In particular, $\exp \mathfrak{g}_x = (G_x)^\circ$ for all $x \in G$. Note that for a general Lie subalgebra $\mathfrak{h}$ of the Lie algebra of a Lie group G, the subset $\exp \mathfrak{h}$ is neither closed in G, nor a subgroup of G. ☆

0.3.9 The Adjoint Action

Analogous conclusions as in Section 0.3.2 can be drawn for the adjoint action Ad of G on its Lie algebra $\mathfrak{g}$. Actually the situation is somewhat simpler here, due to the fact that $\mathfrak{g}$ is a vector space, on which moreover G acts by means of linear transformations.

(a) The orbit space $(\mathrm{Ad}\, G)\backslash\mathfrak{g}$ is a Hausdorff topological space. If we delete the origin, it has a natural *conic structure*, that is, a proper and free action of the multiplicative group $\mathbf{R}_{>0}$. Functions on it are the $\mathrm{Ad}\, G$-invariant functions on $\mathfrak{g}$, they are called the *class functions on* $\mathfrak{g}$ *with respect to the group G*. Note that if G is connected, then $\mathrm{Ad}\, G$ is equal to the Lie subgroup of $\mathbf{GL}(\mathfrak{g})$ generated by $e^{\mathrm{ad}\, \mathfrak{g}}$, the *adjoint group* of $\mathfrak{g}$, which is denoted by $\mathrm{Ad}\, \mathfrak{g}$. In this case we simply talk about the *class functions of* $\mathfrak{g}$. On $\mathfrak{g}$ we can also talk about the polynomial class functions, in addition to their C^k counterparts. This can be used to turn $(\mathrm{Ad}\, G)\backslash\mathfrak{g}$ into a real affine algebraic variety.

The action of G by conjugation in a suitable conjugacy-invariant open neighborhood of 1 in G is equivalent to the adjoint action of G on a corresponding $\mathrm{Ad}\, G$-invariant open neighborhood of 0 in $\mathfrak{g}$, via the logarithmic chart. By means of multiplications by nonzero scalars, which commute with the adjoint action, the adjoint action can completely be identified with the conjugation action near 1 in G. However, in the other direction, the action of conjugation in G not always can be recovered completely from the adjoint representation, not even if G is connected. See the discussion in Section 0.3.16 of the examples of $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$. The local equivalence implies that centralizers in G of elements in G near 1 are equal to those of elements in $\mathfrak{g}$ near 0. This is formulated somewhat more precisely in the following lemma; actually the compactness of G does not play any role in it.

Lemma 0.3.10. *Let U be an* $\mathrm{Ad}\, G$*-invariant subset of* $\mathfrak{g}$ *on which the exponential mapping is injective. Then* $G_X = G_{\exp X}$*, for all* $X \in U$*. In particular,* $G_{\exp X} = G_{tX}$*, if* $t \in \mathbf{R} \setminus \{0\}$ *is sufficiently small.*

(b) We now turn to the description of the slices at $X \in \mathfrak{g}$. The analogue of (20) for the adjoint representation is

$$X \in \mathfrak{g}_X, \quad \text{or } [\, X, X\,] = 0.$$

One may also view this as a consequence of the existence of an Abelian Lie subgroup of G with Lie algebra equal to $\mathbf{R} \cdot X$.

In view of the remarks above about the relation between the adjoint action and the action of conjugation in G, Proposition 0.3.3 immediately implies

Proposition 0.3.11. *For every* $X \in \mathfrak{g}$*, we have* $\mathrm{T}_X(\mathrm{Ad}\, G\,(X)) = \mathrm{im}(\mathrm{ad}\, X)$*. Near* X*, the Lie subalgebra* $\mathfrak{g}_X = \mathrm{T}_1(G_X) = \ker(\mathrm{ad}\, X)$ *is a slice at* X *for the adjoint representation of* G*, and it is locally the only one.*

The theory of Sections 0.1.14 - 0.1.21 now leads to the following conclusions. In an open, $\mathrm{Ad}\, G$-invariant neighborhood of X in $\mathfrak{g}$, the adjoint action is analytically equivalent to the G-action on $G \times_{G_X} U$, where U is an $\mathrm{Ad}\, G_X$-invariant neighborhood of 0 in $\mathfrak{g}_X$, and the action of G_X on $\mathfrak{g}_X$ is the adjoint one. The C^k class functions near X are identified with the $\mathrm{Ad}\, G_X$-invariant C^k functions on $\mathfrak{g}_X$, near 0.

(c) The Lie algebra $\mathfrak{g}$ has an $\mathrm{Ad}\, G$-invariant inner product.

(d) From Lemma 0.3.4, we get the following:

Lemma 0.3.12. (i) *For every* $X \in \mathfrak{g}$*, the fixed point set* $\mathfrak{z}_{\mathfrak{g}}(G_X)$ *of* $\mathrm{Ad}\, G_X$ *in* $\mathfrak{g}$ *is contained in* $\mathfrak{g}_X$*, so is equal to* $\mathfrak{z}(G_X)$*, the Lie algebra of* $\mathrm{Z}(G_X)$*.*

(ii) *The normalizer* $\mathrm{N}(G_X)$ *of* G_X *in* G *has the same dimension as* G_X*, and* $\mathrm{N}(G_X)/G_X$ *is a finite group.*

This time, Theorem 0.2.8 takes the following form. For each $X \in \mathfrak{g}$, the orbit type

$$\mathfrak{g}_X^{\sim} = \{\, Y \in \mathfrak{g} \mid \mathfrak{g}_Y = \mathrm{Ad}\, g(\mathfrak{g}_X), \ \text{for some } g \in G\,\}$$

is equal to the local action type $\mathfrak{g}_X^{\approx}$, according to (b). It is a locally closed real-analytic submanifold of $\mathfrak{g}$, of codimension equal to $\dim \mathfrak{g}_X/\mathfrak{z}(G_X)$.

The set $\mathfrak{g}_X^{\sim} \cap \mathfrak{z}(G_X)$ is an open subset of the **Abelian** Lie subalgebra $\mathfrak{z}(G_X)$ of $\mathfrak{g}_X$, containing the point X. The finite group $\mathrm{N}(G_X)/G_X$ acts freely on it, and the quotient is naturally identified with $(\mathrm{Ad}\, G)\backslash\mathfrak{g}_{(\mathrm{Ad}\, G)X}^{\sim}$; hence the orbit type stratum in the orbit space is a real-analytic manifold of dimension equal to $\dim \mathfrak{z}(G_X)$.

The adjoint action of G induces a G-equivariant analytic diffeomorphism

$$G/G_X \times_{\mathrm{N}(G_X)/G_X} (\mathfrak{g}_X^{\sim} \cap \mathfrak{z}(G_X)) \xrightarrow{\sim} \mathfrak{g}_X^{\sim}.$$

The partitioning of $\mathfrak{g}_X^{\sim}$ into $\mathrm{Ad}\, G$-orbits defines a real-analytic fibration

$$\mathfrak{g}_X^{\sim} \to (\mathrm{Ad}\, G)\backslash\mathfrak{g}_{(\mathrm{Ad}\, G)X}^{\sim},$$

with fiber G/G_X.

(e) Using the identification of the orbit types with those near the origin, by means of the conic structure, we obtain from Section 0.2.10 the following results. There are only finitely many orbit types in $\mathfrak{g}$, and each of these has only finitely many connected components; they define a Whitney stratification in $\mathfrak{g}$.

If $X, Y \in \mathfrak{g}$, $Y \in (\mathfrak{g}_X^\sim)^c \setminus \mathfrak{g}_X^\sim$, then $\mathrm{Ad}\, u(\mathfrak{g}_X) \subset \mathfrak{g}_Y$, for some $u \in G$, and $\dim \mathfrak{z}(G_Y) < \dim \mathfrak{z}(G_X)$, and therefore a fortiori $\operatorname{codim} \mathfrak{g}_Y^\sim = \dim G_Y/\mathfrak{z}(G_Y) > \dim G_X/\mathfrak{z}(G_X) = \operatorname{codim} \mathfrak{g}_X^\sim$.

(f) The element $X \in \mathfrak{g}$ is of *principal orbit type for the adjoint action* if and only if $\mathfrak{g}_X^\sim$ is open, or $\dim \mathfrak{z}(G_X) = \dim \mathfrak{g}_X$, or *the adjoint representation of G_X on its Lie algebra $\mathfrak{g}_X$ is trivial.* This implies that $\mathfrak{g}_X$ is Abelian.

Because $\mathfrak{g}$ is connected, there is only one principal orbit type in $\mathfrak{g}$, denoted by $\mathfrak{g}^{\mathrm{princ}}$. This implies that **all** G_X, for $X \in \mathfrak{g}^{\mathrm{princ}}$ **are conjugate to each other**. The complement of $\mathfrak{g}^{\mathrm{princ}}$ in $\mathfrak{g}$ is closed and composed of finitely many locally closed, connected, real-analytic submanifolds of $\mathfrak{g}$ (the connected components of the other orbit types), of codimension ≥ 1 in $\mathfrak{g}$.

(g) X is a *regular element of* $\mathfrak{g}$, if all nearby adjoint orbits have the same dimension, that is, $\dim \mathfrak{g}_Y = \dim \mathfrak{g}_X$, for all Y near X in $\mathfrak{g}$. Using the slice $\mathfrak{g}_X$ at X, we obtain as in Section 0.3.2.(g) that X is regular in $\mathfrak{g}$ if and only if $\mathfrak{g}_X$ **is Abelian**. Note that this condition is formulated in terms of the Lie algebra structure of $\mathfrak{g}$ only, the reason being that regularity is defined in terms of the infinitesimal action, ad, of Ad. The set $\mathfrak{g}^{\mathrm{reg}}$ of regular elements in $\mathfrak{g}$ is connected, because the vector space $\mathfrak{g}$ is connected. In Corollary 0.3.15, we shall actually see that $\mathfrak{g}^{\mathrm{reg}} = \mathfrak{g}^{\mathrm{princ}}$.

(h) Let $X \in \mathfrak{g}^{\mathrm{princ}}$, and let $\mathfrak{c}$ be the connected component of X in $\mathfrak{g}^{\mathrm{princ}} \cap \mathfrak{z}(G_X)$. Then $\mathrm{Ad}\, \mathrm{N}(G_X)\,(\mathfrak{c}) = \mathfrak{g}^{\mathrm{princ}} \cap \mathfrak{z}(G_X)$.

Furthermore, $\mathrm{N}(\mathfrak{c}) := \{\, g \in G \mid \mathrm{Ad}\, g(\mathfrak{c}) = \mathfrak{c} \,\}$ is an open subgroup of $\mathrm{N}(G_X)$ containing G_X, so that $\mathrm{N}(\mathfrak{c})/G_X$ is a finite group which acts freely on $\mathfrak{c}$; the quotient space can be identified with $\mathrm{Ad}\, G \backslash \mathfrak{g}^{\mathrm{princ}}$. The adjoint action of G induces a G-equivariant analytic diffeomorphism: $G/G_X \times_{\mathrm{N}(\mathfrak{c})/G_X} \mathfrak{c} \xrightarrow{\sim} \mathfrak{g}^{\mathrm{princ}}$.

Finally, $\mathfrak{g} = \mathrm{Ad}\, G(\mathfrak{c}^{\mathrm{c}})$.

0.3.13 Connectedness of Centralizers

In general it is a nontrivial problem to decide whether the set of solutions of certain equations is connected, and this applies also to the centralizer groups appearing in Sections 0.3.2 and 0.3.9. We note here that, in a connected, compact Lie group, the centralizer of any element in the Lie algebra is connected, indeed:

Theorem 0.3.14. *Let G be a connected, compact Lie group. Then*

(i) $x \in (G_x)^\circ$*, for every* $x \in G$.

(ii) G_X *is connected, for every* $X \in \mathfrak{g}$.

(iii) G_S *is connected, for every torus* $S \subset G$.

Corollary 0.3.15. *Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}^{\mathrm{princ}} = \mathfrak{g}^{\mathrm{reg}}$; that is, there are no exceptional orbits in $\mathfrak{g}$. Also, if $x \in G$, then G_x is a torus if and only if $x \in G^{\mathrm{princ}}$. All these tori G_x, $x \in G^{\mathrm{princ}}$, are conjugate to each other.*

Theorem 0.3.14 leads to several simplifications of the results in Sections 0.3.2 and 0.3.9, for connected G. For instance, if $X \in \mathfrak{g}$, then $\mathfrak{z}(G_X) = \mathfrak{z}(\mathfrak{g}_X)$. Also

$$\mathrm{N}(G_X) = \mathrm{N}(\mathfrak{g}_X) := \{\, y \in G \mid \mathrm{Ad}\, y(\mathfrak{g}_X) = \mathfrak{g}_X \,\},$$

the *normalizer of* $\mathfrak{g}_X$ *in* G. The groups $\mathrm{N}(G_X)/G_X$ (and $\mathrm{N}(G_x)/G_x$, for $x \in G^{\mathrm{princ}}$) coincide with the component groups of $\mathrm{N}(G_X)$ (and $\mathrm{N}(G_x)$, respectively, for $x \in G^{\mathrm{princ}}$).

For $X \in \mathfrak{g}^{\mathrm{reg}}$, the quotient $\mathrm{N}(T)/T$ is called the *Weyl group* of the Abelian subalgebra $\mathfrak{t} = \mathfrak{g}_X$, here $T = \exp \mathfrak{t} = G_X$. We will return to this in the Sections 0.4.1, 0.4.9, and 0.4.12, where the use of a little more linear algebra leads to a considerable further clarification.

0.3.16 The Group of Rotations and its Covering Group

In $\mathbf{SO}(3)$ the conjugacy classes of all elements $x \neq 1$ are two-dimensional, so all these elements are regular. Let R_a denote the rotation about the vertical axis through the angle $a \in \,]\,0, \pi\,]$. If the element $x \in \mathbf{SO}(3)$ commutes with R_a, then it has to leave invariant the vertical axis, that is the eigenspace of R_a for the eigenvalue 1. On it, it acts as the identity or as the reflection -1. In the horizontal plane, x then acts as a rotation, or as an orientation-reversing orthogonal linear transformation. As is easily verified, the second case is excluded if $a < \pi$; the centralizer of R_a in $\mathbf{SO}(3)$ then is equal to the circle group T of all rotations about the vertical axis. It follows that all R_a with $a \in \,]\,0, \pi\,[$ are of principal orbit type.

The normalizer $\mathrm{N}(T)$ of T in $\mathbf{SO}(3)$ consists of all $x \in \mathbf{SO}(3)$ that leave the vertical axis invariant; then it is equal to the disjoint union of T and sT, where we can take for s the element that sends the standard basis vectors e_1, e_2, e_3 to e_2, e_1, $-e_3$, respectively. $\mathrm{N}(T)$ has therefore two connected components, the Weyl group $\mathrm{N}(T)/T$ is the commutative group of two elements. Because the centralizer of R_π is equal to $\mathrm{N}(T)$, the element R_π is not of principal orbit type, its conjugacy class therefore is the only exceptional orbit. We may describe $\mathbf{SO}(3)^{\mathrm{princ}}$ as an open ball with the center deleted, and fibered by the conjugacy classes, which are the concentric spheres. Furthermore, the conjugacy class of R_π may be described as a real projective plane; the fact that it is even topologically different from a sphere causes that it cannot take part in the fibration by spheres, and this confirms again the exceptional nature of the conjugacy class of R_π.

$\mathrm{N}(T)$ itself is a quite interesting noncommutative group, with commutative identity component T. On T, the conjugation action of $\mathrm{N}(T)$ is by a reflection, the orbits consist of two points, except for the fixed points R_0 and R_π. On the other hand, using that $R_a \cdot s = s \cdot R_{-a}$ and $s^2 = 1$, we get $R_a \cdot (s \cdot R_b) \cdot R_a^{-1} = s \cdot R_{b-2a}$, so $R_a \in T$ acts on $s \cdot T$ as a translation over $-2a$. That is, the connected component $s \cdot T$ is a single orbit for the action by conjugation. This may also serve as a warning that in nonconnected compact Lie groups, the regular orbits can easily have different dimensions in different connected components of the group.

If we take $G = \mathbf{SU}(2)$, then $G^{\mathrm{princ}} = G^{\mathrm{reg}} = \mathbf{SU}(2) \setminus \{-1, 1\}$, which is fibered by the conjugacy classes (2-dimensional spheres). Both for $\mathbf{SO}(3)$ and for $\mathbf{SU}(2)$, the orbit space is homeomorphic to a segment on the real axis, for $\mathbf{SO}(3)$ however, one of the end points represents an exceptional orbit, whereas for $\mathbf{SU}(2)$ both endpoints represent singular orbits (actually elements of the center).

The Lie algebras of $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$ are isomorphic, the adjoint group acts on them as the rotation group. Clearly the regular = principal set is equal to the complement of the origin, and fibered by concentric spheres. The orbit space is homeomorphic to $[\,0, \infty\,[$, with the conic structure on $]\,0, \infty\,[$. It is also easy here to determine for which elements in the Lie algebra the conclusion in Lemma 0.3.10 does not hold.

STRUCTURE THEORY

Johan A.C. Kolk

(1)	Tu	June	8	[LG]: Sections 3.5 - 3.7, 3.10
(2)	We	June	9	[LG]: Sections 3.13 - 3.14
(3)	Th	June	10	[LG]: Sections 4.0, 4.9
(4)	We	June	16	[RI]: Appendix C

(1)	Roots and root spaces, Compact Lie algebras, Maximal tori, The Weyl group as a reflection group
(2)	Integration, The Weyl Integration Theorem
(3)	Introduction, Weyl's character formula
(4)	Universal enveloping algebra, Poincaré–Birkhoff–Witt Theorem, Verma modules

In this course we shall treat results that are foundational for the other courses. We begin with a review of the structure of compact semisimple Lie groups and Lie algebras, with attention to maximal tori, root space decompositions, Weyl groups and the Weyl integration formula. Here we shall see a strong interaction with the course *Group Actions*. We also study examples of noncompact Lie groups, in particular, $\mathbf{SL}(n, \mathbf{R})$, and decompositions of such groups. Next comes the abstract representation theory of compact Lie groups and the classification by highest weight of the finite-dimensional representations. The algebraic tools associated for this, the universal enveloping algebra and the Poincaré–Birkhoff–Witt Theorem, will also be studied, as well as the related theory of Verma modules.

0.4 Structure Theory: First Lecture

"Digging roots and lifting weights" (André Weil)

0.4.1 Roots and Root Spaces

Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G. It will be convenient to study $\operatorname{ad} X \in \operatorname{Lin}(\mathfrak{g}, \mathfrak{g})$, for $X \in \mathfrak{g}$, by passing to the complexification $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{g}$. Note that $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \oplus i\,\mathfrak{g}$ (as vector spaces over $\mathbf{R}$), and that $\mathfrak{g}_{\mathbf{C}}$ is a complex Lie algebra, with the unique complex bilinear mapping $[\,.\,,.\,] : \mathfrak{g}_{\mathbf{C}} \times \mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$ that extends the Lie bracket of $\mathfrak{g}$.

Every $A \in \operatorname{Lin}(\mathfrak{g}, \mathfrak{g})$ has a unique complex linear extension $\mathfrak{g}_{\mathbf{C}} \to \mathfrak{g}_{\mathbf{C}}$, which will also be denoted by A. Conversely, a map $A \in \operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ arises in this way, if and only if $A(\mathfrak{g}) \subset \mathfrak{g}$.

Now let $X \in \mathfrak{g}$. For every $t \in \mathbf{R}$, we have

$$\mathrm{e}^{t\,\operatorname{ad} X} = \mathrm{e}^{\operatorname{ad} tX} = \operatorname{Ad} \exp tX \in \operatorname{Ad} G,$$

which is a compact subset of $\operatorname{Lin}(\mathfrak{g}, \mathfrak{g})$, as the image of the compact group G under the continuous mapping Ad. It follows that the $\mathrm{e}^{t\,\operatorname{ad} X}$, for $t \in \mathbf{R}$, form a bounded subset of $\operatorname{Lin}(\mathfrak{g}, \mathfrak{g})$; and the same holds for the subset of $\operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ formed by the complex linear extensions.

The Jordan normal form theorem for $\operatorname{ad} X \in \operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ says that $\mathfrak{g}_{\mathbf{C}}$ can be written as the direct sum of complex linear subspaces $\mathfrak{g}_j$ with the following properties. Each $\mathfrak{g}_j$ is invariant under $\operatorname{ad} X$, and on it, $\operatorname{ad} X$ is of the form

$$\operatorname{ad} X|_{\mathfrak{g}_j} = c_j \mathrm{I} + N_j,$$

where $c_j \in \mathbf{C}$ and $N_j \in \operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_j, \mathfrak{g}_j)$ is nilpotent, that is, $(N_j)^m = 0$ for some $m \in \mathbf{Z}_{>0}$. For any linear mapping of the form $L = c\,\mathrm{I} + N$, with $c \in \mathbf{C}$ and $N^m = 0$, we have the formula

$$\mathrm{e}^{tL} = \mathrm{e}^{tc} \sum_{k=0}^{m-1} \frac{t^k}{k!} N^k.$$

Assuming that the vector space on which L acts is nonzero, we have $\ker N \neq 0$. On $\ker N$, the operator e^{tL} acts as multiplication by e^{tc}; so if $\{\, \mathrm{e}^{tL} \mid t \in \mathbf{R} \,\}$ is bounded, then c must be purely imaginary. Multiplying by the numbers e^{-tc}, for $t \in \mathbf{R}$, which remain bounded, we see by downward induction on k that also necessarily $N^k = 0$, for $k = m, \ldots, 1$, or $N = 0$. We have proved:

Lemma 0.4.2. *For each X in the Lie algebra $\mathfrak{g}$ of a compact Lie group,* $\operatorname{ad} X \in \operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ *is diagonalizable, with only purely imaginary eigenvalues.*

If $X, Y \in \mathfrak{g}$, and $[X, Y] = 0$, then $\operatorname{ad} X$ and $\operatorname{ad} Y$ commute, because ad is a homomorphism $\mathfrak{g} \to \operatorname{Lin}(\mathfrak{g}, \mathfrak{g}) \subset \operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$. This implies that $\operatorname{ad} Y$ leaves invariant the eigenspaces of $\operatorname{ad} X$. Any of these, say E, then decomposes further as the direct sum of the eigenspaces of $\operatorname{ad} Y|_E$. In this way $\mathfrak{g}_{\mathbf{C}}$ is the direct sum of the complex linear subspaces which are the common eigenspaces of $\operatorname{ad} X$ and $\operatorname{ad} Y$.

Assume now that $\mathfrak{t}$ is an *Abelian subspace* of $\mathfrak{g}$, that is, $\mathfrak{t}$ is a real linear subspace of $\mathfrak{g}$ such that $[X, Y] = 0$ whenever $X, Y \in \mathfrak{t}$. Applying the result above successively to a basis of $\mathfrak{t}$, we then get a direct sum decomposition of $\mathfrak{g}_{\mathbf{C}}$ into finitely many complex linear subspaces $\mathfrak{g}_j \neq 0$, and real-linear functions $\alpha_j : \mathfrak{t} \to \mathbf{C}$ (actually taking purely imaginary values only), such that

$$[X, Y] = \alpha_j(X) Y, \quad \text{for every } X \in \mathfrak{t},\ Y \in \mathfrak{g}_j.$$

The construction is such that $\alpha_j \neq \alpha_k$, whenever $j \neq k$.

A more intrinsic notation is obtained by defining, for every $\mathbf{R}$-linear function $\alpha : \mathfrak{t} \to \mathbf{C}$, the "common eigenspace for $\mathfrak{t}$"

$$\mathfrak{g}_\alpha := \mathfrak{g}_{\alpha^\mathfrak{t}} := \{\, Y \in \mathfrak{g}_\mathbf{C} \mid [\, X, Y\,] = \alpha(X)Y, \quad \text{for all } X \in \mathfrak{t}\,\}. \tag{21}$$

These complex linear subspaces of $\mathfrak{g}_\mathbf{C}$ are nonzero for only finitely many $\alpha \in (\mathfrak{t}^*)_\mathbf{C}$ (and then actually $\alpha(\mathfrak{t}) \subset i\mathbf{R}$), and $\mathfrak{g}_\mathbf{C}$ is equal to the direct sum of the nonzero $\mathfrak{g}_\alpha$. Also,

$$\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{-\alpha}, \tag{22}$$

where $^-$ denotes the complex conjugation $X + iY \mapsto X - iY$, for $X,\ Y \in \mathfrak{g}$. In particular, $\mathfrak{g}_{-\alpha}$ has the same dimension as $\mathfrak{g}_\alpha$; and the space $\mathfrak{g}_0$, which contains $\mathfrak{t}$, is invariant under complex conjugation, that is, $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{g}) + i(\mathfrak{g}_0 \cap \mathfrak{g})$.

$\mathfrak{t}$ is said to be a *maximal Abelian subspace* of $\mathfrak{g}$ if it is an Abelian subspace of $\mathfrak{g}$, and $\mathfrak{s} = \mathfrak{t}$ for every Abelian subspace $\mathfrak{s}$ of $\mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{s}$. Because $\mathfrak{t} + \mathbf{R} \cdot Y$ is an Abelian subspace of $\mathfrak{g}$ if $Y \in \mathfrak{g}_0 \cap \mathfrak{g}$, this amounts to requiring that $\mathfrak{t}$ is an Abelian subspace of $\mathfrak{g}$ such that

$$\mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{t}, \quad \mathfrak{g}_0 = \mathfrak{t} \oplus i\,\mathfrak{t} =: \mathfrak{h}.$$

The space $\mathfrak{h}$ then is a maximal Abelian subspace of $\mathfrak{g}_\mathbf{C}$, such that $\operatorname{ad} X \in \operatorname{Lin}_\mathbf{C}(\mathfrak{g}_\mathbf{C}, \mathfrak{g}_\mathbf{C})$ is diagonalizable for every $X \in \mathfrak{h}$. Such a subspace is said to be a *Cartan subalgebra* of the complex Lie algebra $\mathfrak{g}_\mathbf{C}$. The $\alpha \in (\mathfrak{t}^*)_\mathbf{C}$ extend to complex linear forms on $\mathfrak{h}$, also denoted by α; and $[\, X, Y\,] = \alpha(X)Y$, for every $X \in \mathfrak{h}$, $Y \in \mathfrak{g}_\alpha$. In other words, we may identify $(\mathfrak{t}^*)_\mathbf{C}$ with $\mathfrak{h}^*$, the complex dual of $\mathfrak{h}$.

The $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq 0$ **and** $\alpha \neq 0$ are called the *roots*, or *root forms* of the Cartan subalgebra $\mathfrak{h}$, and the $\mathfrak{g}_\alpha$ are the corresponding *root spaces*. The set of roots will be denoted by R. (The terminology comes from the computation of $\alpha(X)$ as the root of the eigenvalue equation $\det(\operatorname{ad} X - c\ \mathrm{I}) = 0$, $c \in \mathbf{C}$.)

Combined with (22), the direct sum decomposition

$$\mathfrak{g}_\mathbf{C} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha, \tag{23}$$

has as a real counterpart

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in P} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}.$$

Here $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}$, the real part of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, has real dimension equal to $2 \dim_\mathbf{C} \mathfrak{g}_\alpha$; and we need only to sum over a subset P of R which contains one of each pair $\alpha, -\alpha$ in R.

The $i^{-1}\alpha$, $\alpha \in R$, are nonzero real linear forms on $\mathfrak{t}$, so the $\ker \alpha \cap \mathfrak{t}$, with $\alpha \in R$, are real hyperplanes in $\mathfrak{t}$, called the *root hyperplanes* in $\mathfrak{t}$. An element $X \in \mathfrak{t}$ is said to be *regular* in $\mathfrak{t}$, if $\alpha(X) \neq 0$, for all roots α, that is, if it belongs to

$$\mathfrak{t}^{\mathrm{reg}} := \mathfrak{t} \setminus \bigcup_{\alpha \in R} \ker \alpha,$$

the complement in $\mathfrak{t}$ of the finitely many root hyperplanes. The elements of $\mathfrak{t}$ which lie on a root hyperplane are said to be *singular*. It is easy to verify that $\mathfrak{t}^{\mathrm{reg}} = \mathfrak{t} \cap \mathfrak{g}^{\mathrm{reg}}$, cf. Theorem 0.4.10.(ii) below, which explains this terminology. From the description in Corollary 0.4.16 below of the action of the Weyl group $W = \operatorname{Ad} \mathrm{N}(\mathfrak{t})|_\mathfrak{t} = \mathrm{N}(T)/T$ on $\mathfrak{t}$, it follows also that $\mathfrak{t}^{\mathrm{reg}}$ is also equal to the set where W acts freely; that is, $\mathfrak{t}^{\mathrm{reg}}$ is the principal orbit type for the action of W on $\mathfrak{t}$. (Warning: for the action of a finite group, every point is regular; so here it is essential to use the more refined concept of orbit types.)

Now let $\mathfrak{c}$ be a connected component of $\mathfrak{t}^{\mathrm{reg}}$. For each $\alpha \in R$, the real-valued continuous function $i^{-1}\alpha|_{\mathfrak{t}}$ has no zeros in $\mathfrak{c}$, so the connectedness of $\mathfrak{c}$ implies that it is either everywhere > 0 or everywhere < 0 on $\mathfrak{c}$. That is,

$$R = P \cup (-P), \quad P \cap (-P) = \emptyset, \tag{24}$$

if we take

$$P = P(\mathfrak{c}) := \{\alpha \in R \mid i^{-1}\alpha > 0 \text{ on } \mathfrak{c}\}, \tag{25}$$

and if we write $-P = \{-\alpha \mid \alpha \in P\}$. Conversely, if P is a subset of R satisfying (24), then the set

$$\mathfrak{c} = \mathfrak{c}(P) := \{X \in \mathfrak{t} \mid i^{-1}\alpha(X) > 0, \text{ for all } \alpha \in P\} \tag{26}$$

is an open, convex polyhedral cone in $\mathfrak{t}$, contained in $\mathfrak{t}^{\mathrm{reg}}$. It is nonvoid, and therefore equal to a connected component of $\mathfrak{t}^{\mathrm{reg}}$, if and only if the convex cone in $i\mathfrak{t}^*$ generated by P is *proper*, that is, if P satisfies

$$\sum_{\alpha \in P} c_\alpha \alpha = 0, \quad c_\alpha \geq 0, \text{ for all } \alpha \in P \qquad \Longrightarrow \qquad c_\alpha = 0, \text{ for all } \alpha \in P. \tag{27}$$

A subset P of R, satisfying (24) and (27), is said to be a *choice of positive roots*. The connected components of $\mathfrak{t}^{\mathrm{reg}}$ are called the *Weyl chambers* in $\mathfrak{t}$. We have proved:

Lemma 0.4.3. *Each of the relations (25), and (26), respectively, defines a bijective correspondence between the set of choices P of positive roots in R and the set of Weyl chambers $\mathfrak{c}$ in $\mathfrak{t}$.*

If one has fixed the choice of a Weyl chamber $\mathfrak{c}$, then it is customary to write

$$P = R^+, \quad \mathfrak{c} = \mathfrak{t}^+,$$

motivated by (25), (26), respectively.

0.4.4 Compact Lie Algebras

As an intermezzo, we give an algebraic characterization of those Lie algebras which arise as the Lie algebra of a compact Lie group.

For our purposes, a convenient tool is the bilinear form κ, defined on any Lie algebra $\mathfrak{g}$ by

$$\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y), \quad X, Y \in \mathfrak{g}.$$

Here tr denotes the trace of $\mathrm{ad}\, X \circ \mathrm{ad}\, Y \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{g})$. Then κ is called the *Killing form* of $\mathfrak{g}$; and it will be denoted by $\kappa_{\mathfrak{g}}$ or $\kappa_{\mathfrak{g}}^{\mathbf{R}}$, if there is danger of confusion. This bilinear form is **symmetric**, because in general $\mathrm{tr}(A \circ B) = \mathrm{tr}(B \circ A)$, for linear endomorphisms A and B. Because the real trace of a linear mapping is equal to the complex trace of its complex linear extension, the Killing form extends to a complex bilinear form on $\mathfrak{g}_{\mathbf{C}}$ by means of the formula

$$\kappa(X, Y) = \mathrm{tr}_{\mathbf{C}}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y), \quad X, Y \in \mathfrak{g}_{\mathbf{C}},$$

where now $\mathrm{ad}\, X \circ \mathrm{ad}\, Y$ is considered as an element of $\mathrm{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$. Note however that the Killing form of $\mathfrak{g}_{\mathbf{C}}$, if we consider $\mathfrak{g}_{\mathbf{C}}$ as a Lie algebra over $\mathbf{R}$, is equal to $2\,\mathrm{Re}\,\kappa_{\mathfrak{g}}^{\mathbf{R}}$; this because the trace of multiplication by $c \in \mathbf{C}$, considered as a real linear mapping: $\mathbf{C} \to \mathbf{C}$, is equal to $2\,\mathrm{Re}\, c$.

Now let $\mathfrak{g}$ satisfy the conclusion of Lemma 0.4.2, and let $\mathfrak{t}$ be a maximal Abelian subspace of $\mathfrak{g}$. Then, for $X \in \mathfrak{t}$,

$$\kappa\,(X, X) = \operatorname{tr}((\operatorname{ad} X)^2) = \sum_{\alpha \in R} \big(\alpha(X)\big)^2 \dim_{\mathbf{C}} \mathfrak{g}_\alpha,$$

cf. (23) and (21). Because $\alpha(X) \in i\mathbf{R}$, for all $\alpha \in R$, it follows that $\kappa\,(X, X) \leq 0$; and $\kappa\,(X, X) = 0$ if and only if $\alpha(X) = 0$ for all $\alpha \in R$. This in turn is the case if and only if X belongs to $\mathfrak{z}$, the center of $\mathfrak{g}$. For any $X \in \mathfrak{g}$, we have $[\,X, X\,] = 0$, so $\mathbf{R} \cdot X$ is an Abelian subspace of $\mathfrak{g}$; and this in turn is contained in a maximal Abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$. The conclusion is therefore that κ is negative semi-definite on $\mathfrak{g}$, and $\ker \kappa = \mathfrak{z}$.

It is also obvious that

$$\kappa_{\mathfrak{g}} = \pi^* \kappa_{\mathfrak{g}/\mathfrak{z}},$$

if π denotes the canonical projection: $\mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$. In particular, the Killing form of $\mathfrak{g}/\mathfrak{z}$ is negative definite in our situation.

For the next lemma, we note that the automorphism group $\operatorname{Aut}\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$ is a closed Lie subgroup of $\mathbf{GL}(\mathfrak{g})$; with Lie algebra equal to the space $\operatorname{Der}\mathfrak{g}$ of *derivations* of $\mathfrak{g}$, the linear mappings $D : \mathfrak{g} \to \mathfrak{g}$ such that

$$D([\,X, Y\,]) = [\,D(X), Y\,] + [\,X, D(Y)\,], \quad \text{for all } X, Y \in \mathfrak{g}.$$

This can also be written as

$$\operatorname{ad} D(X) = [\,D, \operatorname{ad} X\,] \text{ in } \operatorname{Lin}(\mathfrak{g}, \mathfrak{g}), \quad \text{for all } X \in \mathfrak{g}.$$

We also recall that the adjoint group $\operatorname{Ad}\mathfrak{g}$ of $\mathfrak{g}$ is the connected Lie subgroup of $\operatorname{Aut}\mathfrak{g}$ with Lie algebra $\operatorname{ad}\mathfrak{g} \subset \operatorname{Der}\mathfrak{g}$. Summarizing:

$$\begin{array}{ccccc}
e^{\operatorname{ad}\mathfrak{g}} = \operatorname{Ad}\mathfrak{g} & \hookrightarrow & \operatorname{Aut}\mathfrak{g} & \hookrightarrow & \mathbf{GL}(\mathfrak{g}) \\
\uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} & & \uparrow{\scriptstyle \exp} \\
\operatorname{ad}\mathfrak{g} & \hookrightarrow & \operatorname{Der}\mathfrak{g} & \hookrightarrow & \operatorname{Lin}(\mathfrak{g}, \mathfrak{g})
\end{array}$$

Lemma 0.4.5. *Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate Killing form κ; this implies that $\mathfrak{z} = 0$. Then $\operatorname{ad}\mathfrak{g} = \operatorname{Der}\mathfrak{g}$, and $\operatorname{Ad}\mathfrak{g} = (\operatorname{Aut}\mathfrak{g})^\circ$, a closed Lie subgroup of $\mathbf{GL}(\mathfrak{g})$. If moreover κ is definite, then $\operatorname{Aut}\mathfrak{g}$, and therefore also $\operatorname{Ad}\mathfrak{g}$, is compact; and κ is negative definite.*

In the following theorem we also encounter the *derived Lie algebra* $[\,\mathfrak{g}, \mathfrak{g}\,]$ of $\mathfrak{g}$; it is defined as the linear span of the $[\,X, Y\,]$, for $X, Y \in \mathfrak{g}$.

Theorem 0.4.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{R}$. Then the following conditions are equivalent*

(i) *$\mathfrak{g}$ is equal to the Lie algebra of a compact Lie group G.*

(ii) *$\operatorname{Ad}\mathfrak{g}$ is compact.*

(iii) *For each $X \in \mathfrak{g}$, the $e^{t \operatorname{ad} X}$, with $t \in \mathbf{R}$, form a bounded subset of $\operatorname{Lin}(\mathfrak{g}, \mathfrak{g})$.*

(iv) *For each $X \in \mathfrak{g}$, we have $\operatorname{ad} X \in \operatorname{Lin}_{\mathbf{C}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ is diagonalizable and has only purely imaginary eigenvalues.*

(v) *The Killing form of* $\mathfrak{g}$ *is negative semi-definite, and its kernel is equal to* $\mathfrak{z}$*, the center of* $\mathfrak{g}$*.*

(vi) *There exists an inner product* β *on* $\mathfrak{g}$ *such that, for all* $X, Y, Z \in \mathfrak{g}$*, we have*

$$\beta((\operatorname{ad} X)Y, Z) + \beta(Y, (\operatorname{ad} X)Z) = 0.$$

(vii) $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$*, for a Lie subalgebra* $\mathfrak{g}'$ *of* $\mathfrak{g}$*, on which the Killing form is negative definite.*

Finally, if $\mathfrak{g}'$ *is as in (vii), then* $[\,\mathfrak{g}, \mathfrak{g}\,] = \mathfrak{g}'$*.*

A Lie algebra is said to be *compact*, if any of the equivalent characterizations in Theorem 0.4.6 holds. Note that if $\mathfrak{z} \neq 0$, then obviously not **every** connected Lie group with Lie algebra equal to $\mathfrak{g}$ is compact. Also, $\operatorname{Aut}\mathfrak{g} = \operatorname{Aut}\mathfrak{g}' \times \mathbf{GL}(\mathfrak{z})$, $\operatorname{Der}\mathfrak{g} = \operatorname{Der}\mathfrak{g}' \times \operatorname{Lin}(\mathfrak{z}, \mathfrak{z}) = \operatorname{Ad}\mathfrak{g} \times \operatorname{Lin}(\mathfrak{z}, \mathfrak{z})$; so in this case the conclusion of Lemma 0.4.5 does not hold either. Finally, one can describe $\operatorname{Ad}\mathfrak{g}$ as the identity component of the compact algebraic group $\{\, \Phi \in \operatorname{Aut}\mathfrak{g} \;\big|\; \Phi|_Z = \text{ identity on } \mathfrak{z} \,\} = \operatorname{Aut}(\mathfrak{g}/\mathfrak{z})$.

One may show that if G is a connected compact Lie group, and the Lie algebra of G has zero center, then the fundamental group of G is finite. This can be applied to the adjoint group of a compact Lie algebra $\mathfrak{g}$. For any connected Lie group G with Lie algebra equal to $\mathfrak{g}/\mathfrak{z}$, the mapping Ad defines a covering: $G \to \operatorname{Ad}\mathfrak{g}$. Because the fiber is a subgroup of the fundamental group of $\operatorname{Ad}\mathfrak{g}$, it follows that G is compact.

For any Lie algebra $\mathfrak{g}$, the Killing form is nondegenerate if and only if $\mathfrak{g}$ is a so-called *semisimple* Lie algebra. Summarizing these remarks, we get the following:

Corollary 0.4.7. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra over* $\mathbf{R}$*. Then the following conditions are equivalent:*

(i) $\mathfrak{g}$ *has negative definite Killing form.*

(ii) $\mathfrak{g}$ *is compact and semisimple.*

(iii) $\mathfrak{g}$ *is compact and has zero center.*

(iv) *Every connected Lie group with Lie algebra isomorphic to* $\mathfrak{g}$ *is compact.*

In any of these cases, $\mathfrak{g} = [\,\mathfrak{g}, \mathfrak{g}\,]$*.*

A *complex torus* is defined as a connected, compact and Abelian complex Lie group. That is, a Lie group isomorphic to $(\mathbf{C}^n, +)/\Delta$, where Δ is a discrete subgroup of $(\mathbf{C}^n, +)$ which contains a real basis of $\mathbf{C}^n \simeq \mathbf{R}^{2n}$. Note that as a real Lie group, $\mathbf{C}^n/\Delta$ is isomorphic to the standard $2n$-dimensional torus $(\mathbf{R}/\mathbf{Z})^{2n}$, but two Δ's only yield isomorphic complex Lie groups if they can be mapped to each other by an element of $\mathbf{SL}(n, \mathbf{C})$.

Corollary 0.4.8. *A compact, connected, complex Lie group is a complex torus.*

0.4.9 Maximal Tori

From now on G will denote a connected, compact Lie group, with Lie algebra equal to $\mathfrak{g}$.

In this situation we know already several salient facts, see Proposition 0.3.5, Corollary 0.3.7, and Section 0.3.13. The concepts of root and root space, however, lead to considerable further clarification, as we want to show now.

A torus in G, that is, a connected, compact, Abelian subgroup of G, is said to be a *maximal torus* in G, if for any torus T' in G such that $T \subset T'$, we have $T = T'$. A subset S of G is said to be a *maximal Abelian subset* of G, if $S \subset \mathrm{Z}_G(S)$ and if, for any $S' \subset G$ such that $S \subset S' \subset \mathrm{Z}_G(S')$, we have $S = S'$. Clearly a torus in G is a maximal torus in G if it is a maximal Abelian subset of G, but there exist examples of maximal Abelian subsets that are no tori. For instance, the three rotations through the angle π, with axes equal to the three coordinate axes in $\mathbf{R}^3$, form a maximal Abelian subset of $\mathbf{SO}(3)$, as is not hard to verify.

We begin with some simple observations.

Theorem 0.4.10 (Maximal Torus Theorem). (i) *For any $X \in \mathfrak{g}^{\mathrm{reg}}$, $\mathfrak{t} := \mathfrak{g}_X$ is a maximal Abelian subspace of $\mathfrak{g}$. Furthermore, $T := G_X$ is a torus in G with Lie algebra equal to $\mathfrak{t}$, and T is a maximal Abelian subset of G. The same conclusions are valid if we replace X by $x \in G^{\mathrm{princ}}$.*

(ii) *All maximal Abelian subspaces $\mathfrak{t}$ of $\mathfrak{g}$, and maximal tori T in G, arise as $\mathfrak{g}_X$, and G_X, respectively, for some $X \in \mathfrak{g}^{\mathrm{reg}}$. Also, $\mathfrak{g}^{\mathrm{reg}} \cap \mathfrak{t} = \mathfrak{t}^{\mathrm{reg}}$, and it equals the complement of the root hyperplanes in $\mathfrak{t}$, for any maximal Abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$.*

(iii) *The equation $T = \exp \mathfrak{t}$ defines a bijective correspondence between the maximal Abelian subspaces $\mathfrak{t}$ of $\mathfrak{g}$ and the maximal tori T in G. Every connected Abelian subgroup of G is contained in a maximal torus in G, and every Abelian subset of $\mathfrak{g}$ is contained in a maximal Abelian subspace of $\mathfrak{g}$.*

(iv) *All maximal tori in G are conjugate to each other, and $\mathrm{Ad}\, G$ acts transitively on the set of maximal Abelian subspaces of $\mathfrak{g}$. Each element of G is conjugate to an element of a given maximal torus, and $\mathrm{Ad}\, G(\mathfrak{t}) = \mathfrak{g}$ for any maximal Abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$.*

The common dimension of the maximal tori in G, and of the maximal Abelian subspaces of $\mathfrak{g}$, is called the *rank* of G, and of $\mathfrak{g}$, respectively. It can also be defined as the dimension of the principal orbit type in the orbit space $(\mathbf{Ad}\, G)\backslash G$, and $(\mathrm{Ad}\, G)\backslash \mathfrak{g}$, respectively.

Theorem 0.4.11 (Weyl's Covering Theorem). *Let T be a maximal torus in G, with the maximal Abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$ as its Lie algebra. Then*

(i) *The Weyl group $W := \mathrm{N}(T)/T = \mathrm{N}(\mathfrak{t})/T = \mathrm{Ad}\, \mathrm{N}(\mathfrak{t})|_{\mathfrak{t}}$ is finite. The action of G by conjugation in G, and by the adjoint action in $\mathfrak{g}$, induces a G-equivariant real-analytic diffeomorphism*

$$\begin{array}{ccc} G/T \times_W (G^{\mathrm{reg}} \cap T) & \xrightarrow{\ \sim\ } & G^{\mathrm{reg}}, \\ & \searrow^{\text{associated f.b. with fiber } G/T} \quad \swarrow & \\ & (\mathbf{Ad}\, G)\backslash G^{\mathrm{reg}} & \end{array} \tag{28}$$

and

$$G/T \times_W \mathfrak{t}^{\mathrm{reg}} \overset{\sim}{\to} \mathfrak{g}^{\mathrm{reg}}, \tag{29}$$

respectively.

(ii) *W acts transitively on the set of connected components of $G^{\text{reg}} \cap T$, and also on the set of Weyl chambers in $\mathfrak{t}$.*

(iii) $G^{\text{reg}} \cap T = \{ x \in T \mid G_x \subset \mathrm{N}(T) \}$*, so* $G^{\text{princ}} \cap T$ *is precisely the subset of* $G^{\text{reg}} \cap T$ *on which W acts freely.*

The main argument in the proof of Theorem 0.4.11 is as follows. The set

$$U := G/T \times (G^{\text{reg}} \cap T)$$

is precisely the open subset of $G/T \times T$ on which the mapping

$$\Gamma : G/T \times T \to G \qquad \text{given by} \qquad \Gamma(gT, t) = gtg^{-1}$$

has a bijective tangent mapping. Also, (ii) implies that Γ is surjective; because G^{reg} is conjugacy invariant, it follows that $\Gamma(U) = G^{\text{reg}}$.

If $gtg^{-1} = hsh^{-1}$, for $g, h \in G$, and $s, t \in G^{\text{reg}} \cap T$, then, writing $k = h^{-1}g$, we get $s = ktk^{-1}$. It follows that $G_s = kG_tk^{-1}$, and also, on taking Lie algebras, $\mathfrak{g}_s = \mathrm{Ad}\, k(\mathfrak{g}_t)$. Because $s, t \in T$ and T is Abelian, G_s and G_t both contain T; so $\mathfrak{g}_s$ and $\mathfrak{g}_t$ both contain $\mathfrak{t}$. Because $x \mapsto \dim \mathfrak{g}_x$ is constant on connected components of G^{reg}, and G^{reg} is connected, cf. 0.3.2.(g), we get $\dim \mathfrak{g}_s = \dim \mathfrak{t} = \dim \mathfrak{g}_t$; hence $\mathfrak{g}_s = \mathfrak{t} = \mathfrak{g}_t$, and therefore $\mathfrak{t} = \mathrm{Ad}\, k(\mathfrak{t})$. Thus $k \in \mathrm{N}(\mathfrak{t}) = \mathrm{N}(T)$, or $kT \in W$.

Because W acts analytically on both G/T (from the right) and on the open subset $G^{\text{reg}} \cap T$ of T, the left hand side in (28) is a real-analytic manifold, "W-fold" covered by the set U. The mapping Γ induces a real-analytic, G-equivariant mapping from it to G^{reg}, and we have just shown that this mapping is bijective and has bijective tangent mappings. The inverse function theorem now implies that the inverse is real-analytic as well.

A generalization of (28) to arbitrary proper group actions can be found in Proposition 0.2.24.(ii). The basic fact that simplified the situation here is that G_x is connected for elements x of principal orbit type.

The diffeomorphisms (28) and (29) immediately lead to the **Weyl integration formula** in the group G, and the Lie algebra $\mathfrak{g}$, respectively, see Theorem 0.5.4.

0.4.12 The Weyl Group as a Reflection Group

A positive root $\alpha \in P$ is said to be simple if it cannot be written as the sum of two positive roots. Crossing a wall of a Weyl chamber $\mathfrak{c}$, corresponding to a simple root α, one enters an adjacent Weyl chamber $\mathfrak{c}'$. As a consequence of Weyl's Covering Theorem 0.4.11, there is exactly one $s \in W$, such that $s(\mathfrak{c}) = \mathfrak{c}'$. In order to describe the action of this Weyl group element, we need some more insight in the structure of the root spaces $\mathfrak{g}_\alpha$.

Lemma 0.4.13. *Let G be a connected, compact Lie group of rank equal to one. Then G is isomorphic to the circle, to* $\mathbf{SO}(3)$*, or to* $\mathbf{SU}(2)$*.*

Theorem 0.4.14. *Let G be a connected, compact Lie group, $\mathfrak{t}$ a maximal Abelian subspace of the Lie algebra $\mathfrak{g}$ of G, and α a root of $\mathfrak{t}$. Then:*

(i) $\dim_{\mathbf{C}} \mathfrak{g}_\alpha = 1$*, and* $\mathfrak{g}_{k\alpha} = 0$*, if* $k \in \mathbf{C}$ *and* $k \neq -1,\ 0,\ 1$.

(ii) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h} = \mathfrak{g}_0$, *and* $\mathfrak{g}^{(\alpha)} :=$ *the real part of* $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *is a* 3*-dimensional Lie subalgebra of* $\mathfrak{g}$, *isomorphic to* $\mathfrak{so}(3, \mathbf{R}) = \mathfrak{su}(2)$.

(iii) $G^{(\alpha)} := \exp \mathfrak{g}^{(\alpha)}$ *is a compact Lie subgroup of* G, *isomorphic to either* $\mathbf{SO}(3)$ *or* $\mathbf{SU}(2)$.

(iv) If $\alpha^\vee$ *denotes the unique element of* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *such that* $\alpha(\alpha^\vee) = 2$, *then* $i\alpha^\vee \in \mathfrak{t}$ *and* $\exp 2\pi i \alpha^\vee = 1$, *while* $\beta(\alpha^\vee) \in \mathbf{Z}$, *for all* $\beta \in R$.

(v) There exists $g \in G^{(\alpha)}$ *such that* $\operatorname{Ad} g$ *leaves the root hyperplane* $\ker \alpha \cap \mathfrak{t}$ *pointwise fixed, and maps* $i\alpha^\vee$ *to* $-i\alpha^\vee$.

Remark 0.4.15. The conclusion $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}_0$ also follows from:

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}, \text{ for any } \alpha, \beta \in R. \tag{30}$$

(30) follows immediately from the Jacobi identity, and the argument works for any complex Lie algebra. ✰

The complex linear mapping:

$$s_\alpha : X \mapsto X - \alpha(X)\alpha^\vee : \mathfrak{h} \to \mathfrak{h}$$

leaves $\ker \alpha$ pointwise fixed and maps $\alpha^\vee$ to $-\alpha^\vee$, so it is equal to the complex linear extension of $\operatorname{Ad} g|_{\mathfrak{t}}$ with g as in Theorem 0.4.14.(v). Because $\operatorname{Ad} g(\mathfrak{t}) = \mathfrak{t}$, we get $g \in \mathrm{N}(\mathfrak{t}) = \mathrm{N}(T)$, or: $s_\alpha|_{\mathfrak{t}}$ **belongs to the Weyl group** $W = W(\mathfrak{g}, \mathfrak{t})$.

We also note that $\alpha^\vee$ is orthogonal to $\ker \alpha$, with respect to every $\operatorname{Ad} \mathfrak{g}$-invariant bilinear form β on $\mathfrak{g}_{\mathbf{C}}$. Indeed, if $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, and $H \in \mathfrak{h}$, then $\beta([X, Y], H) = -\beta(Y, [X, H]) = \beta(Y, \alpha(H)X) = 0$, whenever $\alpha(H) = 0$. For this reason, s_α is referred to as the *orthogonal reflection in the root hyperplane* $\ker \alpha$. Note also that $(s_\alpha)^2 = 1$.

The reflection $s_\alpha = s_\alpha^{-1}$ is the restriction to $\mathfrak{h}$ of an automorphism Φ of the complex Lie algebra $\mathfrak{g}_{\mathbf{C}}$. In general, if $\Phi \in \operatorname{Aut} \mathfrak{g}_{\mathbf{C}}$, and $\Phi(\mathfrak{h}) = \mathfrak{h}$, then we have for any $X \in \mathfrak{h}$, $Y \in \mathfrak{g}_\beta$, that $[X, \Phi(Y)] = \Phi([\Phi^{-1}(X), Y]) = \Phi(\beta(\Phi^{-1}(X))Y) = \beta(\Phi^{-1}(X))\Phi(Y)$. That is,

$$\Phi(\mathfrak{g}_\beta) = \mathfrak{g}_{\Phi\beta},$$

if we write, as usual, $(\Phi^{-1})^*(\beta) = \Phi\beta$. In particular,

$$s_\alpha \beta \in R, \text{ whenever } \alpha, \beta \in R.$$

That is, R forms a *reduced root system* in $V = i\mathfrak{t}^* \cap \mathfrak{z}^0$, in the sense of the algebraic theory of root systems, with the $\alpha^\vee$ as the corresponding coroots. Conversely, one may show that every reduced root system is equal to the root system of a compact semisimple Lie algebra. Moreover, one can prove that two such Lie algebras are isomorphic if and only if their root systems are isomorphic. So the classification of the reduced root systems can also be viewed as a classification of the compact semisimple Lie algebras.

Corollary 0.4.16. *The Weyl group* $W = \operatorname{Ad} \mathrm{N}(\mathfrak{t})|_{\mathfrak{t}} = \mathrm{N}(T)/T$ *is equal to the group* W_R *generated by the orthogonal reflections in the root hyperplanes. That is, it can be identified with the Weyl group of the root system* R, *and of the dual root system* $R^\vee$ *of the coroots, respectively.*

0.5 Structure Theory: Second Lecture

0.5.1 Integration

This section contains a general discussion of invariant densities, especially on homogeneous spaces.

A *density* (that is, an unoriented volume form) θ on a manifold M is an assignment, to each local coordinate chart κ (from an open subset V_κ of M onto an open subset U_κ of $\mathbf{R}^m$, with $m = \dim M$), of a function θ_κ on U_κ, with the substitution rule

$$\theta_\kappa(z) = \theta_{\kappa'}(\kappa' \circ \kappa^{-1}(z)) \cdot |\det D(\kappa' \circ \kappa^{-1})(z)|, \tag{31}$$

if κ' is another coordinate chart, and $z \in U_\kappa \cap \kappa(V_{\kappa'})$.

One can also express this by saying that θ is a section of the line bundle $D(M)$ over M (the density bundle), whose fiber over $x \in M$ is equal to the (1-dimensional) space of all functions θ_x on $\bigwedge^m \mathrm{T}_x M$, such that $\theta_x(cv) = |c|\theta_x(v)$, for all $c \in \mathbf{R}$ and $v \in \bigwedge^m \mathrm{T}_x M$.

θ is said to be C^k, and positive, if each θ_κ is C^k, and everywhere positive, respectively. Using C^k partitions of unity, we see that every (paracompact, Hausdorff) C^k manifold M admits positive densities of class C^k, here $0 \le k \le \infty$.

If f is a C^k function with compact support on M then, using a C^k partition of unity, we can write f as a finite sum

$$f = \sum f_\kappa, \quad \text{with } f_\kappa \in \mathrm{C}^k_c(M) \text{ and support of } f_\kappa \subset V_\kappa. \tag{32}$$

Now assume that θ is continuous. The *integral of f against the density θ* is then defined as

$$\int_M f \cdot \theta = \sum \int_{U_\kappa} f_\kappa(\kappa^{-1}(y))\theta_\kappa(y)\, dy. \tag{33}$$

Using the formula for substitutions of variables in an integral,

$$\int_U f(x)\, dx = \int_V f(\Phi(y)) \cdot |\det D\Phi(y)|\, dy,$$

where Φ is a C^1 diffeomorphism from an open subset V of $\mathbf{R}^m$ to another open subset U of $\mathbf{R}^m$, and f a continuous function on U, with compact support contained in U, one sees that (31) guarantees that the right hand side in (33) is independent of the way we have written f as in (32).

The standard theory of Lebesgue integration now consists of defining the space $\mathrm{L}^1(M, \theta)$ of *Lebesgue integrable functions on M, with respect to the density θ*, as the completion of $\mathrm{C}_c(M)$ with respect to the integral norm $f \mapsto \int |f| \cdot \theta$. Similarly, the space $\mathrm{L}^p(M, \theta)$ is defined as the completion of $\mathrm{C}_c(M)$ with respect to the norm $f \mapsto (\int |f|^p \cdot \theta)^{1/p}$. This definition allows one to prove many results first only in the space $\mathrm{C}_c(M)$, and then extending them to these completions by continuity. (The more tricky part of Lebesgue theory then of course is to identify the elements of the completion with ordinary functions on M.)

The familiar formula for substitutions in an integral now immediately gets the following generalization to manifolds. If Φ is a diffeomorphism from a manifold N onto M, and N is provided with the standard positive density τ, then

$$\int_M f \cdot \theta = \int_N \Phi^*(f \cdot \theta) = \int_N f \circ \Phi \cdot \Phi^*\theta = \int_N (f \circ \Phi) \cdot J_\Phi \cdot \tau,$$

where the *absolute Jacobian* J_Φ is defined as the positive function $(\Phi^*\theta)/\tau$. In turn, $J_\Phi(y) = |\det(L^{-1} \circ \mathrm{T}_y \Phi)|$, if L is an auxiliary linear mapping: $\mathrm{T}_y N \to \mathrm{T}_x M$, with $x = \Phi(y)$, such that $L^*\theta_x = \tau_y$. By the principle above, $f \in \mathrm{L}^1(M, \theta)$ if and only if $(f \circ \Phi) \cdot J_\Phi \in \mathrm{L}^1(N, \tau)$; and if this is the case, then the integrals agree.

If θ is positive, then $\mu_\theta : f \mapsto \int f \cdot \theta$ is a *positive Radon measure* on M, that is, a linear form μ on $\mathrm{C}_c(M)$, the space of continuous functions with compact support on M, such that moreover $\mu(f) > 0$, whenever $f \in \mathrm{C}_c(M)$, $f \geq 0$, $f \neq 0$.

Every nowhere zero, continuous volume form Ω on M (that is, differential form of degree equal to the dimension of M) gives rise to a positive, continuous density $\theta = |\,\Omega\,|$ on M, by means of the definition

$$\kappa^*(\theta_\kappa dx_1 \wedge \cdots \wedge dx_m) = \Omega, \quad \text{on } V_\kappa,$$

where we only allow local coordinate charts κ for which the resulting functions θ_k are positive. Note that, given the continuous, nowhere zero volume form Ω, there exists an atlas $\mathcal{A}$ of such coordinate charts κ, and that

$$\det D(\kappa' \circ \kappa^{-1})(z) > 0, \text{ if } z \in U_\kappa \cap \kappa(V_{\kappa'}), \quad \text{for } \kappa, \kappa' \in \mathcal{A}.$$

The choice of such an atlas is said to be an *orientation* of the manifold M. The manifold M is called *orientable* if it has such an atlas, and this is equivalent to the existence of a nowhere vanishing continuous volume form on M.

If a group G acts on M by means of diffeomorphisms, then one has an induced action on $\mathrm{C}_c(M)$ defined by $(g, f) \mapsto (g^{-1})^*(f)$, where we use the standard notation

$$\big(\Phi^*(f)\big)(x) = f(\Phi(x))$$

for the *pull back* of a function f by a mapping Φ. (Actually this is a special case of the pull back of differential forms used above.) The adjoint of Φ^*, acting on measures, is called the *push forward* by Φ, and will be denoted by Φ_*. Then $(g, \mu) \mapsto g_*\mu$ defines an action of G on the space of measures on M. All such induced actions of G on spaces of functions, and their dual spaces of distributions, are always by means of continuous linear transformations in these function, and distribution, spaces, respectively, that is, these are representations in the sense of Example 0.1.9. In other words, the usually nonlinear action of G on the finite-dimensional manifold M induces a linear action of G on the infinite-dimensional spaces of functions or distributions defined on M. Now the measure μ on M is said to be G*-invariant* if $g_*\mu = \mu$ for all $g \in G$, and similarly the density θ is said to be invariant if μ_θ is G-invariant. It is also easy to verify that, for a volume form Ω on M, the density $\theta = |\,\Omega\,|$ is G-invariant, if and only if $g^*\Omega = \pm\Omega$, for all $g \in G$.

Let G be a Lie group acting transitively (and smoothly) on a manifold M. According to the assertion following Corollary 0.1.10, there is a G-equivariant diffeomorphism from the quotient space G/H onto M, for a suitable closed Lie subgroup H of G; here the action of G on G/H is by means of multiplications from the left. If we denote the Lie algebra of G, and H, by $\mathfrak{g}$, and $\mathfrak{h}$, respectively, then the tangent space of G/H at the base point $b = 1H$ is identified with $\mathfrak{g}/\mathfrak{h}$, in such a way that the tangent mapping, at b, of the canonical projection $\Pi : G \to G/H$ is equal to the canonical projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Here $\Pi : G \to G/H$ is a (real-analytic) principal fiber bundle, with structure group H, acting on G by means of multiplications from the right.

The stabilizer group of the base point $b = 1H$ is equal to H, it acts on $\mathrm{T}_b(G/H) = \mathfrak{g}/\mathfrak{h}$ via the mapping $(h, X + \mathfrak{h}) \mapsto \mathrm{Ad}\, h(X) + \mathfrak{h}$, this is obtained by differentiating

$$hxH = hxh^{-1}H, \quad \text{with } h \in H, x \in G,$$

with respect to x at $x = 1$, in the direction of $X \in \mathfrak{g}$.

It follows that $h^{-1} \in H$ acts on the space of volume forms Ω_b on $\mathfrak{g}/\mathfrak{h}$ (the antisymmetric m-linear forms on $\mathfrak{g}/\mathfrak{h}$, with $m = \dim \mathfrak{g}/\mathfrak{h}$) via multiplication by

$$\det(\operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{h}} = \big(\det(\operatorname{Ad} h)|_{\mathfrak{g}}\big)/\big(\det(\operatorname{Ad} h)|_{\mathfrak{h}}\big).$$

The volume form Ω_b on $\mathfrak{g}/\mathfrak{h} = \mathrm{T}_b(G/H)$ extends to a *G-invariant volume form* Ω on G/H (that is, a differential form on G/H of degree $m = \dim G/H$), if and only if Ω_b remains fixed under the action of H. If this is the case, then Ω is uniquely determined by Ω_b, and is real-analytic. Clearly, Ω is nonzero everywhere, and zero everywhere, if and only if $\Omega_b \neq 0$, and $\Omega_b = 0$, respectively. In turn, the existence of a nonzero G-invariant volume form Ω on G/H is equivalent to the condition that

$$\det(\operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{h}} = 1, \quad \text{for all } h \in H. \tag{34}$$

The existence of a G-invariant density θ, and measure μ_θ, respectively, on G/H is equivalent to the slightly weaker condition

$$|\det(\operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{h}}| = 1, \quad \text{for every } h \in H. \tag{35}$$

Note that (35) is equivalent to (34), if and only if G/H is orientable, or $\det(\operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{h}} > 0$ for all $h \in H$; and this in turn is the case if H is connected. Using convolutions, one can show that any G-invariant distribution u on G/H is of the form μ_θ for a smooth G-invariant density θ on G/H, which in turn is locally equal to $|\Omega|$ for a locally G-invariant volume form Ω on G/H. This implies that u is uniquely determined up to a constant factor. If G/H is orientable, then $u = \theta_{|\Omega|}$ for a G-invariant volume form Ω on G/H, and we are back in the situation we started out with.

In general the function $h \mapsto |\det(\operatorname{Ad} h)_{\mathfrak{g}/\mathfrak{h}}| : H \to \mathbf{R}_{>0}$ is a continuous homomorphism from H to the multiplicative group $\mathbf{R}_{>0}$. So if H **is compact**, the image is a compact subgroup of $\mathbf{R}_{>0}$, which can only be $\{1\}$, and the conclusion is that G/H carries a G-invariant positive density.

A special case occurs when $H = \{1\}$, that is, $G/H = G$, viewed as a G-homogeneous space via the action by left multiplications. In this case any nonzero volume form Ω_1 on $\mathfrak{g}$ gives rise to a unique left-invariant volume form Ω on G; we get a corresponding left-invariant density, measure, and orientation, respectively on G. The left-invariant measure is also called the (left-) *Haar measure* on G, the corresponding density is usually denoted by dx. If G is compact, then the unique Haar measure such that

$$\int_G 1\, dx = 1,$$

is the one that is used in the procedure of averaging over G, and is called the *normalized Haar measure on G*.

If Ω is a left-invariant volume form on G then, for each $g \in G$, we have

$$\mathrm{L}(x)^*\, \mathrm{R}(g)^*\Omega = \big(\mathrm{R}(g) \circ \mathrm{L}(x)\big)^*\Omega = \big(\mathrm{L}(x) \circ \mathrm{R}(g)\big)^*\Omega = \mathrm{R}(g)^*\, \mathrm{L}(x)^*\Omega = \mathrm{R}(g)^*\Omega,$$

that is $\mathrm{R}(g)^*\Omega$ is left-invariant as well. It follows that it is a constant multiple of Ω. Evaluating $\mathrm{R}(g)^*\Omega = \mathrm{R}(g)^*\, \mathrm{L}(g^{-1})^*\Omega = (\mathbf{Ad}\, g^{-1})^*\Omega$ at 1, we get

$$\mathrm{R}(g)^*\Omega = \det \operatorname{Ad} g^{-1} \cdot \Omega, \quad \text{for } g \in G. \tag{36}$$

In particular, there exists a *bi-invariant* (that is, a left-invariant and right-invariant) volume form on G, if and only if $\det \operatorname{Ad} g = 1$ for all $g \in G$; that is, the adjoint representation maps into the special linear group of $\mathfrak{g}$. For the existence of a bi-invariant density, and measure, respectively on G, the condition reads

$$|\det \operatorname{Ad} g| = 1, \quad \text{for all } g \in G;$$

in this case the group G is said to be *unimodular*. As before, every compact Lie group is unimodular. This implies also that

$$\int_G f(x^{-1})\,dx = \int_G f(x)\,dx, \quad \text{for } f \in \mathrm{C}_c(G).$$

Now let $\pi : M \to B = G\backslash M$ be a principal fiber bundle with the Lie group G as structure group, cf. (7). For any positive, continuous density dg on G, and any $f \in \mathrm{C}_c(M)$, we get a new continuous function $\int_G f : x \mapsto \int_G f(gx)\,dg$, obtained from f by "*integration over the fiber*". Clearly $\int_G f$ is G-invariant for every $f \in \mathrm{C}_c(M)$ if and only if dg is right-invariant. If this is the case, $\int_G f$ can be regarded as a function on B, this actually defines a continuous linear mapping $\int_G : \mathrm{C}_c(M) \to \mathrm{C}_c(B)$.

For any positive, continuous density db on B, the product densities of these two on the local trivializations piece together to a positive, continuous density dx on M, such that, for all $f \in \mathrm{C}_c(M)$,

$$\int_M f(x)\,dx = \int_{G\backslash M}\left(\int_G f(gx)\,dg\right)d(Gx) = \int_B\left(\int_G f\right)(b)\,db. \tag{37}$$

The proof is by observing that, on the domain $G \times U$ of a retrivialization $\delta : (g, b) \mapsto (g\chi(b), b)$ (see Theorem 0.1.6), we have

$$\begin{aligned}\int_{G\times U} f &= \int_U\left(\int_G f(g,b)\,dg\right)db = \int_U\left(\int_G f(g\chi(b), b)\,dg\right)db\\ &= \int_U\left(\int_G (\delta^* f)(g,b)\,dg\right)db = \int_{G\times U}\delta^* f.\end{aligned}$$

Next, applying in the integrations over G the substitution of variables $g = h^{-1}g'$, and using the analogue of (36) for right-invariant densities on G, we obtain that the density dx on M satisfies

$$\int_M f(hx)\,dx = |\det \mathrm{Ad}\, h^{-1}| \int_M f(x)\,dx, \quad \text{for } h \in G. \tag{38}$$

So this density dx on M is G-invariant if and only if G is unimodular. If conversely the positive, continuous density dx on M satisfies (38) and dg is a right-invariant density on G, then there is a unique positive, continuous density db on B such that (37) holds. In this situation we write

$$dx = dg\,db, \quad \text{and} \quad db = \frac{dx}{dg}$$

Remark 0.5.2. (a) There exists a G-invariant positive, continuous density dx on M and a positive, continuous density db on B such that (37) hold if and only if the mappings $\chi : U \to G$, which appear in the retrivializations, can be chosen such that they only take values in the kernel of the homomorphism $g \mapsto |\det \mathrm{Ad}\, g| : G \to \mathbf{R}_{>0}$.

(b) If we start out with a left-invariant density on G, then we end up with the formula

$$\int_M f(x)\,dx = \int_{G\backslash M}\left(\int_G f(g^{-1}x)\,dg\right)d(Gx), \tag{39}$$

and the density dx on M again satisfies (38).

(c) In the case that $\pi : M \to B$ is a covering, which is a special case of a principal fiber bundle, the group G is discrete (0-dimensional), and it is customary to use the the counting measure on G. Then G is unimodular, so dx is G-invariant, and (37) takes the form

$$\int_G f(x)\,dx = \int_B \Big(\sum_{g\in G} f(gx)\Big)\,d(Gx).$$

(d) In the principal fibration $G \to G/H$, two actions are considered on G: the action of G by means of left multiplications, and the action $h \mapsto \mathrm{R}(h)^{-1}$ of H by means of right multiplications. Taking the left-invariant density on H, (39) and (38) read

$$\int_G f(x)\,dx = \int_{G/H}\left(\int_H f(xh)\,dh\right)d(xH), \tag{40}$$

and

$$\int_G f(xh)\,dx = |\det(\mathrm{Ad}\,h)|_{\mathfrak{h}}|\int_G f(x)\,dx, \quad \text{for } h \in H,$$

for any densities dx, and $d(xH)$, respectively, for which (40) holds. It is clear now that dx is left-invariant, if and only if $d(xH)$ is G-invariant, which in turn is equivalent to (34). One gets all the invariances one could wish for, if both H and G are unimodular.

☆

0.5.3 The Weyl Integration Theorem

If G is a compact Lie group, and H is a compact subgroup, then the invariant positive densities on them, which always exist, are both left- and right-invariant, and the quotient density on G/H is G-invariant as well. See the preceding Remark 0.5.2.(d).

Theorem 0.5.4 (Weyl's Integration Formula). *Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$, T a maximal torus in G with Lie algebra $\mathfrak{t}$, $W = \mathrm{N}(T)/T$ the corresponding Weyl group. Let dx, and dt, be invariant positive densities on G, and T, respectively, and provide G/T with the quotient density $d(xT) = dx/dt$. On $\mathfrak{g}$, and $\mathfrak{t}$, we take the constant densities, equal to $(dx)_1$, and $(dt)_1$, respectively. Then we have the following:*

(i) *The mapping that assigns to $f \in \mathrm{C}(G)$ the function*

$$F : (xT, t) \mapsto f(xtx^{-1})|\det(\mathrm{Ad}\,t - \mathrm{I})_{\mathfrak{g}/\mathfrak{t}}| \quad \textit{on } G/T \times T,$$

extends to a topological isomorphism from $\mathrm{L}^1(G)$ onto $\mathrm{L}^1(G/T \times T)^W$, the space of W-invariant functions on $G/T \times T$. Here $sT \in W$, with $s \in \mathrm{N}(T)$, acts on (xT, t) by sending it to $(xs^{-1}T, sts^{-1})$. Moreover, if $f \in \mathrm{L}^1(G)$, then

$$\int_G f(x)\,dx = \#(W)^{-1}\int_T\left(\int_{G/T} f(xtx^{-1})\,d(xT)\right)|\det(\mathrm{I} - \mathrm{Ad}\,t)_{\mathfrak{g}/\mathfrak{t}}|\,dt. \tag{41}$$

(ii) *The mapping that assigns to* $\phi \in \mathrm{C}_c(\mathfrak{g})$ *the function*

$$\Phi : (gT, X) \mapsto \phi(\mathrm{Ad}\, g(X)) |\det(\mathrm{ad}\, X)_{\mathfrak{g}/\mathfrak{t}}| \quad \textit{on } G/T \times \mathfrak{t}$$

extends to a topological isomorphism: $\mathrm{L}^1(\mathfrak{g}) \xrightarrow{\sim} \mathrm{L}^1(G/T \times \mathfrak{t})^W$, *where now* $sT \in W$ *acts on* (gT, X) *by sending it to* $(gs^{-1}T, \mathrm{Ad}\, s(X))$. *Moreover, if* $\phi \in \mathrm{L}^1(\mathfrak{g})$, *then*

$$\int_{\mathfrak{g}} \phi(X)\, dX = \#(W)^{-1} \int_{\mathfrak{t}} \left(\int_{G/T} \phi(\mathrm{Ad}\, g(X))\, d(gT) \right) |\det(\mathrm{ad}\, X)_{\mathfrak{g}/\mathfrak{t}}|\, dX. \tag{42}$$

Observing that the determinant of a real linear mapping is equal to the determinant of its complex linear extension, we see from the root space decomposition (21), (23), that the Jacobian which appears in (42) can be expressed in terms of the roots. For this purpose, we introduce, for any choice of positive roots P, the function

$$\varpi = \varpi_P : X \mapsto \prod_{\alpha \in P} \alpha(X) : \mathfrak{t} \to i^{\#(P)}\mathbf{R}.$$

Then, using (24) and the facts that $-\alpha(X) = \overline{\alpha(X)}$ (notice that $\alpha(X)$ is purely imaginary) and $\dim_{\mathbf{C}} \mathfrak{g}_\alpha = 1$ for every $\alpha \in R$ (Theorem 0.4.14.(i)), we get

$$\det(\mathrm{ad}\, X)_{\mathfrak{g}/\mathfrak{t}} = \varpi(X)\overline{\varpi(X)}, \quad \text{for } X \in \mathfrak{t}.$$

Using the covering $\exp : \mathfrak{t} \to T$, we easily can give an explicit formula for the Jacobian factor appearing in (41). Writing $t \in T$ as $t = \exp X$, with $X \in \mathfrak{t}$, we recall that $\mathrm{Ad}\, t = \mathrm{Ad} \exp X = \mathrm{e}^{\mathrm{ad}\, X}$ acts on $\mathfrak{g}_\alpha$ as multiplication by the scalar

$$t^\alpha := \mathrm{e}^{\alpha(X)}.$$

The fact that t^α lies on the unit circle in $\mathbf{C}$ corresponds to $t^{-\alpha} = (t^\alpha)^{\mathrm{conj}}$, so now the Jacobian in (41) is given by

$$\det(\mathrm{I} - \mathrm{Ad}\, t)_{\mathfrak{g}/\mathfrak{t}} = \delta(t)\overline{\delta(t)},$$

where we have used the function

$$\delta := \delta_P : t \mapsto \prod_{\alpha \in P} (1 - t^{-\alpha}) : T \to \mathbf{C}.$$

Note that in both cases the determinants themselves already are positive, so that we actually did not need the absolute value signs in (41), (42).

Under the assignment of Theorem 0.5.4 above, the conjugacy invariant functions f on G, and the Ad-invariant functions ϕ on $\mathfrak{g}$, correspond bijectively to functions F, and Φ, respectively that do not depend on the first variable xT and that are Weyl group invariant as a function of the second variable. Because all these spaces are closed linear subspaces of the corresponding function spaces, Theorem 0.5.4 now leads immediately to the following:

Corollary 0.5.5. *Set* $c := \int_{G/T} d(xT) = \int_G dx / \int_T dt$, $c' = (c/\#(W))^{1/2}$. *Then*

(i) *The assignment* $f \mapsto (f|_T)|\delta|^2$ *defines a topological linear isomorphism from* $\mathrm{L}^1(G)^{\mathbf{Ad}\, G}$, *the space of Lebesgue integrable class functions on* G, *onto the space* $\mathrm{L}^1(T)^W$ *of Lebesgue* *integrable functions on T that are Weyl group invariant. For $f \in \mathrm{L}^1(G)^{\mathbf{Ad}\,G}$, we have the integral formula*

$$\int_G f(x)\,dx = \#(W)^{-1} c \int_T f(t)\,|\,\delta(t)\,|^2\,dt. \tag{43}$$

Secondly, the assignment $f \mapsto c'\delta(f|_T)$ defines a (unitary) linear isomorphism from the Hilbert space $\mathrm{L}^2(G)^{\mathbf{Ad}\,G}$ of square integrable class functions on G onto the Hilbert space $\mathrm{L}^2(T)^W$ of Weyl group invariant, square integrable functions on T.

(ii) *The assignment $\phi \mapsto (\phi|_{\mathfrak{t}})|\,\varpi\,|^2$ defines a topological isomorphism: $\mathrm{L}^1(\mathfrak{g})^{\mathrm{Ad}\,G} \xrightarrow{\sim} \mathrm{L}^1(\mathfrak{t})^W$; moreover, if $\phi \in \mathrm{L}^1(\mathfrak{g})^{\mathrm{Ad}\,G}$, then*

$$\int_{\mathfrak{g}} \phi(X)\,dX = \#(W)^{-1} c \int_{\mathfrak{t}} \phi(H)\,|\,\varpi(H)\,|^2\,dH. \tag{44}$$

The assignment $\phi \mapsto c'\varpi(\phi|_{\mathfrak{t}})$ defines a unitary transformation from $\mathrm{L}^2(\mathfrak{g})^{\mathrm{Ad}\,G}$ onto $\mathrm{L}^2(\mathfrak{t})^W$.

In a more metric approach, one would start with a left- and right-invariant Riemannian structure β on G and take the densities on G, T, G/T that assign the value 1 to any orthonormal basis in a tangent space. These Riemannian structures are, via the mapping $\beta \mapsto B = \beta_1$, in bijective correspondence with the Ad G-invariant inner products B on $\mathfrak{g}$. (If $\mathfrak{z} = 0$, we could take B equal to minus the Killing form, cf. Section 0.4.4. Also, if $\mathfrak{g}$ is simple, then this is the only choice up to a positive factor, but in general there is more freedom.)

The factor $c = \mathrm{vol}(G/T)$, which appears in (43) and in (44), can be determined explicitly once we can evaluate the integrals in the left and right hand side for a suitable invariant and integrable function f, and ϕ, respectively. One could for instance take, in (44), ϕ equal to the characteristic function of the unit ball in $\mathfrak{g}$; then one is left with the computation of the integral of the polynomial $\varpi(H)\overline{\varpi(H)}$ over the unit ball in $\mathfrak{t}$.

Instead, we take $\phi(X) = \mathrm{e}^{-\langle X, X\rangle/2}$; in the notation we use the inner product B to identify $\mathfrak{g}$, and $\mathfrak{t}$, respectively, with its dual, and we also write $\mu(X) = \langle X, \mu\rangle$, for a linear form μ. Then, as is well-known, the left hand side of (44) is equal to $(2\pi)^{\dim \mathfrak{g}/2}$. On the other hand, for any **polynomial** f, the function

$$F_t : X \mapsto (4\pi t)^{-\dim \mathfrak{t}/2} \int_{\mathfrak{t}} \mathrm{e}^{-\langle X-Y, X-Y\rangle/4t} f(Y)\,dY$$

is a polynomial of degree $\leq \deg f$, depending smoothly on $t > 0$. This function F_t satisfies the differential equation

$$\frac{\partial F_t}{\partial t} = \Delta F_t, \quad \text{with the boundary condition} \quad \lim_{t \to 0, t > 0} F_t = f.$$

Here Δ denotes the Laplace operator with respect to the given inner product, $\Delta = \sum_j \frac{\partial^2}{\partial Y_j^2}$ on any orthonormal basis. Now Δ leaves the finite-dimensional space of polynomials of degree $\leq \deg f$ invariant, and because it decreases degrees, actually acts on this space as a nilpotent operator. It follows that $F_t = \sum_{k \geq 0} (k!)^{-1} (t\Delta)^k f$, and therefore, taking $t = \frac{1}{2}$, we get

$$\int_{\mathfrak{t}} \mathrm{e}^{-\langle Y, Y\rangle/2} f(Y)\,dY = (2\pi)^{\dim \mathfrak{t}/2} \sum_{k \geq 0} \frac{1}{k!} \big((\frac{1}{2}\Delta)^k f\big)(0),$$

where both sums are finite. In our case, $f : Y \mapsto \prod_{\alpha \in R} \alpha(Y)$ is a homogeneous polynomial of degree $2p$, if we write $p = \#(P) = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{t})$. This implies that only the term with $k = p$ gives a nonzero contribution. We therefore arrive at

$$(2\pi)^p \#(W) = cd, \quad \text{where } p = \#(P),$$

and

$$d = (p!\,2^p)^{-1} \sum_{\sigma} \langle \alpha_{\sigma(1)}, \alpha_{\sigma(2)} \rangle \ldots \langle \alpha_{\sigma(2p-1)}, \alpha_{\sigma(2p)} \rangle.$$

Here $j \mapsto \alpha_j$ is an enumeration of R, and the sum is over all permutations of $\{1, \ldots, 2p\}$.

This formula is explicit, but quite cumbersome in practical computations. A simpler formula can be given by using the fact that ϖ is W-anti-invariant, actually

$$\varpi(Y) = \frac{1}{p!} \sum_{s \in W} \det s \big((s\rho)(Y) \big)^p, \quad \text{with} \quad \rho = \frac{1}{2} \sum_{\alpha \in P} \alpha.$$

This follows from

$$\prod_{\alpha \in P} \big(\mathrm{e}^{\alpha(X)/2} - \mathrm{e}^{-\alpha(X)/2} \big) = \sum_{s \in W} (\det s)\, \mathrm{e}^{s \cdot \rho(X)},$$

by inserting $X = tY$ and comparing the coefficients of t^p in the Taylor expansion. Furthermore, each W-anti-invariant polynomial has ϖ as a factor, because anti-invariance of f under s_α implies that $f = 0$ on $\ker \alpha$, for each $\alpha \in P$.

Because Δ commutes with W, we have that $\Delta\varpi$ is W-anti-invariant, and because $\deg \Delta\varpi < \deg \varpi$, the conclusion is that $\Delta\varpi = 0$. Because

$$\frac{1}{2} \Delta \big(f^2 \big) = \sum_j \Big(\frac{\partial f}{\partial Y_j} \Big)^2 + f \Delta f$$

for any function f, and because Δ commutes with all constant coefficient linear partial differential operators, the conclusion is that

$$\begin{aligned}
\frac{1}{p!} \Big(\frac{1}{2} \Delta \Big)^p (\varpi^2) &= \frac{1}{p!} \sum_j \Big(\frac{\partial^p \varpi}{\partial Y_{j(1)} \ldots \partial Y_{j(p)}} \Big)^2 = \frac{1}{p!} \sum_j \Big(\sum_{s \in W} \operatorname{sgn} s \frac{1}{p!} \frac{\partial^p (s\rho)}{\partial Y_{j(1)} \ldots \partial Y_{j(p)}} \Big)^2 \\
&= \frac{1}{p!} \sum_j \Big(\sum_{s \in W} \operatorname{sgn} s\, (s\rho)_{j(1)} \ldots (s\rho)_{j(p)} \Big)^2 \\
&= \frac{1}{p!} \sum_j \sum_{s, s'} \operatorname{sgn} s \operatorname{sgn} s' (s\rho)_{j(1)} \ldots (s\rho)_{j(p)} (s'\rho)_{j(1)} \ldots (s'\rho)_{j(p)} \\
&= \frac{1}{p!} \sum_{s, s'} \operatorname{sgn} s \operatorname{sgn} s' \langle s\rho, s'\rho \rangle^p = \frac{1}{p!} \sum_{s, s'} \operatorname{sgn}(s^{-1}s') \langle \rho, s^{-1}s'\rho \rangle^p \\
&= \frac{1}{p!} \#(W) \sum_s \operatorname{sgn} s \big(s\rho(B^{-1}\rho) \big)^p = \#(W) \varpi(B^{-1}\rho),
\end{aligned}$$

where $B^{-1}\rho$ is the element of $\mathfrak{h}$ such that $\mu(B^{-1}\rho) = \langle \rho, \mu \rangle$ for all $\mu \in \mathfrak{h}^*$. Also, j ranges over all mappings from $\{1, \ldots, p\}$ to $\{1, \ldots, \dim \mathfrak{t}\}$, and $(s\rho)_k$ equals the k-th coordinate of $s\rho$ with respect to the chosen orthonormal basis in $\mathfrak{t}$.

We have now arrived at

$$\operatorname{vol}(G/T) = \frac{(2\pi)^{\#(P)}}{(-1)^p} \prod_{\alpha \in P} \langle \rho, \alpha \rangle.$$

If one takes, for $\mathfrak{z} = 0$, $B = -\kappa$, then also the factors -1 in this formula get nicely absorbed in the product.

0.6 Structure Theory: Third Lecture

0.6.1 Representations of Compact Lie Groups: Introduction

In this introduction, we give the basic definitions of representation theory, followed by a summary of the main results for compact (Lie) groups.

Let G be a Lie group, and V a finite-dimensional vector space. A *representation of G in V* is defined as a continuous homomorphism $\pi : G \to \mathbf{GL}(V)$. Because every continuous homomorphism between Lie groups is real-analytic, one may as well require π to be real-analytic. Another way of saying this is that π defines a real-analytic action of G on V by means of linear transformations. The adjoint representation is a representation of G in its own Lie algebra $\mathfrak{g}$, this example we already have met numerous times.

If M is a manifold, and $A : (g, x) \mapsto A(g)(x) : G \times M \to M$ is a C^k action of G on M (with $0 \le k \le \omega$), then, for each $f \in \mathrm{C}^k(M)$, the function $A(g)(f) : x \mapsto f(g^{-1}x)$ is another C^k function on M. The mapping $A(g) : f \mapsto A(g)(f)$ is a linear mapping from $\mathrm{C}^k(M)$ onto itself. Furthermore, the mapping $A : (g, f) \mapsto A(g)(f)$ is continuous: $G \times \mathrm{C}^k(M) \to \mathrm{C}^k(M)$.

Because such induced actions on function spaces form a central theme in the theory of representations, and because, at least for compact groups, the differentiable structure will not be used for quite some time, the definition of a representation has been generalized in the following way. A *topological group* is a group G that at the same time is a Hausdorff topological space, in such a way that the multiplication: $(g, h) \mapsto gh$, and the inversion: $g \mapsto g^{-1}$, is continuous: $G \times G \to G$, and $G \to G$, respectively. If V is a topological vector space (which usually will be locally convex and complete), then a *representation of the topological group G in the topological vector space V* is defined as a homomorphism $\pi : G \to \mathbf{GL}(V)$, such that the mapping: $(g, v) \mapsto \pi(g)(v)$ is continuous: $G \times V \to V$. We will also write $V = V_\pi$.

An example is provided by the action of G on $\mathrm{C}(G)$ (or $\mathrm{C}_c(G)$, the space of compactly supported continuous functions on G), induced by the action of G on itself by left and right multiplications respectively. These are called the *left* and *right regular representation* of G, denoted by L and R respectively. (Note that the action of G on itself by right multiplication is given by $(g, x) \mapsto xg^{-1}$, so the induced action of the function space is given by $(g, f) \mapsto (x \mapsto f(xg).)$ One has also the representation of $G \times G$ on $\mathrm{C}(G)$, induced by the left-right action of $G \times G$ on G.

A representation π of G in a complete, locally convex, topological vector space V is called *irreducible* if there are no $\pi(G)$-invariant, closed linear subspaces U of V, other than $U = 0$ or $U = V$. (A subspace U is said to be *$\pi(G)$-invariant* if $\pi(g)(U) \subset U$, for every $g \in G$.) The representation is said to be *completely reducible* if, for every $\pi(G)$-invariant, closed linear subspace U of V, there is another $\pi(G)$-invariant, closed linear subspace U' of V, such that $V = U \oplus U'$. Note that all linear subspaces of V are automatically closed, if V is finite-dimensional.

The representations $\sigma : G \to \mathbf{GL}(U)$ and $\tau : G \to \mathbf{GL}(V)$ respectively are said to be *equivalent* if there is a topological linear isomorphism L from U onto V, such that $L \circ \sigma(g) = \tau(g) \circ L$, for all $g \in G$, that is,

$$\begin{array}{ccc} U & \xrightarrow{\sigma(g)} & U \\ L\big\downarrow\wr & & \wr\big\downarrow L \\ V & \xrightarrow{\tau(g)} & V \end{array} .$$

The equivalence class of the representation π will be denoted by $[\pi]$. The set of equivalence classes of irreducible representations of G is called the *dual* $\widehat{G}$ of G.

If the representation $\pi : G \to \mathbf{GL}(V)$ is finite-dimensional, then

$$\chi_\pi : g \mapsto \operatorname{tr} \pi(g)$$

is a continuous, conjugacy-invariant function (class function) on G, called the *character* of the representation. Note that $\chi_\pi(1) = \dim V$, called the *dimension* d_π of the representation π. Clearly, $\chi_\sigma = \chi_\tau$ if σ, τ are equivalent finite-dimensional representations of G.

We are now ready to formulate our goals regarding the representation theory of a **compact** topological group G. We will restrict ourselves here to **complex** representations, that is, homomorphisms from G to the group of complex linear transformations of a complex vector space V, because for those the formulation is somewhat simpler than for the real ones.

(i) G carries a unique left-invariant measure: $f \mapsto \int_G f(g)\, dg$, such that $\int_G dg = 1$. It is automatically right invariant, and is also called *averaging over G*.

(ii) Every irreducible representation π of G is finite-dimensional. The **Peter-Weyl theorem** states that the characters of the irreducible representations of G form a countable orthonormal basis of the Hilbert space of square-integrable, complex-valued, conjugacy-invariant functions on G. In particular, inequivalent irreducible representations have different characters (actually orthogonal to each other with respect to the L^2-inner product in the function space).

(iii) Let σ be a representation of G in the complete, locally convex, topological vector space U, and let π be an irreducible representation of G. The *π-isotypical subspace* U_π of U is defined as the sum of all d_π-dimensional, $\sigma(G)$-invariant linear subspaces V of U, such that $g \mapsto \sigma(g)|_V$ is equivalent to π. Then

$$E_\pi := \mathrm{d}_\pi \int_G \chi_\pi(g^{-1})\sigma(g)\, dg$$

is a continuous linear projection from U onto U_π. The U_π, for $[\pi] \in \widehat{G}$, are closed linear subspaces of U, and their sum

$$U^{\mathrm{fin}} = \sum_{[\pi]\in\widehat{G}} U_\pi$$

is direct. It is an immediate consequence of the Peter-Weyl theorem that U^{fin} is dense in U. The space U^{fin} is also equal to the space of *G-finite vectors* in U, that is, the $u \in U$ such that the $\sigma(g)u$, for $g \in G$, span a finite-dimensional linear subspace of U. Finally, if U_π is finite-dimensional, then it can actually be written as a direct sum of copies of π; the number, $\dim U_\pi / \mathrm{d}_\pi$, of these is called the *multiplicity* $[\sigma : \pi]$ of $[\pi]$ in σ.

(iv) For the right regular representation R in $\mathrm{C}(G)$ (or in $\mathrm{L}^2(G)$), the multiplicity of $[\pi] \in \widehat{G}$ in R is equal to d_π. The π-isotypical subspace M_π is also equal to the $\pi^\vee$-isotypical subspace for the left regular representation L of G in $\mathrm{C}(G)$. Here $\pi^\vee : g \mapsto \pi(g^{-1})^* : G \to \mathbf{GL}(V_\pi^*)$ is the *contragredient* or *dual* representation of π. The space M_π is irreducible for the left-right action of $G \times G$ on $\mathrm{C}(G)$. By (iii), the direct sum $M = \oplus_{[\pi]\in\widehat{G}} M_\pi$, which is orthogonal with respect to the L^2-inner product, is dense in $\mathrm{C}(G)$. The space M of functions of finite type is also called the *space of matrix coefficients*. If H is a closed subgroup of G, then the $\pi^\vee$-isotypical subspace of $\mathrm{C}(G/H) = \mathrm{C}(G)^H$ is the space M_π^H of $\mathrm{R}(H)$-fixed elements in M_π; so the multiplicity of $\pi^\vee$ in $\mathrm{C}(G/H)$ is $\leq \mathrm{d}_\pi$. The decomposition

$$\mathrm{C}(G/H) = \Big(\bigoplus_{[\pi]\in\widehat{G}} M_\pi^H\Big)^{\mathrm{c}}$$

is called the *Fourier decomposition* of the homogeneous G-space G/H; the elements of the M_π^H here play the role of the harmonic oscillations in the classical Fourier decomposition of the functions on the circle, that is, the periodic functions on $\mathbf{R}$.

(v) The compact group G is a Lie group if and only if it has no arbitrarily small subgroups. If this is the case, then G has the structure of a real affine algebraic set, with the matrix coefficients of the finite-dimensional real representations as the real-valued polynomial functions on G. With this structure, the multiplication, and inversion, is a polynomial mapping: $G \times G \to G$, and $G \to G$, respectively, making G into a *real affine algebraic group*. It has a natural *complexification* $G_{\mathbf{C}}$, and these complexifications of the compact Lie groups are precisely the so-called reductive complex affine algebraic groups.

(vi) For a connected, compact Lie group G, the conjugacy-invariant functions are determined by their restrictions to a maximal torus T in G. The **Weyl character formula** is an explicit formula for the restrictions to T of the characters of the irreducible representations π of G; and an explicit formula for the dimensions $d_\pi = \chi_\pi(1)$ follows. The irreducible representations themselves may be constructed by means of a complex structure on G/T, which is called a *flag manifold* associated with G.

0.6.2 Weyl's Character Formula

In this section, G is a connected, compact Lie group.

Fix a maximal torus T in G, with Lie algebra equal to the maximal Abelian subspace $\mathfrak{t}$ of $\mathfrak{g}$. Every element of G is conjugate to an element of T, cf. Theorem 0.4.10. Because characters χ_π of irreducible representations π of G are conjugacy-invariant functions on G, χ_π is determined by its restriction to T.

For the determination of $\chi_\pi|_T$, we start with the observation that $\pi|_T : t \mapsto \pi(t) : T \to \mathbf{GL}(V)$ is a finite-dimensional representation of T, which is a direct sum of finitely many one-dimensional representations, with characters $.^\mu : t \mapsto t^\mu$. Here μ is a linear form: $\mathfrak{t} \to i\mathbf{R}$, such that $\mu(\Lambda) \subset 2\pi i\mathbf{Z}$, where Λ denotes the lattice $\ker \exp|_\mathfrak{t}$ in $\mathfrak{t}$. By a slight abuse of notation, we will use the letter μ both for such a linear form and for (the equivalence class of) the corresponding irreducible representation of T, also called a *weight of T*. The set of these μ will accordingly be denoted by $\widehat{T}$. So we have

$$\chi_\pi(t) = \sum_{\mu \in \widehat{T}} m_\mu t^\mu, \quad \text{for } t \in T, \tag{45}$$

where the sum is only over finitely many of the $\mu \in \widehat{T}$, and

$$m_\mu := [\pi|_T : \mu]$$

is a positive integer for each of the occurring $\mu \in \widehat{T}$, the *multiplicity* of μ in π.

The next observation is that $\mathrm{N}(T)$, the normalizer of T in G, acts on T by conjugation; so the function $\chi_\pi|_T$ on T is invariant under the action on T of the Weyl group $W = \mathrm{N}(T)/T$. Now $s^*(.^\mu) = .^{s^*(\mu)}$, where on the right hand side we have used the action of $s \in W$ on $\mathfrak{t}^*$. The identity $\chi_\pi|_T = s^*(\chi_\pi|_T)$ therefore takes the form

$$\sum_{\mu \in \widehat{T}} m_\mu t^\mu = \sum_{\mu \in \widehat{T}} m_\mu t^{s^*(\mu)} = \sum_{\mu \in \widehat{T}} m_{s\cdot\mu} t^\mu,$$

where we have written $s \cdot \mu = (s^{-1})^*(\mu)$ for the natural action of s on the weights of T. Because the t^μ, for $\mu \in \widehat{T}$, are linearly independent (they form an L^2-orthonormal system of functions on T, as can be verified directly), we conclude that

$$m_{s\cdot\mu} = m_\mu, \quad \text{for all } s \in W, \mu \in \widehat{T}. \tag{46}$$

The third and last fact about χ_π that will be used is that the irreducibility of π implies

$$\int_G \chi_\pi(x)\overline{\chi_\pi(x)}\,dx = 1,$$

this follows from the orthogonality relations for characters. Applying to the left hand side Weyl's integral formula (43) for conjugacy-invariant functions, we get

$$\int_T \delta(t)\overline{\delta(t)}\,\chi_\pi(t)\overline{\chi_\pi(t)}\,dt = \#(W); \tag{47}$$

or $\langle\phi, \phi\rangle = \#(W)$, if we write

$$\phi := \delta(\chi_\pi|_T).$$

Here

$$\delta(t) = \delta_P(t) = \prod_{\alpha\in P}(1 - t^{-\alpha}), \tag{48}$$

for a choice P of positive roots, which we assume fixed from now on. Note that for each root α, the mapping $t \mapsto t^\alpha$ is a character of T, namely the one of the one-dimensional representation $t \mapsto (\mathrm{Ad}\,t)|_{\mathfrak{g}_\alpha}$. In other words, each root is a weight of T. It is also clear that the weights form an additive subgroup of $i\,\mathfrak{t}^*$. Combining this with (45), we get

$$\phi(t) = \sum_{\mu\in\widehat{T}} c_\mu t^\mu;$$

here again the sum is finite, and $c_\mu \in \mathbf{Z}$, for all $\mu \in \widehat{T}$, although this time we expect also negative integers. Using that the characters of T form an orthonormal system, we see that (47) now takes the form

$$\sum_{\mu\in\widehat{T}} {c_\mu}^2 = \#(W). \tag{49}$$

Because the ${c_\mu}^2$ are nonnegative integers, not more than $\#(W)$ of them can be nonzero.

In order to understand the consequences of the Weyl group invariance of $\chi_\pi|_T$ for the coefficients c_μ, we need to investigate how δ_P behaves under the action of $s \in W$.

If s is given by conjugation with $x \in \mathrm{N}(T)$, then, by definition,

$$(s\cdot\delta_P)(t) = \delta_P(s^{-1}(t)) = \delta_P(x^{-1}tx).$$

Now $\mathrm{Ad}(x^{-1}tx) = \mathrm{Ad}\,x^{-1} \circ \mathrm{Ad}\,t \circ \mathrm{Ad}\,x$ acts on $\mathfrak{g}_\alpha$ as multiplication by the same scalar as $\mathrm{Ad}\,t$ does on $(\mathrm{Ad}\,x)(\mathfrak{g}_\alpha) = \mathfrak{g}_{s\cdot\alpha}$, which is equal to $t^{s\cdot\alpha}$. So $s\cdot\delta_P = \delta_{s\cdot P}$. Here we have used the following general fact. If $\Phi \in \mathrm{Aut}\,\mathfrak{g}$, and $\Phi(\mathfrak{t}) = \mathfrak{t}$, then we have, for any $X \in \mathfrak{t}$, $Y \in \mathfrak{g}_\alpha$, that $[\,X, \Phi(Y)\,] = \Phi([\,\Phi^{-1}(X), Y\,]) = \Phi(\alpha(\Phi^{-1}(X))Y) = \alpha(\Phi^{-1}(X))\Phi(Y)$. Or, with the usual notation $(\Phi^{-1})^*(\alpha) = \Phi\alpha$,

$$\Phi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\Phi\alpha}.$$

Because $s \cdot R = R$, and R is equal to the disjoint union of P and $-P$, cf. (24), we get that $s \cdot P$ is equal to the disjoint union of $P \cap s \cdot P$ and $-(P \setminus s \cdot P)$. It follows that

$$\delta_{s \cdot P}(t) = \delta_P(t) \prod_{\alpha \in P \setminus s \cdot P} \frac{1 - t^\alpha}{1 - t^{-\alpha}}.$$

Now

$$1 - t^\alpha = -t^\alpha\,(1 - t^{-\alpha}), \quad \text{for } \alpha \in R,$$

so

$$\delta_{s \cdot P}(t) = \delta_P(t)\,(-1)^{\#(P \setminus s \cdot P)}\; t^{\Sigma(s)}, \qquad \text{where} \qquad \Sigma(s) = \sum_{\alpha \in P \setminus s \cdot P} \alpha.$$

On the basis of Corollary 0.4.16, one may show that s^{-1} can be written as a composition of $\#(P \setminus s \cdot P)$ many reflections in root hyperplanes. Because each of these reflections has determinant equal to -1, it follows that $(-1)^{\#(P \setminus s \cdot P)} = \det s^{-1} = \det s$.

A neat expression for $\Sigma(s)$ can be obtained by introducing

$$\rho := \frac{1}{2} \sum_{\alpha \in P} \alpha;$$

the so-called *half the sum of the positive roots*. The idea is that

$$s \cdot \rho = \frac{1}{2} \sum_{\alpha \in P} s \cdot \alpha = \frac{1}{2} \sum_{\beta \in s \cdot P} \beta = \frac{1}{2} \left(\sum_{\alpha \in P \cap s \cdot P} \alpha - \sum_{\alpha \in P \setminus s \cdot P} \alpha \right) = \rho - \Sigma(s),$$

or $\Sigma(s) = \rho - s \cdot \rho$. Our desired transformation formula for δ_P now takes the form

$$(s \cdot \delta_P)(t) = \det s\; t^{\rho - s \cdot \rho}\, \delta_P(t), \quad \text{for } s \in W. \tag{50}$$

Remark 0.6.3. The example of $\mathbf{SO}(3)$ shows that the linear form ρ is not always a weight of T. However, the $\rho - s \cdot \rho$, for $s \in W$, always are weights of T, and we shall only use these if we consider functions on T. ✰

The Weyl group invariance of $\chi_\pi|_T$ implies that (50) holds with δ replaced by ϕ, that is,

$$\sum_{\theta \in \widehat{T}} c_\theta t^{s^*(\theta)} = s^*(\phi) = \det s\; t^{\rho - s^*(\rho)} \phi = \det s \sum_{\mu \in \widehat{T}} c_\mu t^{\mu + \rho - s^*(\rho)}.$$

Substituting $s^*(\theta) = \mu + \rho - s^*(\rho)$, or $\theta = s \cdot \mu + s \cdot \rho - \rho$, and using the linear independence of the $.^\mu$, for $\mu \in \widehat{T}$, we get

$$c_{s \cdot \mu + s \cdot \rho - \rho} = \det s\; c_\mu, \quad \text{for } \mu \in \widehat{T}, s \in W. \tag{51}$$

In other words, the coefficients c_μ behave in an antisymmetric way under the *shifted action*

$$(s, \mu) \mapsto s \cdot \mu + (s \cdot \rho - \rho)$$

of the Weyl group W on the lattice $\widehat{T}$ of weights of T.

A weight μ of T is called a *weight* of π, if $m_\mu \neq 0$. Let us, temporarily, say that $\lambda \in \widehat{T}$ is a *highest weight* of π, if it is a weight of π, and if $\mu \in \widehat{T}$ is not a weight of π, whenever

$\mu = \lambda + \Sigma(Q)$, with Q a nonvoid subset of P. Here we have written $\Sigma(Q) = \sum_{\alpha \in Q} \alpha$. Later we will give equivalent characterizations which agree more with the conventional definition of highest weights, see Proposition 0.6.8.

Because the set of weights of π is finite, and the convex cone in $i\mathfrak{t}^*$ generated by P is proper, cf. (27), there exists at least one highest weight λ. Otherwise we could continue adding nonzero sums of positive roots to weights of π, leading to infinitely many different weights of π.

Working out the product in (48), we get

$$\delta_P(t) = \sum_{Q \subset P} (-1)^{\#(Q)}\, t^{-\Sigma(Q)}, \qquad \text{or} \qquad \phi(t) = \sum_{\mu, Q} (-1)^{\#(Q)}\, m_\mu\, t^{\mu - \Sigma(Q)},$$

or, comparing coefficients,

$$c_\theta = \sum_Q (-1)^{\#(Q)}\, m_{\theta + \Sigma(Q)}.$$

Again using that the convex cone generated by P is proper, the only possibility that $\Sigma(Q) = 0$ is that Q is void. This means that $c_\lambda = m_\lambda$, if λ is a highest weight.

Furthermore, if $s \in W$, and $s \cdot \lambda + s \cdot \rho - \rho = \lambda$, then the fact that $m_{s \cdot \lambda} = m_\lambda > 0$, cf. (46), implies that $\rho - s \cdot \rho = \Sigma(P \setminus s \cdot P) = 0$. That is, $s \cdot P = P$, or $s = 1$, because the Weyl group acts freely on the set of Weyl chambers. It follows that the shifted action of W is free on λ; or the orbit of λ has $\#(W)$ many elements. Applying (49) and the sentence following it, combined with (51), we find that $c_\lambda^2 = 1$, or $c_\lambda = m_\lambda = 1$, and $c_\theta = 0$, if θ is not in the orbit of λ for the shifted Weyl group action. We have proved:

Theorem 0.6.4 (Weyl's character formula). *For each irreducible representation π of G, there is exactly one highest weight $\lambda \in \widehat{T}$, which has multiplicity equal to 1. At $t \in T \cap G^{\mathrm{reg}}$, the character of π is given by the formula*

$$\chi_\pi(t) = \frac{\sum_{s \in W} \det s\; t^{s \cdot \lambda + s \cdot \rho - \rho}}{\prod_{\alpha \in P} (1 - t^{-\alpha})}. \tag{52}$$

Remark 0.6.5. If π is the trivial representation, then $\chi_\pi = 1$, $\lambda = 0$, so (52) then yields

$$\delta_P(t) := \prod_{\alpha \in P} (1 - t^{-\alpha}) = \sum_{s \in W} \det s\; t^{s \cdot \rho - \rho}, \quad \text{for } t \in T. \tag{53}$$

If we substitute $t = \exp X$, for $X \in \mathfrak{t}$, and multiply the numerator and the denominator in (52) both by $\mathrm{e}^{\rho(X)}$, using the formula (53) for the denominator, then we get

$$\chi_\pi(\exp X) = \frac{\Phi_{\lambda+\rho}(X)}{\Phi_\rho(X)}, \quad \text{for } X \in \exp^{-1}(G^{\mathrm{reg}}) \cap \mathfrak{t}, \tag{54}$$

where we have written, for any $\theta \in \mathfrak{t}^*_{\mathbf{C}}$,

$$\Phi_\theta(X) = \sum_{s \in W} \det s\; \mathrm{e}^{s \cdot \theta(X)}, \quad \text{for } X \in \mathfrak{t}.$$

The advantage of this form is that the functions Φ_θ are obviously W-anti-invariant on $\mathfrak{t}$, the slight disadvantage is that the numerator and denominator in (54) only are the pull back (under the exponential mapping) of single-valued functions on T if ρ is a weight of T. ✰

Theorem 0.6.6 (Weyl's dimension formula). *If the irreducible representation π of G has highest weight λ, then its dimension is given by*

$$\mathrm{d}_\pi = \chi_\pi(1) = \sum_{\mu \in \widehat{T}} m_\mu = \prod_{\alpha \in P} \frac{\langle \alpha, \lambda + \rho \rangle}{\langle \alpha, \rho \rangle} = \prod_{\alpha \in P} \frac{(\lambda + \rho)(\alpha^\vee)}{\rho(\alpha^\vee)},$$

where $\langle ., . \rangle$ denotes any W-invariant inner product in $i\mathfrak{t}^*$. If $\mathfrak{z} = 0$, then $\{ [\pi] \in \widehat{G} \mid \mathrm{d}_\pi \leq C \}$ is a finite subset of $\widehat{G}$, for each $C > 0$.

Lemma 0.6.7. *Suppose that there is only one positive root α for the group G. If π is an irreducible representation of G with highest weight λ, then $l := \lambda(\alpha^\vee) \in \mathbf{Z}_{\geq 0}$, and*

$$\chi_\pi(t) = \sum_{k=0}^{l} t^{\lambda - k\alpha}.$$

Returning to our general connected, compact Lie group G, let us begin by observing that for any root α, we have $\exp 2\pi i \alpha^\vee = 1$, cf. Theorem 0.4.14.(iv). Hence, if $\mu \in \widehat{T}$, we have $1 = (\exp 2\pi i \alpha^\vee)^\mu = \mathrm{e}^{\mu(2\pi i \alpha^\vee)} = \mathrm{e}^{2\pi i \mu(\alpha^\vee)}$, or

$$\mu(\alpha^\vee) \in \mathbf{Z}, \quad \text{for every } \mu \in \widehat{T}, \alpha \in R.$$

If $\mu \in \widehat{T}$, and $\alpha \in R$, then the *α-ladder from μ to $s_\alpha \mu$* is defined as the set

$$\{ \mu - k\alpha \mid k \in \mathbf{Z}, 0 \leq k \leq \mu(\alpha^\vee) \},$$

if $\mu(\alpha^\vee) \geq 0$; and as the same set with α replaced by $-\alpha$, if $\mu(\alpha^\vee) < 0$. From Lemma 0.6.7, we now get:

Proposition 0.6.8. *(i) Let π be a finite-dimensional representation of G, and let μ be a weight of π. Then, for any root α, the α-ladder from μ to $s_\alpha \mu$ consists of weights of π.*

(ii) If π is irreducible with highest weight λ, then $\lambda(\alpha^\vee) \geq 0$, for all positive roots α.

(iii) For a weight λ of π, the following conditions (a)–(c) are equivalent:

(a) λ is the highest weight of π.

(b) If $\alpha \in P$, then $\lambda + \alpha$ is not a weight of π.

(c) For any weight μ of π, we have $\mu = \lambda - \sum_{\alpha \in P} n_\alpha \alpha$, for some $n_\alpha \in \mathbf{Z}_{\geq 0}$.

Remark 0.6.9. Because each irreducible summand of $\pi|_H$ yields its own α-ladder, and the α-ladders are either disjoint (and parallel), or stacked on top of each other with their middles at the same point in $i\mathfrak{t}^*$, it follows that the multiplicities along an α-ladder form a symmetric function under the reflection s_α, which moreover is monotonously nondecreasing towards the middle point. ✰

In $\widehat{T}$, one introduces a *partial ordering* $\preceq$ by writing $\mu \preceq \theta$ if and only if $\mu = \theta - \sum_{\alpha \in P} n_\alpha \alpha$, for some $n_\alpha \in \mathbf{Z}_{\geq 0}$. The property that $\mu \preceq \theta$ and $\theta \preceq \mu$ can only happen if $\mu = \theta$; this follows from the fact that the convex cone generated by P is proper, cf. (27). The customary definition is to call a weight λ of T a *highest weight* of π, if it is a maximal element of the set of weights of π, with respect to the partial order $\preceq$; this is just condition (c) in Proposition 0.6.8.(iii).

A weight μ of T is called *dominant* if $\mu(\alpha^\vee) \geq 0$, for all $\alpha \in P$. That is, if μ belongs to the closure of the positive Weyl chamber $\{ \mu \in i\mathfrak{t}^* \mid \mu(\alpha^\vee) > 0, \text{ for all } \alpha \in P \}$ in $i\mathfrak{t}^*$, with respect to the dual root system of the $\alpha^\vee$, $\alpha \in R$, cf. the discussion following Corollary 0.4.16.

We note that if λ is dominant, then $s \cdot \lambda \preceq \lambda$, for all $s \in W$, whereas conversely the condition that $s_\alpha \cdot \lambda \preceq \lambda$, for all $\alpha \in S$, already implies that λ is dominant. This is the origin of the name “dominant”.

The following theorem contains a converse to Proposition 0.6.8.(ii).

Theorem 0.6.10. *The mapping that assigns to each irreducible representation of G its highest weight, induces a bijection from $\widehat{G}$ onto the set of dominant weights of T.*

With a slight abuse of notation, we shall write $[\pi] = [\pi(\lambda)]$, and $\chi_\pi = \chi_\lambda$, if λ is the highest weight of π. That is, χ_λ is the conjugacy-invariant function on G, whose restriction to T is given by the right hand side in (52).

Remark 0.6.11. Using an $\mathrm{Ad}\, G = \mathrm{Ad}\, \mathfrak{g}$-invariant inner product on $\mathfrak{g}$, we get an identification of $\mathfrak{g}$ with $i\mathfrak{g}^* \subset \mathfrak{g}_{\mathbf{C}}^*$, which intertwines the adjoint action with the *coadjoint* (that is, the contragredient of the adjoint) action on $\mathfrak{g}_{\mathbf{C}}^*$. Under this identification, $\mathfrak{t}$ is mapped to a linear subspace of $i\mathfrak{g}^*$, which can be identified with $i\mathfrak{t}^*$. The adjoint orbits have a unique intersection point with the closure of the Weyl chamber; this implies that each coadjoint orbit in $i\mathfrak{g}^*$ has a unique intersection point λ with the dominant chamber in $i\mathfrak{t}^*$. In this way, one may think of the equivalence classes of irreducible representations of G as being parametrized by certain coadjoint orbits, a picture that is less dependent on choices of maximal Abelian subspaces and Weyl chambers. In the Borel-Weil theorem, representations are constructed in each equivalence class, in terms of geometric data related to the coadjoint orbits (which may be identified with the spaces G/G_λ). It is a *philosophy of A.A. Kirillov* that, in very great generality, irreducible representations of Lie groups are parametrized by, or even constructed geometrically from, coadjoint orbits. ✰