Answers to the Mid-term exam SCI 211, November 1, 2002

1a) Substituting

$$\cos x = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\,x} + \mathrm{e}^{-\mathrm{i}\,x}\right), \quad \sin x = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}\right)$$

we obtain that

$$(\cos x)^2 \sin x = \frac{1}{8\mathrm{i}}\left(\mathrm{e}^{3\,\mathrm{i}\,x} + \mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x} - \mathrm{e}^{-3\,\mathrm{i}\,x}\right) = \frac{1}{4}\sin(3x) + \frac{1}{4}\sin x.$$

1b) This is a Fourier series as in (1.2) with $p = 2\pi$, $b_1 = 1/4$, $b_3 = 1/4$, and all the other coefficients equal to zero. Therefore Parseval's identity (2.21) yields that

$$\int_{-\pi}^{\pi} (\cos x)^4 (\sin x)^2 \, \mathrm{d}x = 2\pi\,\frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2\right) = \frac{\pi}{8}.$$

2)

$$\begin{aligned}
\widehat{f}(\omega) &:= \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}\,\mathrm{d}t = \int_0^{\infty} t\,\mathrm{e}^{-2t}\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}\,\mathrm{d}t \\
&= \int_0^{\infty} t\,\mathrm{e}^{-(2+\mathrm{i}\,\omega)\,t}\,\mathrm{d}t = \frac{1}{2+\mathrm{i}\,\omega}\int_0^{\infty} \mathrm{e}^{-(2+\mathrm{i}\,\omega)\,t}\,\mathrm{d}t \\
&= \frac{1}{(2+\mathrm{i}\,\omega)^2}.
\end{aligned}$$

Here we have substituted the definition of f in the second identity. Furthermore we used

$$\mathrm{e}^{-(2+\mathrm{i}\,\omega)\,t} = -\frac{1}{2+\mathrm{i}\,\omega}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-(2+\mathrm{i}\,\omega)\,t},$$

in order to perform a partial integration in the fourth identity, where there are no boundary terms because $t\,\mathrm{e}^{-(2+\mathrm{i}\,\omega)\,t}$ vanishes when $t = 0$ and when $t \to \infty$.

3) Applying (4.7) with $\nu = 2$, we see that we have to translate the graph of $\widehat{f}$ to the left and to the right over a distance 2, add the functions and multiply by 1/2.

4a) The function g has to satisfy $\partial g(x,\,y)/\partial x = x$ and $\partial g(x,\,y)/\partial y = y$. The first equation yields that $g(x,\,y) = x^2/2 + h(y)$, and then the second equation implies that $h(y) = y^2/2 + c$, in which c is a constant, which we can take equal to zero.

4b) According to (7.2) the line integral is equal to

$$g(\gamma(T)) - g(\gamma(0)) = \frac{1}{2}\left(\frac{1}{(1+T)^2} - 1\right),$$

in which we have used that $(\cos t)^2 + (\sin t)^2 = 1$. When $T \to \infty$, the integral converges to $-1/2$.