Mid-term exam SCI 211, October 30, 2003

1 Compute the length of one cycle of the cycloid $(t - \sin t,\ 1 - \cos t)$, $0 \le t \le 2\pi$. Hint: at some point make the substitution of variables $t = 2s$ in the integral.

2 Let $V(x,\, y) = \ln\left(x^2 + y^2\right)$, for $(x, y) \neq (0, 0)$. Prove that $\Delta V = 0$ in the complement of the origin in the plane.

3 Define the function f and g on the real axis by

$$f(t) = \mathrm{e}^{-t^2/2} \quad \text{and} \quad g(t) = \mathrm{e}^{-t^2/2} \cos(5t), \quad \text{respectively.}$$

Let $\phi(\omega)$ and $\chi(\omega)$ denote the Fourier transform of f and g, respectively.

a) Express $\chi(\omega)$ in terms of the function ϕ. Give an explicit formula for $\phi(\omega)$ and $\chi(\omega)$.

b) Make a sketch of the graphs of the functions ϕ and χ on the interval $-10 \le \omega \le 10$, both in one frame, thereby clearly indicating which is ϕ and which is χ.

4 Let the function $f : \mathbf{R} \to \mathbf{R}$ be determined by the properties that

$$f(x) = \frac{\pi}{8}\, x\, (\pi - x) \quad \text{when} \quad 0 \le x \le \pi,$$

that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$, and that $f(x + 2\pi) = f(x)$ for all $x \in \mathbf{R}$.

a) Make a sketch of the graph of $f(x)$ for $-2\pi \le x \le 2\pi$.

b) Prove that

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{(2l+1)^3} \sin((2l+1)\, x), \quad x \in \mathbf{R}.$$

c) Prove that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - \sin(x))^2 \,\mathrm{d}x = \frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{(2l+1)^6}.$$

d) **(bonus)** Prove that $\langle f - \sin,\, \sin\rangle = 0$, hence $\langle f,\, \sin\rangle = \langle \sin,\, \sin\rangle$, and therefore $\langle f - \sin,\, f - \sin\rangle = \langle f,\, f\rangle - \langle \sin,\, \sin\rangle$. Use this to show that the left hand side in c) is equal to $\pi^6/1920 - 1/2 = 0.00072\ldots$.