Solutions for the final exam UCU SCI 211, December 2001

1a) By the chain rule, $\frac{d}{dt}(x(t)^2 + y(t)^2) = 2x(t)\frac{dx(t)}{dt} + 2y(t)\frac{dy(t)}{dt} = 2x(t)y(t) + 2y(t)\,(-x(t)) = 0$. The function $x(t)^2 + y(t)^2$ is constant, because its derivative is zero. This constant equals $x(0)^2 + y(0)^2 = 1$.

Remark. The derivative $v(t) = r'(t) = (y(t), -x(t))$ of the radius-vector $r(t)$ is perpendicular to it (because $\langle r(t), r'(t)\rangle = 0$. This implies the conservation law for the length of the radius-vector: the square of this length equals $\langle r(t), r(t)\rangle$, the derivative of which is $\langle r'(t), r(t)\rangle + \langle r(t), r'(t)\rangle = 2\langle r(t), r'(t)\rangle = 0$.

1b) Initial conditions: $x_0 = 1$, $y_0 = 0$; step: $x_{n+1} = x_n + hy_n$, $y_{n+1} = y_n - hx_n$. Now we prove by induction on n that $x_n^2 + y_n^2 = (1+h^2)^n$. Base: $x_0^2 + y_0^2 = 1^2 + 0^2 = 1$. Step: suppose that $x_n^2 + y_n^2 = (1+h^2)^n$ for all $n \le k$. For $n = k+1$ we have the following: $x_{k+1}^2 + y_{k+1}^2 = (x_k + hy_k)^2 + (y_k - hx_k)^2 = x_k^2 + 2hx_ky_k + h^2y_k^2 + y_k^2 - 2hy_kx_k + h^2x_k^2 = (x_k^2 + y_k^2)(1+h^2)$. By the induction hypotesis, this is equal to $(1+h^2)^k(1+h^2) = (1+h^2)^{k+1}$, which proves the step of induction.

Remark. The vector $r_{n+1} = (x_{n+1}, y_{n+1})$ can be viewed as the hypotenuse of the right triangle with two other sides $r_n = (x_n, y_n)$ and $\delta r_n = h(y_n, -x_n)$. By the Pythagoras theorem, $r_{n+1}^2 = r_n^2 + (\delta r_n)^2 = r_n^2 + h^2 r_n^2 = (1+h^2)r_n^2$. Make a picture!

1c) According to the previous step, $(x_N^2 + y_N^2)^{1/2} = (1+h^2)^{N/2} = (1 + (t/N)^2)^{N/2}$. Let $f(x) = (1+x)^{N/2}$. Then $f'(x) = \frac{N}{2}(1+x)^{\frac{N}{2}-1} > N/2$ for $x > 0$. Therefore $(x_N^2 + y_N^2)^{1/2} = f\left(\frac{t^2}{N^2}\right) > f(0) + \frac{N}{2}\frac{t^2}{N^2} = 1 + \frac{t^2}{2N}$. On the other hand, $\ln(1+a) < a$ for $a > 0$ (because $\ln'(1+x) = 1/(1+x) < 1$ and $\ln(1+x) = 0$ when $x = 0$; by the way, $\ln(1+a) < a$ for $-1 < a < 0$, too). Thus $\ln\left((x_N^2 + y_N^2)^{1/2}\right) = \frac{N}{2}\ln\left(1 + (t/N)^2\right) < \frac{N}{2}\frac{t^2}{N^2} = \frac{t^2}{2N}$, or equivalently $(x_N^2 + y_N^2)^{1/2} < e^{t^2/2N}$.

Combining these estimates of $(x_N^2 + y_N^2)^{1/2}$ with the fact that $x(t)^2 + y(t)^2 = 1$, we get $(x_N^2 + y_N^2)^{1/2} - (x(t)^2 + y(t)^2)^{1/2} > 1 + \frac{t^2}{2N} - 1 = \frac{t^2}{2N}$ and $(x_N^2 + y_N^2)^{1/2} - (x(t)^2 + y(t)^2)^{1/2} < e^{t^2/2N} - 1$.

2a) The wave propagation speed is $c = 1$, so we have

$$u(x,t) = \frac{1}{2}\big(f(x-t) + f(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds.$$

2b) See Figure 1.

Remarks. Since $g(x) = 0$ for all x, we have $u(x,t) = \frac{1}{2}\big(f(x-t) + f(x+t)\big)$. To plot $u(x,t)$ with some fixed t, shift a copy of the plot of $f(x)$ by t to the right (this gives $f(x-t)$) and to the left (this gives $f(x+t)$) and draw the average (for any x) of the plots obtained. Details are shown on the second graph in the interval $-4 < x < -1$. The green dashed line represents $f(x - 1/2)$ and the red dotted line is the plot of $f(x + 1/2)$.

First graph corresponds to $t = 0$ and shows 2-periodic extension of the function x^2 from the interval $[-1, 1]$. The graph for $t = 2$ is the same as for $t = 0$ because $u(x,2) = \frac{1}{2}\big(f(x-2) + f(x+2)\big) = \frac{1}{2}\big(f(x) + f(x)\big) = f(x)$ due to 2-periodicity of $f(x)$. For the third graph we have $u(x,1) = \frac{1}{2}\big(f(x-1) + f(x+1)\big) = \frac{1}{2}\big(f(x-1) + f(x-1)\big) = f(x-1)$ (because of 2-periodicity), so it is a horisontal shift of the first graph

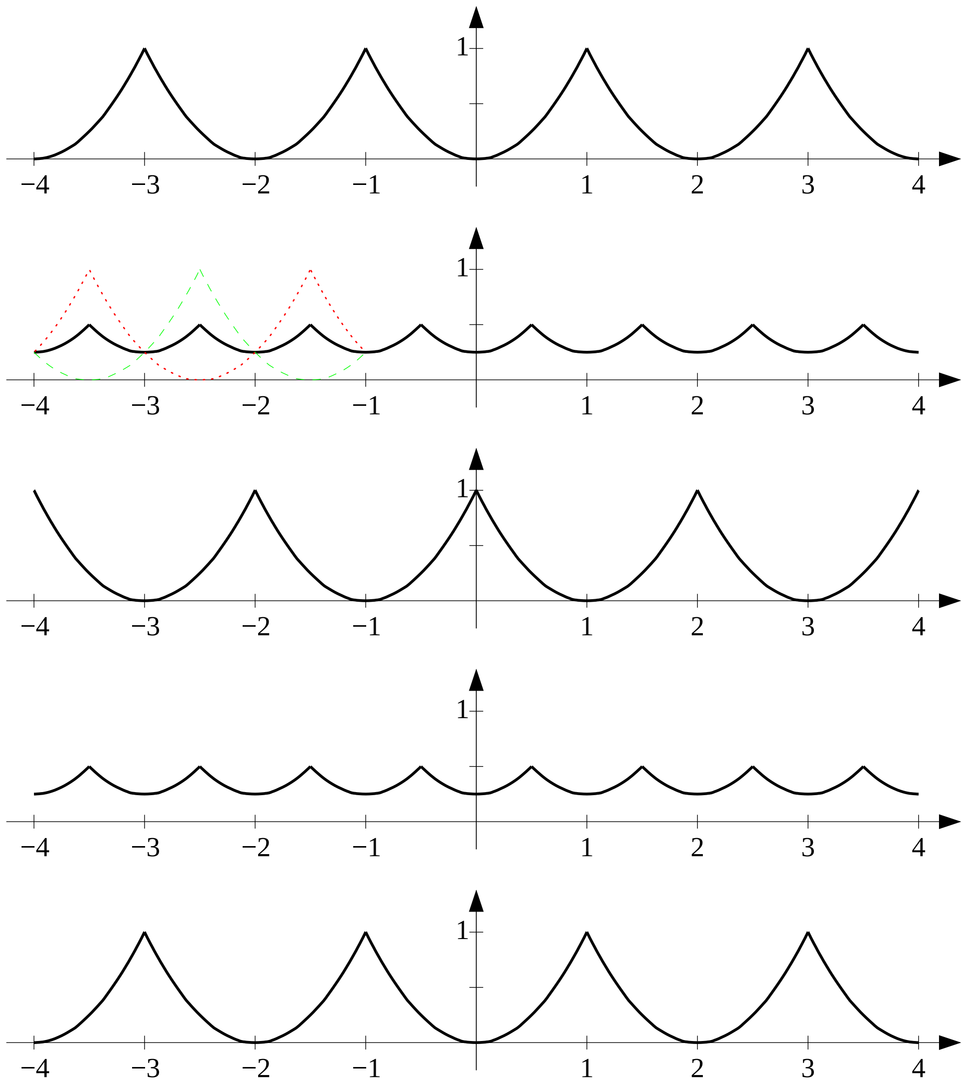

FIGURE 1. Graphs of $u(x,t)$ for $t = 0,\ 1/2,\ 1,\ 3/2,\ 2$

by 1. The second and the fourth graphs coincide, because $f(x-1/2) = f(x+3/2)$ and $f(x+1/2) = f(x-3/2)$ by periodicity, therefore $u(x,1/2) = u(x,3/2)$.

2c) According to the d'Alembert formula, $u(x,t) = \frac{1}{2}\big(f(x-t)+f(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\big(f(x-t)+f(x+t)\big) + \frac{1}{2}\big(f(x+t)-f(x-t)\big) = f(x+t)$ (compare with Exercise 3.3 from the notebook PartialDE.nb). In this case we have a standing wave running to the left with constant speed 1.

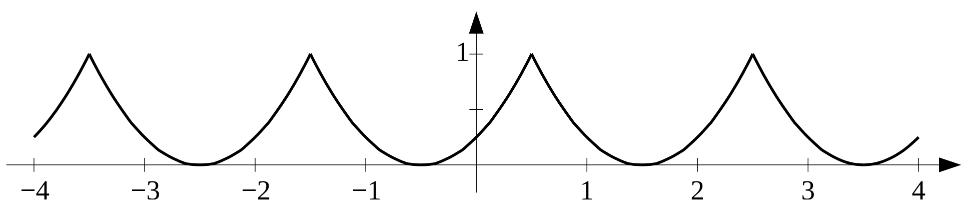

FIGURE 2. Graph of $u(x,1/2)$ for Problem 2c)

If $f(x)$ is as in b) and $g(x)$ is a 2-periodic function such that $g(x) = 2x$ for $-1 < x < 1$, then $g(x) = df(x)/dt$. Thus $u(x,t) = f(x+t)$, in particular, $u(x,1/2) =$

$f(x+1/2)$. See Figure 2 for the graph.

3a) Using the given expression of the Laplace operator in polar coordinates, we can rewrite the diffusion equation in the following form:

$$\frac{\partial U(r,\theta,t)}{\partial t} = \frac{\partial^2 U(r,\theta,t)}{\partial r^2} + \frac{1}{r}\frac{\partial U(r,\theta,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U(r,\theta,t)}{\partial \theta^2}, \quad 0 < r < 1,\ t > 0.$$

Differentiating with respect to t the Fourier series $U(r,\theta,t) = \sum_{k\in\mathbb{Z}} U_k(r,t)e^{ik\theta}$, we get the Fourier series for the left hand side of the equation: $\frac{\partial U(r,\theta,t)}{\partial t} = \sum c_k e^{ik\theta}$ with $c_k = c_k(r,t) = \frac{\partial U_k(r,t)}{\partial t}$. The Fourier series for the right hand side of the equation is $\sum c'_k e^{ik\theta}$ with

$$c'_k = c'_k(r,t) = \frac{\partial^2 U_k(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial U_k(r,t)}{\partial r} - \frac{k^2}{r^2}U_k(r,t).$$

Here we use that $e^{ik\theta}$ does not depend on r and t, so it is treated as a constant when differentiating with respect to r or t (in the left hand side and in the first two summands in the right hand side); $U_k(r,t)$ does not depend on θ and is treated as a constant when differentiating with respect to θ (the last summand in the right hand side, where $\frac{\partial^2}{\partial\theta^2}e^{ik\theta} = (ik)^2 e^{ik\theta} = -k^2 e^{ik\theta}$).

Since the left hand side of the equation equals its right hand side, the Fourier coefficients are equal, too: $c_k(r,t) = c'_k(r,t)$ for all $k \in \mathbb{Z}$, $r \in (0,1)$, $t > 0$. This gives (1). The boundary condition says that $U = 0$ on the boundary of the unit disk, i.e., $U(1,\theta,t) = 0$. Then $U_k(1,t) = 0$ for all $k \in \mathbb{Z}$, $t \geq 0$ (the $U_k(1,t)$ are the Fourier coefficients of $U(1,\theta,t) = 0$), which proves (2). The Fourier coefficients of the 2π-periodic functions $\theta \mapsto U(r,\theta,0)$ and $\theta \mapsto F(r,\theta)$ are equal due to the initial condition $U(r,\theta,0) = F(r,\theta)$; these Fourier coefficients are $U_k(r,0)$ and $F_k(r)$ with any $k \in \mathbb{Z}$ and $0 \leq r \leq 1$, whence (3).

3b) The nth order Bessel function $J_n(\rho)$ satisfies the following differential equation: $\rho^2 J''_n(\rho) + \rho J'_n(\rho) + (\rho^2 - n^2)J_n(\rho) = 0$ (see eq. (11.16) in the Guide Book). Below we will use it in the form

$$J''_n(\rho) + \frac{1}{\rho}J'_n(\rho) - \frac{n^2}{\rho^2}J_n(\rho) = -J_n(\rho).$$

If $U_{k,m}(r,t) = e^{-R^2 t}J_n(Rr)$ (with $R = R_{n,m}$), then the left hand side of (1) is $\frac{\partial U_{k,m}(r,t)}{\partial t} = -R^2 U_{k,m}(r,t)$. Further, put $\rho = Rr$; then $1/r = R/\rho$, $1/r^2 = R^2/\rho^2$, and $d\rho/dr = R$. Now we have $\frac{\partial U_{k,m}(r,t)}{\partial r} = e^{-R^2 t}RJ'_n(\rho)$ (by the chain rule, with $R = d\rho/dr$) and $\frac{\partial^2 U_{k,m}(r,t)}{\partial r^2} = e^{-R^2 t}R^2 J''_n(\rho)$. Consequently, the right hand side of (1) equals

$$R^2 e^{-R^2 t}\left(J''_n(\rho) + \frac{1}{\rho}J'_n(\rho) - \frac{n^2}{\rho^2}J_n(\rho)\right) = -R^2 e^{-R^2 t}J_n(\rho) = -R^2 U_{k,m}(r,t),$$

which is nothing but the left hand side of (1). Thus equation (1) is satisfied.

For $r = 1$ we have $U_{k,m}(1,t) = e^{-R^2 t}J_n(R) = 0$ for all $k \in \mathbb{Z}$ and $t \geq 0$, because $R = R_{n,m}$ is a zero of the function J_n. This yields (2). Finally, if $F_k(r) = J_n(R_{n,m}r)$, then for $t = 0$ we get $U_{k,m}(r,0) = J_n(R_{n,m}r) = F_k(r)$ for all $k \in \mathbb{Z}$ and $r \in [0,1]$, which proves (3).