EXERCISES II

Exercise 2.6. *Let $M = \mathbb{R}$. Find a canonical transformation of T^*M -equipped with its canonical symplectic structure- which is not induced by a diffeomorphism of M.*

Let (M, σ) be a symplectic manifold. We say that a submanifold $N \hookrightarrow M$ is **isotropic** (resp. **coisotropic, lagrangian, symplectic**), if for every $m \in N$ its tangent space $T_m N$ is an **isotropic** (resp. **coisotropic, lagrangian, symplectic**) subspace of the symplectic vector space $(T_m M, \sigma_m)$.

Exercise 2.7. *Let (M, σ) be a symplectic manifold and consider its cotangent bundle with the canonical symplectic structure.*

i. *Prove that the fibers of T^*M are lagrangian submanifolds.*
ii. *Let $N \hookrightarrow T^*M$ be a submanifold which can be parametrized as the graph of a section $\alpha\colon M \to T^*M$. The section α is then a 1-form. Show that N is lagrangian if and only if α is closed (i.e. $d\alpha = 0$).*

Exercise 2.8.

i. *Let $(D^2 \backslash \{0\}, \sigma_0)$ be the punctured open unit disk in $\mathbb{R}^2$ equipped with the restriction the canonical symplectic structure. Find a diffeomorphism*
$$\phi\colon D^2\backslash\{0\} \to D^2\backslash\{0\}$$
such that
   (a) *ϕ sends circles with center the origin to circles with center the origin.*
   (b) *ϕ sends circles approaching the outer boundary component to circles approaching the puncture.*
   (c) *$\phi^*\sigma_0 = \sigma_0$.*

   Hint: Use polar coordinates, and then recall that a symplectic form in $\mathbb{R}^2$ is an area form, and hence induces an orientation.
ii. *Let $B^{2n}\backslash\{0\}$ be the punctured unit ball in $\mathbb{R}^{2n}$. Let ϕ be any self-diffeomorphism of $B^{2n}\backslash\{0\}$ which sends points approaching the outer boundary component to points approaching the puncture. Then the manifold*
$$B^{2n} \coprod B^{2n} / \sim \phi,$$
where the equivalence relation amounts to identifying $x \in B^{2n}\backslash\{0\}$ in the first ball with $\phi(x)$ in the second, is known to be homeomorphic to the sphere S^n.

   Let σ_0 be the restriction to $B^{2n}\backslash\{0\}$ of the canonical symplectic form in $\mathbb{R}^{2n}$. Show that for $n > 1$ one cannot find ϕ as above so that $\phi^*\sigma_0 = \sigma_0$.

Exercise 2.9. *Consider the map*

$$\begin{aligned} f\colon \mathbb{C}^n\backslash\{0\} &\longrightarrow \mathbb{C}\backslash\{0\} \\ (z_1, \dots, z_n) &\longmapsto z_1^2 + \cdots + z_n^2 \end{aligned}$$

We endow $\mathbb{C}^n\backslash\{0\}$ with the restriction of the canonical symplectic structure σ_0. Show that

i. *p is a surjective submersion.*
ii. *Each fiber of p is a symplectic submanifold of $(\mathbb{C}^n\backslash\{0\}, \sigma_0)$.*

Exercise 2.10. *Let (M,σ) be a symplectic manifold and $p\colon M \to \Sigma$ a surjective submersion with the following properties:*

- *The fibers are compact and σ restricts to each of them to a symplectic structure (fibers are symplectic submanifolds).*

At each $m \in M$, the tangent space to the corresponding fiber $T_m^V M$ -also called **vertical tangent subspace**- *is symplectic. Therefore, its symplectic annihilator H_m is a plane transversal to the fiber. This distribution of planes defines an* **Ehresmann connection***: given any path $\gamma\colon [0,1] \to \Sigma$ and a point m in the fiber over $\gamma(0)$, there exists a unique path $\tilde{\gamma}\colon [0,1] \to M$ such that*

- $\tilde{\gamma}(0) = m$.
- $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$.
- $p \circ \tilde{\gamma} = \gamma$.

The path $\tilde{\gamma}$ is called the **horizontal lift** *(with respect to H) of γ at m.*

Now if we let m vary over $p^{-1}(\gamma(0))$ we obtain a diffeomorphism

$$H_\gamma\colon p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1)),$$

the so called **parallel transport over** γ.

Show that the parallel transport preserves the induced symplectic structures on the fibers.

Hint: Recall from construction on the last paragraph of page 13 of the notes, that the flow of a vector field in the kernel of a closed 2-form (in general of any closed p-form), preserves the 2-form.

We have seen that any symplectic manifold (M,σ) admits compatible almost complex structures. Let $\mathcal{J}(M,\sigma)$ denote the space of almost complex structures compatible with σ. They are a subset of the topological vector space of sections of the bundle $End(TM) \to M$ of endomorphisms of the tangent bundle (we use the compact open topology). Therefore they inherit a topology.

One can prove that $\mathcal{J}(M,\sigma)$ is contractible. The first step is looking again at the linear case.

Exercise 2.11. *Let (E,σ) be a symplectic vector space. We fix any $L \in \mathcal{L}(E,\sigma)$ and consider the following smooth map:*

$$\begin{array}{ccc} \mathcal{J}(E,\sigma) & \longrightarrow & \mathcal{L}_{L,0} \\ J & \longmapsto & JL \end{array}$$

Show that this map is well defined and surjective. Identify the fiber over $L' \in \mathcal{L}_{L,0}$ with the set of positive definite symmetric $n \times n$ matrices (here $2n$ is the dimension of E).

The map in exercise 2.11 is actually a smooth surjective submersion. The fibers are diffeomorphicm to vector spaces, and so is $\mathcal{L}_{L,0}$. This implies that $\mathcal{J}(E,\sigma)$ itself is diffeomorphic to a vector space, and hence contractible.

The linear construction of exercise 2.11 can be performed fiberwise (i.e. on each tangent space (T_mM,σ_m). This is seen to imply that $\mathcal{J}(M,\sigma)$ is contractible. One consequence of the contractibility of $\mathcal{J}(M,\sigma)$ is that the tangent bundle of a symplectic manifold has well defined Chern classes; even more, any vector bundle each of whose fibers carries a linear symplectic structure which varies smoothly, has well defined Chern classes (notice that in the above arguments closedness of the symplectic form was not used at all).

Exercise 2.12. *Let (M,σ) be a compact symplectic manifold. The second De Rham cohomology group $H^2_{dR}(M)$ is a finite dimensional vector space. Its rank is by definition the second Betti number of M, and it is denoted by β_2.*

i. *Show that the subset of cohomology classes in* $H^2_{dR}(M) \simeq \mathbb{R}^{\beta_2}$ *which admit a symplectic representative is an open subset.*

A cohomology class $[\alpha] \in H^2_{dR}(M)$ *is called* **integral**, *if for any closed 2-cycle* $[C] \in H_2(M;\mathbb{Z})$ *we have*

$$\int_C \alpha \in \mathbb{Z}$$

It is known that one can always find a basis of $H^2_{dR}(M)$ *all whose vectors are integral cohomology classes.*

ii. *Show that* M *admits symplectic structures whose associated cohomology class is integral.*