Virtual counts on surfaces *Seoul, January 2015*

Abstract

This series of lectures is divided into three parts. In part one, I review a sheaf theory developed by R. Pandharipande and R. P. Thomas known as stable pair theory. Stable pair invariants are closely related to both Gromov-Witten and Donaldson-Thomas invariants. In part two, I discuss virtual cycles on Hilbert schemes of curves with points on surfaces. In the absence of points these give rise to the Poincaré invariants of M. Dürr, A. Kabanov and Ch. Okonek. Poincaré invariants are an algebraic version of Seiberg-Witten invariants. In part three, I relate the virtual cycles of part one and part two for the total space of the canonical bundle over a surface. We apply the MNOP correspondence in this setting to obtain characterizations of Severi degrees and Seiberg-Witten invariants.

I would like to thank Professors B. Kim and M. van Garrel for the opportunity to speak at this winterschool. In these lectures I will discuss the following topics:

1. A sheaf theory known as stable pair theory developed by R. Pandharipande and R. P. Thomas.

2. Virtual cycles on Hilbert schemes on surfaces.

3. Applications to Severi degrees and Poincaré/Seiberg-Witten invariants.

I start with a motivating example which will come back in the final lecture. Consider $\mathbb{P}^2$ and the linear system $|\mathcal{O}(d)|$ of degree d effective divisors. The dimension of this linear system is

$$\dim(|\mathcal{O}(d)|) = \frac{d^2+3d}{2}$$

from which it is easy to see that there is a unique *smooth* curve of degree d passing through $\frac{d^2+3d}{2}$ points in general position.

Consider (possibly reducible) curves in $|\mathcal{O}(d)|$ which are only allowed to have the simplest type of singularities namely *nodes*. Locally, in complex analytic coordinates, a node is given by $xy = 0$. A curve is called δ-nodal if it has exactly δ nodes and no further singularities. The simplest case is $d = 2$ and $\delta = 1$. Such a curve is a product of 2 distinct lines and clearly there are 3 such curves through 4 point in general position. Enumeration of nodal curves on $\mathbb{P}^2$ is a problem with a rich history dating back to the 19th century.

Effective divisors in $|\mathcal{O}(d)|$ passing through a given point $p \in \mathbb{P}^2$ form a codimension 1 linear subspace of $|\mathcal{O}(d)|$. Effective divisors in $|\mathcal{O}(d)|$ having a singularity at a given point $p \in \mathbb{P}^2$ form a codimension 3 linear subspace of $|\mathcal{O}(d)|$ (i.e. defined by $f(p) = \partial_x f(p) = \partial_y f(p) = 0$). Varying p, the locus of divisors on S with a singularity at p is a codimension 1 closed subset of $|\mathcal{O}(d)|$. In the same vein, the locus of effective divisors with at least δ singularities forms a closed subset $\overline{V}_{d,\delta} \subset |\mathcal{O}(d)|$ of codimension δ known as a *Severi variety*. The locus of δ-nodal curves forms an open subset therein.

This implies there are a finite number $N_{d,\delta}$ of δ-nodal curves passing through

$$\frac{d^2+3d}{2} - \delta$$

points in general position and $N_{d,\delta}$ is known as a *Severi degree*. It is equal to the degree of the Severi variety $\overline{V}_{d,\delta}$. The Severi degrees were computed for $\delta = 1, 2, 3$ in the 19th century by A. Cayley and others, but the general determination of Severi degrees remained unsolved until around 1990 by Z. Ran [Ran1] and later L. Caporaso and J. Harris [CH].

Interestingly, modern invariants known as Gromov-Witten invariants (having their origin in string theory) help solve problems of enumerative geometry. Spectacularly, for curves of geometric genus 0, i.e.

$$\frac{d^2 - 3d}{2} + 1 - \delta = 0$$

M. Kontsevich used Gromov-Witten invariants of $\mathbb{P}^2$ to find a recursive formula enumerating all such curves[1]. Gromov-Witten invariants often contain important enumerative information. However, generally Gromov-Witten invariants are very hard to compute. However, nowadays there are two closely related sheaf theories on 3-folds, Donaldson-Thomas and stable pair theory, which have the same content as Gromov-Witten invariants but are often easier to calculate.

1 Stable pair theory *(after Pandharipande-Thomas)*

1.1 Moduli space

I start with the formal definition of stable pairs and then discuss concrete examples. Let X be a smooth variety. A *pair* (F, s) on X consists of:

(i) F is pure dimension 1 sheaf on X. This means F is a coherent sheaf on X and the support

$$\mathrm{Supp}(E) := \{x \in X : \text{the stalk } E_x \neq 0\}$$

of any subsheaf $0 \neq E \subset F$ is 1-dimensional. In particular, the support of F is 1-dimension.

(ii) $s \in H^0(F)$ a section.

A *stable pair* (F, s) on X is a pair for which the cokernel of s is 0-dimensional. One should think of this as a stability condition. In fact, stable pairs are a limit of more general stable objects defined by J. Le Potier [LeP].

Given a stable pair (F, s), one can form an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow Q \longrightarrow 0.$$

We claim the scheme theoretic support of $\mathrm{Im}(s)$ is the scheme theoretic support C_F of F [PT1, Lem. 1.6]. Equivalently $K = I_{C_F}$. The reason is this. Since the question is local, we can work over an open affine subset $\mathrm{Spec}\ R \subset X$. The scheme theoretic support of F is $\mathrm{Ann}(F)$ so the claim is $\mathrm{Ann}(F) = \mathrm{Ann}(s(1))$. Clearly, $\mathrm{Ann}(F) \subset \mathrm{Ann}(s(1))$. Moreover $\mathrm{Ann}(s(1)).F \subset F$ has 0-dimensional support since $\mathrm{coker}(s)$ has 0-dimensional support, so $\mathrm{Ann}(s(1)).F = 0$ by purity and therefore $\mathrm{Ann}(s(1)) \subset \mathrm{Ann}(F)$.

[1] For a nice exposition of Kontsevich formula, see [KV].

Purity of F implies C_F is Cohen-Macaulay; i.e. pure dimensional with no embedded components. Note that Q is a 0-dimensional sheaf supported on C_F. So roughly, we can think of a stable pair as a Cohen-Macaulay curve C_F with a bunch of points on it (the support of Q). Here are some examples:

(i) The simplest example comes from a closed embedding of a 1-dimensional subscheme $\iota : C \hookrightarrow X$. Then $\mathcal{O}_X \to \iota_*\mathcal{O}_C$ is surjective and $\mathcal{O}_C$ is pure if and only if C is Cohen-Macaulay. So Cohen-Macaulay curves give rise to stable pairs.

(ii) More generally, let C be a Cohen-Macaulay curve and $D \subset C$ an effective Cartier divisor. This gives an induced canonical section $s_D \in H^0(\mathcal{O}_C(D))$, which induces a stable pair $\mathcal{O}_X \to \iota_*\mathcal{O}_C \to \iota_*\mathcal{O}_C(D)$. This is the prototype stable pair.

(iii) This example is taken from [PT4]. Let $C_t = \{x = z = 0\}$ and $D_t = \{y = 0, z = t\}$, $t \in \mathbb{C}$. Moreover, for all $t \in \mathbb{C}$ we have a natural map

$$\mathcal{O}_{\mathbb{C}^3} \longrightarrow \mathcal{O}_{C_t} \oplus \mathcal{O}_{D_t}$$

For $t \neq 0$ this is the union of two disjoint lines, hence a stable pair. For $t = 0$ this is a map with cokernel $\mathcal{O}_{\mathbb{C}^3,0}$ hence a stable pair as well. More formally, this fits into a flat family of stable pairs. In general: a *flat family of stable pairs* on X over a base scheme B is a morphism $\mathcal{O}_{X\times B} \to \mathcal{F}$ such that $\mathcal{F}$ is a B-flat coherent sheaf on $X \times B$ and over each closed point $t \in B$, the pull-back $\mathcal{O}_X \to \mathcal{F}_t$ is a stable pair on X. The above provides a flat family of stable pairs where the general element is given by the structure sheaf of a Cohen-Macaulay curve and the limit element is a map with a cokernel. The limit object in this example is not of rank 1 on its scheme theoretic support.

(iv) The previous example shows F need not be $\mathcal{O}_{C_F}$. In the examples so far Q is always the structure sheaf of a 0-dimensional subsheaf on C_F. Even this need not be true in general. Let $C = \{xy = 0\} \subset \mathbb{C}^2$ be the node and let $C_1 = \{y = 0\} \subset C$ and $C_2 = \{x = 0\} \subset C$. Taking $p = (0, 0)$, the maps $\mathcal{O}_{C_i}(-p) \hookrightarrow \mathcal{O}_{C_i}$ induce a stable pair

$$\mathcal{O}_C \longrightarrow \iota_{1*}\mathcal{O}_{C_1}(p) \oplus \iota_{2*}\mathcal{O}_{C_2}(p).$$

At the level of modules, and after some identifications, this is

$$\mathbb{C}[x, y]/(xy) \xrightarrow{(x\ y)} \mathbb{C}[x, y]/(y) \oplus \mathbb{C}[x, y]/(x).$$

The cokernel Q is the 3-dimensional vector space spanned by, for example, $(1, 0)$, $(0, 1)$, $(0, y)$, so is not generated by one element as a $\mathbb{C}[x, y]/(xy)$-module.

Exercise 1.1. Consider three lines C_t, D_t, E_t in $\mathbb{C}^3$ which are disjoint for $t \neq 0$ and come together to form a triple point singularity for $t = 0$. Analyze the cokernel Q of $\mathcal{O}_X \to \mathcal{O}_{C_t} \oplus \mathcal{O}_{D_t} \oplus \mathcal{O}_{E_t}$ for $t = 0$. Is Q the structure sheaf of a 0-dimensional subscheme?

Let X be a smooth projective variety. In order to obtain a well-behaved moduli problem, we consider stable pairs (F, s) on X with fixed homology class $[C_F] = \beta \in$

$H_2(X)$ and holomorphic Euler characteristic $\chi(F) = \chi \in \mathbb{Z}$. Then one can show that the moduli functor of families of stable pairs on X with class β and holomorphic Euler characteristic χ is represented by a projective scheme. More precisely, consider the contravariant functor

$$\underline{P_\chi(X,\beta)} : (\mathrm{Sch}/\mathbb{C})^o \longrightarrow \mathrm{Sets},$$

which associates to a base scheme B the collection of B-flat families, where two such families $(\mathcal{F}, s), (\mathcal{G}, t)$ are identified if there exists an isomorphism $\phi : \mathcal{F} \to \mathcal{G}$ such that $t = \phi \circ s$. It associates to a morphism $f : B \to B'$ pull-back f^* of families along f. It follows from work of Le Potier (who considers a more general setup) that there exists a projective scheme $P_\chi(X,\beta)$ and an isomorphism of functors

$$\underline{P_\chi(X,\beta)} \cong \mathrm{Hom}(\cdot, P_\chi(X,\beta)).$$

The scheme $P_\chi(X,\beta)$ is said to *represent* $\underline{P_\chi(X,\beta)}$ and $P_\chi(X,\beta)$ is called a *fine* moduli space.

We end this section with some comments on stable pairs on surfaces. I mentioned one should *roughly* think of a stable pair as a Cohen-Macaulay curve with points. On surfaces this can be made precise (notwithstanding example (iv)). To see this we first introduce some Hilbert schemes. Let S be a smooth projective surface and $\beta \in H_2(S)$. In these lectures we will be dealing with the following two Hilbert schemes

$$\begin{aligned} S^{[n]} &:= \mathrm{Hilb}^n(S) : \text{Hilbert scheme of zero dimensional subschemes of } S \text{ of length } n, \\ H_\beta &:= \mathrm{Hilb}_\beta(S) : \text{Hilbert scheme of effective divisors on } S \text{ with class } \beta. \end{aligned}$$

The first is the famous Hilbert scheme of points which is smooth, because S is of dimension 2. The second Hilbert scheme need not be smooth. When the first Betti number of S vanishes, i.e. $b_1(S) = 0$, H_β is just a linear system (though not necessarily of expected dimension). When $b_1(S) \neq 0$ it can have a rather complicated structure over the Picard variety. We discuss this in more detail in the next lecture. Both are fine moduli spaces, i.e. their moduli functors are isomorphic to

$$\mathrm{Hom}(\cdot, S^{[n]}), \ \mathrm{Hom}(\cdot, H_\beta).$$

These are special cases of A. Grothendieck's general construction of the Hilbert scheme. A contravariant functor

$$\underline{M} : (\mathrm{Sch}/\mathbb{C})^o \longrightarrow \mathrm{Sets}$$

for which there exists a scheme M and an isomorphism of functors $\underline{M} \cong \mathrm{Hom}(\cdot, M)$ naturally gives rise to a *universal family*. Indeed the identity map $M = M$ gives an element $\xi_M \in \underline{M}(M)$. In this case this leads to families

$$\mathcal{Z} \subset S^{[n]} \times S \longrightarrow S^{[n]}, \qquad \mathcal{C} \subset H_\beta \times S \longrightarrow H_\beta.$$

Using the definitions you will find that for any family over B, there exists a unique map $B \to S^{[n]}$ or H_β such that the family pulls back from the universal family.

Exercise 1.2. Prove this last statement.

We also have a third Hilbert scheme defined by an incidence correspondence

$$\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \subset S^{[n]} \times H_\beta : \text{relative Hilbert scheme of points on the fibres of } \mathcal{C} \to H_\beta,$$

which need not be smooth either. Throughout these lectures, we let $\mathsf{k} := c_1(\mathcal{O}(\omega_S))$, where ω_S is the dualizing line bundle.

Theorem 1.1 (Pandharipande-Thomas)**.** *Let S be a surface, $\beta \in H_2(S)$ a curve class, and let $2h-2 = \beta(\beta+\mathsf{k})$ be the arithmetic genus of divisors with class β. Let $\mathcal{C} \to H_\beta$ be the universal curve. Then*

$$P_\chi(S,\beta) \cong \mathrm{Hilb}^n(\mathcal{C}/H_\beta),$$

where $\chi = 1-h+n$.

For stable pairs on surfaces the support curve $C := C_F$ is in fact Gorenstein because $\omega_C = \omega_S(C)$ is a line bundle by adjunction.

The key Claim is [PT3, Lem. B.2], which we do not prove here. Claim states that a generically locally free sheaf F on a Gorenstein curve C is pure if and only if

$$\mathscr{E}xt^i(F, \mathcal{O}_C) = 0,$$

for all $i > 0$.

Given a stable pair (F,s), let $C := C_F$. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F|_C \longrightarrow Q|_C \longrightarrow 0$$

on C. Dualizing gives a long exact sequence

$$0 \longrightarrow (Q|_C)^* \longrightarrow (F|_C)^* \longrightarrow \mathcal{O}_C \longrightarrow \mathscr{E}xt^1(Q|_C, \mathcal{O}_C) \longrightarrow \mathscr{E}xt^1(F|_C, \mathcal{O}_C) \longrightarrow 0.$$

Here $(Q|_C)^*$ is zero since $\mathcal{O}_C$ is pure and Q is 0-dimensional. By Claim $\mathscr{E}xt^1(F|_C, \mathcal{O}_C) = 0$, so $\mathscr{E}xt^1(Q|_C, \mathcal{O}_C)$ is the structure sheaf of a 0-dimensional closed subscheme $Z \subset C$.

Exercise 1.3. Define the map in the other direction by dualizing $0 \to I_Z \to \mathcal{O}_S \to \mathcal{O}_Z \to 0$ for $Z \subset C$ 0-dimensional. Make sure you land in $P_\chi(S,\beta)$ (check purity using Claim).

1.2 Deformation theory

Back to a smooth projective variety X and the moduli space $P := P_\chi(X,\beta)$. Let $p = (F_0, s_0) \in P_\chi(X,\beta)$ be a closed point. Then the deformation functor associated to (F_0, s_0) is the covariant functor

$$\mathcal{D}_p : \mathrm{Artin} \longrightarrow \mathrm{Sets},$$

which sends to a local Artinian $\mathbb{C}$-algebra A with residue field $\mathbb{C}$ the collection of Spec A flat families which pull-back to $p = (\mathcal{F}_0, s_0)$ over the closed point. It sends a morphism of local $\mathbb{C}$-algebras $f : A \to A'$ to pull-back along the induced morphism of schemes. We should think of $\mathcal{D}_p(A)$ as the collection of infinitesimal deformations of $(\mathcal{F}_0, s_0)$ over the 0-dimensional scheme Spec A. Representability of the moduli functor implies

$$\mathcal{D}_p \cong \mathrm{Hom}(\widehat{\mathcal{O}}_{P,p}, \cdot), \tag{1}$$

where $\widehat{\mathcal{O}}_{P,p}$ is the formal completion of the stalk $\mathcal{O}_{P,p}$. In this context $\widehat{\mathcal{O}}_{P,p}$ is said to *pro-represent* $\mathcal{D}_p$.

Exercise 1.4. Prove the isomorphism (1) assuming representability of the moduli functor for P. Use the fact that any morphism of local $\mathbb{C}$-algebras $\mathcal{O}_{P,p} \to A$ factors through $\widehat{\mathcal{O}}_{P,p}$ (why?).

In particular, for $A = \mathbb{C}[\epsilon]/(\epsilon)^2$ we get

$$\mathcal{D}_p(\mathbb{C}[\epsilon]/(\epsilon^2)) \cong T_P|_p \tag{2}$$

the Zariski tangent space to P at p.

Exercise 1.5. (i) Let X be a $\mathbb{C}$-scheme and $p \in X$ a closed point. Show that the collection of morphisms Spec $\mathbb{C}[\epsilon]/(\epsilon^2)$ to X mapping the closed point to p is in natural bijective correspondence with the elements of the Zariski tangent space $T_X|_p$. Recall $T_X|_p := \mathfrak{m}_p/\mathfrak{m}_p^2$, where $\mathfrak{m}_p$ is the maximal ideal corresponding to p. This is [Har, Exc. II.2.8]. (ii) Now use pro-representability of the deformation functor (or representability of the the moduli functor) to establish (2).

By a very general result on pro-representability of deformation functors due to M. Schlessinger [Sch], this implies the following characterization of deformations. Suppose we have a short exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A' \longrightarrow A \longrightarrow 0$$

of local Artinian $\mathbb{C}$-algebras with residue field $\mathbb{C}$. This sequence is called a *small extension* if $\mathfrak{a} \subset A'$ is a nonzero principal ideal satisfying $\mathfrak{a}\mathfrak{m}' = 0$ where $\mathfrak{m}' \subset A'$ is the maximal ideal. Examples of small extensions are

$$0 \longrightarrow (\epsilon^k) \longrightarrow \mathbb{C}[\epsilon]/(\epsilon^{k+1}) \longrightarrow \mathbb{C}[\epsilon]/(\epsilon^k) \longrightarrow 0,$$

for any $k \geq 0$. Suppose $(\mathcal{F}, s)$ is a deformation of p over Spec A. Then one can ask:

- Does there exist a flat family $(\mathcal{F}', s')$ over Spec A' which extends $(\mathcal{F}, s)$?
- If so, can we characterize all such extensions?

Schlessinger's result gives the answer to the second question: there exists a transitive and faithful action of $T_P|_p \otimes \mathfrak{a}$ on the set of extensions of $(\mathcal{F}, s)$. Similar to Hilbert schemes, for stable pairs the Zariski tangent space to P at p is given by

$$\begin{aligned} T_P|_p &\cong \mathrm{Hom}(I_0^\bullet, F_0), \\ I_0^\bullet &= \{\mathcal{O}_{X \times A_0} \to F_0\}, \end{aligned}$$

where the latter is viewed as a 2-term complex in degrees $0, 1$.

Exercise 1.6. Let $X^{[n]}$ be the Hilbert scheme of n points on a smooth projective variety X. Consult M. Lehn's *Lectures on Hilbert schemes* [Leh] to see that at a closed point

$Z \in X^{[n]}$ there exists a natural isomorphism $T_{X^{[n]}}|_p \cong \mathrm{Hom}(I_Z, \mathcal{O}_Z)$, where $I_Z \subset \mathcal{O}_X$ is the ideal sheaf defining $Z \subset X$. Note that the prove works for any Hilbert scheme (not necessarily of points).

What about the first question regarding obstructions? It turns our there exists a natural class

$$\mathfrak{o}(\mathcal{F}_0, s) \in \mathrm{Ext}^1(I_0^\bullet, \mathcal{F}_0) \otimes \mathfrak{a}$$

such that $\mathfrak{o}(\mathcal{F}_0, s) = 0$ if and only if extensions of $(\mathcal{F}, s)$ exist. This class is known as the *obstruction class*. We say somewhat informally that for any $I^\bullet = \{\mathcal{O}_X \to F\} \in P$

$$\mathrm{Hom}(I^\bullet, F) = \text{deformation space},$$
$$\mathrm{Ext}^1(I^\bullet, F) = \text{obstruction space}.$$

For further details we refer the reader to [HL] and [Sch].

Donaldson-Thomas theory is about the moduli space $\mathrm{Hilb}_{\beta,\chi}(X)$, i.e. the Hilbert scheme of closed subschemes $Y \subset X$ of dimension ≤ 1 such that $[Y] = \beta$ and $\chi(\mathcal{O}_Y) = \chi$. This is a fine projective moduli scheme and

$$\mathrm{Hom}(I_Y, \mathcal{O}_Y) = \text{deformation space},$$
$$\mathrm{Ext}^1(I_Y, \mathcal{O}_Y) = \text{obstruction space},$$

where $I_Y \subset \mathcal{O}_X$ is the ideal sheaf of Y (see [HL, Ch. 2] for a proof). However, for both stable pair and Donaldson-Thomas theory, these deformation and obstruction spaces are no good. The reason is that (in dimension ≥ 3) $\mathrm{Ext}^{\geq 2}(I^\bullet, \mathcal{F})$ and $\mathrm{Ext}^{\geq 2}(I_Y, \mathcal{O}_Y)$ need not vanish.

One key insight in Donaldson-Thomas theory is to take a different viewpoint on a closed subscheme $Y \subset X$. One can view the ideal sheaf I_Y as a rank 1 (stable) torsion free sheaf on X with trivial determinant $\mathcal{O}_X$. This gives a map

$$\mathrm{Hilb}_{\beta,\chi}(X) \to I_\chi(X, \beta).$$

Here the LHS is the Hilbert scheme of ideal sheaves $Y \subset X$ such that $[Y] = \beta$ and $\chi(\mathcal{O}_Y) = \chi$. The RHS is the moduli space of rank 1 stable torsion free sheaves on X with trivial determinant (i.e. ideal sheaves $I_Y \subset \mathcal{O}_X$) such that $[Y] = \beta$ and $\chi(\mathcal{O}_Y) = \chi$. This map turns out to be an isomorphism of schemes. Moreover the deformation and obstruction space of I_Y, *as a sheaf*, are

$$\mathrm{Ext}^1(I_Y, I_Y)_0 = \text{deformation space},$$
$$\mathrm{Ext}^2(I_Y, I_Y)_0 = \text{obstruction space},$$

where $(\cdot)_0$ means trace-free part, i.e. kernel of the map

$$\mathrm{tr} : \mathrm{Ext}^i(I_Y, I_Y) \to H^i(\mathcal{O}_X).$$

By a result of I. V. Artamkin [Art], the obstruction class lies in the kernel of this map (provided $H^1(\mathcal{O}_X) = 0$, which we always assume when X is a 3-fold). I highly recommend his paper for learning about obstruction classes for deformations of sheaves. The short exact sequence

$$0 \longrightarrow I_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

gives a map $\mathrm{Hom}(I_Y, \mathcal{O}_Y) \to \mathrm{Ext}^1(I_Y, I_Y)_0$ which is an isomorphism of the tangent spaces. However, the map between obstructions $\mathrm{Ext}^1(I_Y, \mathcal{O}_Y) \to \mathrm{Ext}^2(I_Y, I_Y)_0$ is in general not an isomorphism. So changing viewpoints from subschemes to ideal sheaves leads to the same moduli space but a different obstruction space. The sheaf deformation and obstruction spaces have several advantages. Firstly, $\mathrm{Hom}(I_Y, I_Y)_0 = 0$ (I_Y is simple). Secondly, if X is 3-dimensional and Calabi-Yau, i.e. $\omega_X \cong \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$, then

$$\begin{aligned}\mathrm{Ext}^3(I_Y, I_Y)_0 &\cong \mathrm{Hom}(I_Y, I_Y)_0^* = 0,\\ \mathrm{Ext}^2(I_Y, I_Y)_0 &\cong \mathrm{Ext}^1(I_Y, I_Y)_0^*.\end{aligned}$$

The latter means the expected dimension of the moduli space at I_Y is zero! We come back to this. *From now on let X 3-dimensional Calabi-Yau.*

Similarly, in stable pair theory Pandharipande and Thomas change Le Potier's deformation and obstruction space. They consider a stable pair $I^\bullet = \{\mathcal{O}_X \to \mathcal{F}\} \in P_\chi(X, \beta)$ as a 2-term complex in the bounded derived category $D^b(X)$ of coherent sheaves on X. The complex $I^\bullet$ is concentrated in degrees $0, 1$ and has trivial determinant. It turns out two stable pairs are isomorphic if and only if the the corresponding complexes are quasi-isomorphic. The exact triangle

$$I^\bullet \longrightarrow \mathcal{O}_X \longrightarrow F$$

induces a map $\mathrm{Hom}(I^\bullet, F) \to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$. As an element of the derived category, the complex $I^\bullet$ has deformations/obstructions governed by

$$\begin{aligned}\mathrm{Ext}^1(I^\bullet, I^\bullet)_0 &= \text{deformation space},\\ \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 &= \text{obstruction space}.\end{aligned}$$

Theorem 1.2 (Pandharipande, Thomas)**.** *The map* $\mathrm{Hom}(I^\bullet, F) \to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ *is an isomorphism of tangent spaces, i.e. the first order deformations of the stable pair* (F, s) *and 2-term complex* $I^\bullet$ *match.*

In fact, they prove more: the deformations match to all order. So by viewing elements of $P_\chi(X, \beta)$ as 2-term complexes, we endow it with the same deformation spaces but different obstruction spaces. By Serre duality, the groups $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ satisfy

$$\begin{aligned}\mathrm{Ext}^3(I^\bullet, I^\bullet)_0 &\cong \mathrm{Hom}(I^\bullet, I^\bullet)_0^* = 0,\\ \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 &\cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_0^*,\end{aligned} \tag{3}$$

so the moduli space has expected dimension zero at $I^\bullet$.

A moment ago I said "from now on let X be a Calabi-Yau 3-fold", yet this lecture series is about virtual counts on *surfaces*. Given a smooth projective surface S, the total space of the canonical bundle $X := \mathrm{Tot}(K_S)$ is a Calabi-Yau 3-fold containing S as its zero section. Although X is non-compact, our discussion up to this point is still valid as long as we consider stable pairs with proper support. The variety X has a natural $\mathbb{C}^*$ action namely scaling of the fibres. This action lifts to an action on the moduli space $P_\chi(X, \beta)$.

Exercise 1.7. Choose a $\mathbb{C}^*$ equivariant structure on $\mathcal{O}_X$ and use it to define the action on $P_\chi(X,\beta)$ at the level of closed points. Next use the moduli functor to define the action as a morphism $\mathbb{C}^* \times P_\chi(X,\beta) \to P_\chi(X,\beta)$ and show it is an action.

All elements of the fixed locus $P_\chi(X,\beta)^{\mathbb{C}^*}$ are stable pairs for which the reduced support C_F^{red} lies in the zero section S. However the scheme theoretic support may not lie in S. For example, any effective divisor in $D \subset S$ thickened infinitesimally into the fibre direction gives rise to a stable pair $\mathcal{O}_X \to \mathcal{O}_{2D}$ satisfying

$$\begin{aligned} 2[D] &= \beta \\ \chi(\mathcal{O}_{2D}) &= \chi(\mathcal{O}_D) + \chi(\mathcal{O}_D \otimes K_S^{-1}). \end{aligned}$$

Consider the morphism

$$\begin{aligned} P_\chi(S,\beta) &\longrightarrow P_\chi(X,\beta)^{\mathbb{C}^*}, \\ I_S^\bullet = \{\mathcal{O}_S \to F\} &\mapsto I_X^\bullet = \{\mathcal{O}_X \to \iota_*\mathcal{O}_S \to \iota_* F\}, \end{aligned}$$

where $\iota : S \subset X$ is inclusion. We claim this is an open immersion, i.e. $P_\chi(S,\beta)$ is a connected component of $P_\chi(X,\beta)^{\mathbb{C}^*}$. This follows from matching the deformation and obstruction spaces of the LHS and RHS. (Strictly speaking on should also show a compatibility between the vanishing of the obstruction classes, but we will not do that here.)

Proposition 1.3. *There exist natural isomorphisms*

$$\mathrm{Ext}^i(I_S^\bullet, F) \cong \mathrm{Ext}^i(I_X^\bullet, \iota_* F)^{\mathbb{C}^*} \cong \mathrm{Ext}^{i+1}(I_X^\bullet, I_X^\bullet)_0^{\mathbb{C}^*}.$$

Although the proof is not so important, I include it here in order to give a flavour of the kind of techniques used in this subject. Consider the exact triangle

$$I_X^\bullet \to \mathcal{O}_X \to \iota_* F.$$

By definition, the trace map gives rise a short exact sequence

$$0 \longrightarrow \mathrm{Ext}_X^i(I_X^\bullet, I_X^\bullet)_0 \longrightarrow \mathrm{Ext}_X^i(I_X^\bullet, I_X^\bullet) \longrightarrow H^i(\mathcal{O}_X) \longrightarrow 0.$$

In fact, these sequences are split: the map $s : \mathcal{O}_X \to R\mathscr{H}om(I_X^\bullet, I_X^\bullet)$ has the property

$$\mathrm{tr} \circ s = \mathrm{rk}(I_X^\bullet).\,\mathrm{id} = \mathrm{id},$$

where we use $\mathrm{rk}(I_X^\bullet) = 1$. We obtain the following commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & H^i(\mathcal{O}_X) \arrow[r, equal] \arrow[d] & H^i(\mathcal{O}_X) \arrow[d] & & \\
\cdots \arrow[r] & \mathrm{Ext}_X^i(I_X^\bullet, I_X^\bullet) \arrow[r] \arrow[d] & \mathrm{Ext}_X^i(I_X^\bullet, \mathcal{O}_X) \arrow[r] \arrow[d] & \mathrm{Ext}_X^i(I_X^\bullet, \iota_* F) \arrow[r] & \cdots \\
 & \mathrm{Ext}_X^i(I_X^\bullet, I_X^\bullet)_0 \arrow[d] & \mathrm{Ext}_X^{i+1}(\iota_* F, \mathcal{O}_X) & & \\
 & 0 & & &
\end{tikzcd}$$

From this diagram we obtain an exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i_X(I^\bullet_X, I^\bullet_X)_0 \longrightarrow \mathrm{Ext}^{i+1}_X(\iota_* F, \mathcal{O}_X) \longrightarrow \mathrm{Ext}^i_X(I^\bullet_X, \iota_* F) \longrightarrow \cdots$$

Excerise 1.8. Prove that this exact sequence arises from the diagram.

All complexes involved are $\mathbb{C}^*$-equivariant so we should view these sequences in the category of representations of $\mathbb{C}^*$. Although ω_X is trivial, $\mathbb{C}^*$-equivariantly it has weight -1, so we write $\omega_X \cong \mathcal{O}_X \otimes \mathfrak{t}^{-1}$. Serre duality on the middle term shows that it has no $\mathbb{C}^*$-fixed part

$$\mathrm{Ext}^i_X(\iota_* F, \mathcal{O}_X) \cong H^{3-i}(\iota_* F)^* \otimes \mathfrak{t}.$$

We conclude $\mathrm{Ext}^{i+1}_X(I^\bullet_X, I^\bullet_X)^{\mathbb{C}^*}_0 \cong \mathrm{Ext}^i_X(I^\bullet_X, \iota_* F)^{\mathbb{C}^*}$. For the second part we use the exact triangle

$$F \otimes N^*_{S/X} \longrightarrow L\iota^* I^\bullet_X \longrightarrow I^\bullet_S,$$

where $\iota : S \subset X$ is inclusion of the zero section and $N_{S/X}$ is the normal bundle of $S \subset X$. The normal bundle is K_S but with weight 1, i.e. $N_{S/X} \cong K_S \otimes \mathfrak{t}$. We obtain the long exact sequence

$$\cdots \to \mathrm{Ext}^i_S(I^\bullet_S, F) \to \mathrm{Ext}^i_X(I^\bullet_X, \iota_* F) \to \mathrm{Ext}^i_S(F, F \otimes K_S) \otimes \mathfrak{t} \to \cdots,$$

where we used adjunction on the middle term. Since the third term has no $\mathbb{C}^*$ fixed part, we conclude $\mathrm{Ext}^i_X(I^\bullet_X, \iota_* F)^{\mathbb{C}^*} \cong \mathrm{Ext}^i_S(I^\bullet_S, \iota_* F)$.

1.3 Stable pair invariants

Let X be a smooth projective Calabi-Yau 3-fold. The spaces $\mathrm{Hom}(I^\bullet, F)$, $\mathrm{Ext}^1(I^\bullet, F)$ and $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$, $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ can be combined in families as follow. Let $\mathbb{I}^\bullet = \{\mathcal{O} \to \mathbb{F}\}$ be the universal stable pair on $P \times X$, where $P := P_\chi(X, \beta)$. Using projection $\pi : P \times X \to P$, we can form

$$R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{F}),\ R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0.$$

Pulled back to a closed point $I^\bullet = \{\mathcal{O}_X \to F\} \in P$, these complexes become $R\,\mathrm{Hom}(I^\bullet, F)$ and $R\,\mathrm{Hom}(I^\bullet, I^\bullet)_0$.

We mentioned in the previous section that we cannot hope that (in dimension ≥ 3) $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{F})$ is quasi-isomorphic to a 2-term complex of locally free sheaves. Pandharipande and Thomas show $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ is quasi-isomorphic to a 2-term complex of locally free sheaves [PT1, Lem. 2.10]. Moreover:

Theorem 1.4 (Pandharipande-Thomas)**.** *The complex $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ gives rise to a perfect obstruction theory on $P_\chi(X, \beta)$.*

I should note that this perfect obstruction theory can be defined on any smooth 3-fold. If you do not know at this point what a perfect obstruction theory is, do not worry. The notion will explained in Cristina's and Tom's lectures and I will also talk about it in the next lecture. What is important now is that a perfect obstruction theory on a proper space M gives rise to a natural cycle

$$[M]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(M)$$

called the *virtual cycle*. Here vd is the *virtual dimension*, which is by definition the rank of the 2-term complex defining the perfect obstruction theory. In our case

$$\mathrm{vd} = \mathrm{rk}\, R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 = \dim \mathrm{Ext}^1_X(I^\bullet, I^\bullet)_0 - \dim \mathrm{Ext}^2_X(I^\bullet, I^\bullet)_0 = 0$$

by Serre duality. Therefore we get a virtual cycle of dimension 0 and one defines

$$\mathrm{PT}_{\chi,\beta}(X) := \int_{[P_\chi(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z},$$

to be a *stable pair invariant* (or *Pandharipande-Thomas invariant*) of X. Note that for a general smooth 3-fold the virtual dimension need not be zero.

Why are they called invariants? The constructions so far can all be done relative to a base. Suppose $\mathcal{X} \to B$ is a smooth projective family of projective Calabi-Yau 3-folds over a smooth base B. The construction of the moduli space of stable pairs works in families: there exists a moduli space $P_\chi(\mathcal{X}/B, \beta)$ of stable pairs (scheme theoretically supported) on the fibres of $\mathcal{X} \to B$. This turns out to have a *relative* perfect obstruction theory with associated virtual cycle

$$[P_\chi(\mathcal{X}/B, \beta)]^{\mathrm{vir}} \in H_{2\dim B}(P_\chi(\mathcal{X}/B, \beta)).$$

For any $t \in B$, let $i_t : P_\chi(\mathcal{X}_b, \beta) \hookrightarrow P_\chi(\mathcal{X}/B, \beta)$ be the fibre over t. Then one can show that [PT1, Thm. 2.15]

$$i_t^![P_\chi(\mathcal{X}/B, \beta)]^{\mathrm{vir}} = [P_\chi(\mathcal{X}_t, \beta)]^{\mathrm{vir}}.$$

This in turn implies *deformation invariance*

$$\mathrm{PT}_{\chi,\beta}(\mathcal{X}_t) = \mathrm{PT}_{\chi,\beta}(\mathcal{X}_{t'}),\ \forall t, t' \in B.$$

1.4 Conjectures

What are the relations of stable pair invariants to other curve counting invariants?

Let X be smooth projective variety and $\beta \in H_2(X)$. For any $g, m \geq 0$, one can consider the moduli stack $\overline{M}_{g,m}(X, \beta)$ of *stable maps* $f : C \to X$, where C is a connected curve of arithmetic genus g with at worst nodal singularities, with m marked smooth points, and satisfying $f_*[C] = \beta$. Stable essentially means the automorphism group of f is finite. The stack $\overline{M}_{g,m}(X, \beta)$ is a proper Deligne-Mumford stack and admits a perfect obstruction theory [Beh, LT1]. When X is a Calabi-Yau 3-fold and there are no marked points, the virtual dimension is zero and the degree of the virtual cycle is by definition a *Gromov-Witten invariant* of X

$$\mathrm{GW}_{g,\beta}(X) := \int_{[\overline{M}_{g,0}(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Q}.$$

Note that the invariant lies in $\mathbb{Q}$ since we are considering classes on a *stack* and therefore automorphisms have to be taken into account. I should stress that this perfect obstruction theory exists for any smooth (or in fact symplectic) variety X and any number of marked points, but in general the virtual dimension need not be zero.

Now let X be a smooth projective Calabi-Yau 3-fold. Then the moduli space of ideal sheaves $I_\chi(X, \beta)$ discussed earlier admits a perfect obstruction theory [Tho]. Serre

duality implies it has virtual dimension zero and the degree of the virtual cycle is by definition a *Donaldson-Thomas invariant* of X

$$\mathrm{DT}_{\chi,\beta}(X) := \int_{[I_\chi(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Similar to stable pair invariants, Donaldson-Thomas invariants can be defined for any smooth projective 3-fold with $H^0(K_X^*) \neq 0$, but in general the virtual dimension need not be zero.

We put the invariants into generating series (keeping X, β fixed):

$$\mathrm{GW}_\beta(X) := \sum_g \mathrm{GW}_{g,\beta}(X) u^{2g-2},$$
$$\mathrm{DT}_\beta(X) := \sum_\chi \mathrm{DT}_{\chi,\beta}(X) q^\chi,$$
$$\mathrm{PT}_\beta(X) := \sum_\chi \mathrm{PT}_{\chi,\beta}(X) q^\chi.$$

These are all formal Laurent series. The famous MNOP conjecture (or GW/DT correspondence) [MNOP1] states:

Conjecture 1.5 (DT/GW correspondence)**.** $\mathrm{DT}'_\beta(X)$ *is a rational function function of* q *invariant under* $q \leftrightarrow q^{-1}$ *and* $\mathrm{DT}'_\beta(X) = \mathrm{GW}'_\beta(X)$ *for* $-q = e^{iu}$.

There are two subtleties. (1) The prime in $\mathrm{GW}'_\beta(X)$ means we consider stable maps with disconnected domain curve but no contracted connected components. (2) $\mathrm{DT}'_\beta(X) = \mathrm{DT}_\beta(X)/\mathrm{DT}_0(X)$ where $\mathrm{DT}_0(X)$ is the point part of Donaldson-Thomas theory. For stable pairs, we have the following conjecture [PT1, Conj. 3.2, 3.3].

Conjecture 1.6 (GW/PT correspondence)**.** $\mathrm{PT}(X)$ *is a rational function function of* q *invariant under* $q \leftrightarrow q^{-1}$ *and* $\mathrm{GW}'_\beta(X) = \mathrm{PT}_\beta(X)$ *for* $-q = e^{iu}$.

Together these conjectures imply a DT/PT correspondence: $\mathrm{DT}'_\beta(X) = \mathrm{PT}_\beta(X)$. This DT/PT correspondence has been proved by T. Bridgeland [Bri] (see also Y. Toda [Tod] for an Euler characteristic version). The GW/DT correspondence was proved in the toric case by [MOOP]. Recently, many new cases (including all complete intersections in products of projective spaces) of the GW/PT correspondence have been proved by Pandharipande and A. Pixton [PP1, PP2].

Finally, we mention the Gopakumar-Vafa invariants, which play an important role in the final lecture. For *any* collection of Laurent series in u with rational coefficients and indexed by non-zero effective curve classes (for instance $\{\mathrm{GW}_\beta(X)\}_{\beta\neq 0 \text{ effective}}$), the system of equations

$$\Big\{\mathrm{GW}_\beta(X) = \sum_{g\geq 0} \sum_{\substack{\beta = d\beta' \\ d>0}} \frac{n^{\mathrm{GW}}_{g,\beta'}}{d} \Big(2\sin(du/2)\Big)^{2g-2}\Big\}_{\beta\neq 0 \text{ effective}}$$

has a unique solution $n^{\mathrm{GW}}_{g,\beta'} \in \mathbb{Q}$. This is simply a statement about formal power series. According to R. Gopakumar and C. Vafa [GV1, GV2], the numbers $n^{\mathrm{GW}}_{g,\beta'}$ have an interpretation in M-theory and are *integer*. This leads to:

Conjecture 1.7. *The numbers $n^{\mathrm{GW}}_{g,\beta'}$ are integer and vanish for $g \gg 0$.*

Guided by the variable substitution $-q = e^{iu}$ and the GW/PT correspondence, one can have an analogous discussion on the stable pair side. Since the above is about *connected* Gromov-Witten invariants, we first formally *define* (following the analogous formula for Gromov-Witten invariants)

$$1 + \sum_{\beta \neq 0 \text{ effective}} \mathrm{PT}_\beta(X) v^\beta = \exp\Big(\sum_{\beta \neq 0 \text{ effective}} \mathrm{PT}^{\mathrm{conn}}_\beta(X) v^\beta \Big).$$

At the moment there does not exist a geometric theory of connected stable pair invariants. For *any* collection of Laurent series in q with integer coefficients and indexed by non-zero effective curve classes (for instance $\{\mathrm{PT}_\beta(X)\}_{\beta \neq 0 \text{ effective}}$), the system of equations

$$\Big\{ \mathrm{PT}^{\mathrm{conn}}_\beta(X) = \sum_{g \in \mathbb{Z}} \sum_{\substack{\beta = d\beta' \\ d > 0}} (-1)^{(d+1)(g-1)} \frac{n^{\mathrm{PT}}_{g,\beta'}}{d} \Big(q^{\frac{d}{2}} + (-1)^{d+1} q^{-\frac{d}{2}} \Big)^{2g-2} \Big\}_{\beta \neq 0 \text{ effective}}$$

has a unique solution $n^{\mathrm{PT}}_{g,\beta'} \in \mathbb{Z}$ [PT1, Lem. 3.11, Thm. 3.19]. Moreover $n^{\mathrm{PT}}_{g,\beta'} = 0$ for $g \gg 0$ [PT1, Lem. 3.12]. Pandharipande and Thomas conjecture [PT1, Conj. 3.14]:

Conjecture 1.8. *The numbers $n^{\mathrm{PT}}_{g,\beta'}$ vanish for $g < 0$.*

The GW/PT correspondence for X, β implies to $n^{\mathrm{GW}}_{g,\beta'} = n^{\mathrm{PT}}_{g,\beta'}$ for all effective $0 \neq \beta' \leq \beta$ and $g \in \mathbb{Z}$ (and therefore trivially Conjecture 1.7). Conversely for fixed X, β, the equalities $n^{\mathrm{GW}}_{g,\beta'} = n^{\mathrm{PT}}_{g,\beta'}$ for all effective $0 \neq \beta' \leq \beta$, $g \in \mathbb{Z}$ and with Conjecture 1.8 imply the GW/PT correspondence for X, β.

Exercise 1.9. Prove these two statements.

2 Virtual cycles on Hilbert schemes of surfaces

2.1 Perfect obstruction theories

I give a brief review on perfect obstruction theories. There is some overlap with the lectures of Cristina and Tom, but this does not harm. Numerous parts of this discussion follow the wonderful overview article [PT4].

Let M be a scheme. In general, M can have several (possibly non-reduced) irreducible components of different dimension, so its fundamental class $[M] \in H_*(M)$ is not a very well-behaved object. The fundamental class is especially badly behaved for most moduli spaces M by Murphy's Law of R. Vakil [Vak]. Let us study the toy model $M = s^{-1}(0) \subset A$, where A is a smooth projective ambient variety of dimension n and s is a section of a rank r vector bundle E on A.

I: Transverse case. We start with the best case. Suppose the section s is transverse, i.e. $\mathrm{codim}(M) = r$ and M is smooth. Then

$$\iota_*[M] = c_r(E),$$

where $\iota : M \hookrightarrow A$ is inclusion. Let $I \subset \mathcal{O}_A$ be the ideal sheaf of M, then $I/I^2|_M$ is locally free and

$$N_{M/A} := (I/I^2|_M)^* \cong E|_M.$$

Moreover we have an exact sequence of locally free sheaves

$$0 \longrightarrow T_M \longrightarrow T_A|_M \longrightarrow N_{M/A} \longrightarrow 0.$$

Exercise 2.1. The section s is regular if $\operatorname{codim}(M) = r$ but M is not necessarily smooth. Which parts of the preceding discussion still go through in this context? Prove your answer.

As an application we get a formula for the topological Euler characteristic of M

$$\begin{aligned} e(M) &= \int_M c_{\text{top}}(T_M) = \int_M \left.\frac{c_\bullet(T_A)}{c_\bullet(E)}\right|_M \\ &= \int_A \iota_*[M]\frac{c_\bullet(T_A)}{c_\bullet(E)} = \int_A c_r(E)\frac{c_\bullet(T_A)}{c_\bullet(E)}, \end{aligned} \tag{4}$$

where $c_\bullet(\cdot)$ denotes total Chern class.

II: Split bundle case. A slightly more complicated case is when $E = E' \oplus E''$ is a direct sum of vector bundles and $s = (s', 0)$ with s' a transverse section of E'. Suppose we perturb 0 to a section ϵ of E'' and let $s_\epsilon := (s', \epsilon)$. Morally we think of $s^{-1}(0)$ and $s_\epsilon^{-1}(0)$ as describing the same moduli space. However

$$s_\epsilon^{-1}(0) = \epsilon|_M^{-1}(0) \subset M.$$

Suppose we perturbed such that $\epsilon|_M$ is a transverse section of $E''|_M$. Since $N_{M/A} \cong E'|_M$, we have

$$[s_\epsilon^{-1}(0)] = c_{r''}(E''|_M) = c_{r-r'}(E|_M/E'|_M) = \{c_\bullet(E|_M)s_\bullet(N_{M/A})\}_{n-r} \in H_{2(n-r)}(M),$$

where $s_\bullet(\cdot) := 1/c_\bullet(\cdot)$ is the Segre class.

III: General case. For a general section, the expression

$$\{c_\bullet(E|_M)s_\bullet(C_{M/A})\}_{n-r} \in H_{2(n-r)}(M)$$

still makes sense when $N_{M/A}$ is replaced by the normal cone $C_{M/A}$, which is defined to be the limit of the graph of $t.s$ inside $E|_M$ as $t \to \infty$. If $M \subset A$ is lci, then $C_{M/A} \cong N_{M/A}$. Inside the expression

$$\left\{\frac{c_\bullet(E|_M)}{c_\bullet(T_A|M)}c_\bullet(T_A|_M)s_\bullet(C_{M/A})\right\}_{n-r} \in H_{2(n-r)}(M),$$

the class $c_F(M) := c_\bullet(T_A|_M)s_\bullet(C_{M/A})$ is known as the *Fulton's canonical class* [Ful, Sie]. One can show $c_F(M)$ does not depend on choice of embedding $M \subset A$ into a smooth ambient variety. It should be thought of as a generalization of $c_\bullet(T_M)$, which is not well-behaved when M is not smooth. The above expression is known as the *virtual*

cycle $[M]^{\mathrm{vir}}$, and $\mathrm{vd} := n - r$ is known as the *virtual dimension.* Denoting inclusion by $\iota : M \to A$, it turns out that

$$\iota_*[M]^{\mathrm{vir}} = c_r(E), \tag{5}$$

regardless of s being transverse. The latter formula will come back in the final lecture.

Exercise 2.2. Show that when s is transverse (or just regular) $[M]^{\mathrm{vir}} = [M]$.

The section s provides us with a commutative diagram (D)

$$\begin{tikzcd} E^*|_M \arrow[r, "d\circ s^*"] \arrow[d, "s^*"'] & \Omega_A|_M \arrow[d, equal] \\ I/I^2|_M \arrow[r, "d"] & \Omega_A|_M, \end{tikzcd}$$

where $s : \mathcal{O}_A \to E$ induces $s^* : E^* \to I$ and d is the Kähler differential. The bottom row is the *truncated cotangent complex* of M (concentrated in degrees $-1, 0$)

$$\mathbb{L}_M := \tau_{\geq -1} L_M^\bullet \cong \{I/I^2|_M \to \Omega_A|_M\}.$$

Up to quasi-isomorphism, $\mathbb{L}_M$ is independent of choice of embedding $M \subset A$ into a smooth ambient variety. The top complex $\mathbb{E} := \{E^*|_M \to \Omega_A|_M\}$ is a 2-term complex of locally free sheaves concentrated in degrees $-1, 0$. The diagram provides a morphism $\phi : \mathbb{E} \to \mathbb{L}_M$ such that $h^{-1}(\phi)$ is surjective and $h^0(\phi)$ is an isomorphism. Note that the rank of $\mathbb{E}$ equals the virtual dimension.

It turns out this diagram tells us two things:

- Firstly, the virtual cycle is given by

  $$[M]^{\mathrm{vir}} = \{c_\bullet(\mathbb{E}^\vee) c_F(M)\}_{\mathrm{vd}},$$

  so only depends on the complex $\mathbb{E}$ up to quasi-isomorphism (or even just its K group class).

- Secondly, the complex $\mathbb{E}$ governs the deformation and obstruction theory of closed points in M.

Let me be more precise about the latter. Given a small extension

$$0 \longrightarrow \mathfrak{a} \longrightarrow A' \longrightarrow A \longrightarrow 0$$

and a map Spec $A \to M$, the associated deformation problem is to determine when lifts to Spec A' exist and to characterize them. Let p be the closed point of this map. It is a very general fact about the cotangent complex that

$$\begin{aligned} h^0(L_M^{\bullet\vee}|_p) \otimes I &: \text{deformation space} \\ h^1(L_M^{\bullet\vee}|_p) \otimes I &: \text{obstruction space.} \end{aligned}$$

From the fact that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ surjective, we conclude

$$\begin{aligned} h^0(\mathbb{E}^\vee|_p) \otimes I &: \text{deformation space} \\ h^1(\mathbb{E}^\vee|_p) \otimes I &: \text{obstruction space.} \end{aligned}$$

K. Behrend and B. Fantechi [BF] captured the previous discussion without reference to a smooth ambient space A.

Definition 2.1 (Behrend-Fantechi)**.** Let M be a scheme and $\mathbb{E}$ a 2-term complex of locally free sheaves on M concentrated in degrees $-1, 0$, and $\phi : \mathbb{E} \to \mathbb{L}_M$ a morphism in $D^b(M)$ such that $h^{-1}(\phi)$ is surjective, $h^0(\phi)$ is an isomorphism. Then $\phi : \mathbb{E} \to \mathbb{L}_M$ is called a *perfect obstruction theory* on M. $\oslash$

One can show every perfect obstruction theory is locally of the toy model form studied above. Given such a perfect obstruction theory, one obtains two pieces of information. Firstly, as above, the deformation and obstruction theory of closed points of $p \in M$ is governed by [BF]

$$h^0(\mathbb{E}^\vee|_p) \otimes I : \text{deformation space}$$
$$h^1(\mathbb{E}^\vee|_p) \otimes I : \text{obstruction space.}$$

Secondly, there exists a natural cycle $[M]^{\text{vir}} \in H_{\text{vd}}(M)$ known as the virtual fundamental class or virtual cycle, where $\text{vd} := \text{rk}(\mathbb{E})$ is the virtual dimension [BF]. When M is quasi-projective it is given by[2]

$$[M]^{\text{vir}} = \{c_\bullet(\mathbb{E})c_F(M)\}_{\text{vd}}.$$

In applications one often has two further properties:

- M is a moduli space so the closed points of M are geometric objects and the deformation and obstruction spaces of the points are the deformation and obstruction spaces of the geometric objects. This is obvious in the case of fine moduli spaces.
- Loosely speaking, often the construction of M exists in families over a base: $\mathcal{M} \to B$. Again loosely speaking, this leads to a virtual cycle $[\mathcal{M}]^{\text{vir}}$ such that for all $b \in B$ we have $i_b^![\mathcal{M}]^{\text{vir}} = [M]^{\text{vir}}$.

Exercise 2.3 (i) Suppose $M = s^{-1}(0) \subset A$ is cut out by a section of a vector bundle E on a smooth projective ambient variety A. Define the obstruction sheaf $\text{Ob} := h^1(\mathbb{E}^\vee)$ and assume it is locally free. Use the 5-lemma to show that $h^0(\mathbb{E}^\vee) \cong T_M$ and M is smooth. (ii) Still assuming Ob is locally free, prove that

$$[M]^{\text{vir}} = c_{\text{top}}(\text{Ob}).$$

2.2 Hilbert schemes of curves on surfaces

Let S be a smooth projective surface and $\beta \in H_2(S)$. Recall from the first lecture that $H_\beta := \text{Hilb}_\beta(S)$ denotes the Hilbert scheme of effective divisors on S with class β. Consider the *Abel-Jacobi map*

$$H_\beta \longrightarrow \text{Pic}^\beta(S),\ C \mapsto [\mathcal{O}(C)],$$

where $\text{Pic}^\beta(S)$ is the Picard variety of isomorphism classes of line bundles L on S with $c_1(L) = \beta$ (more precisely: $c_1(L)$ equals the Poincaré dual of β). The latter is a complex torus of dimension $h^{0,1}(S)$. The fibre over $[L] \in \text{Pic}^\beta(S)$ is the linear system $|L|$. Over the locus of line bundles L with $h^1(L) = h^2(L) = 0$, this is a projective bundle of dimension

$$\chi(L) - 1 = \frac{\beta(\beta - \mathsf{k})}{2} - h^{0,1}(S) + h^{0,2}(S),\ \mathsf{k} := c_1(\omega_S).$$

[2]This formula is due to V. Pidstrygach [Pid] and B. Siebert [Sie].

So the expected dimension of H_β is

$$\frac{\beta(\beta-\mathsf{k})}{2}+h^{0,2}(S).$$

In general, H_β is not of expected dimension and can be singular.

Exercise 2.4. Give an example of S, L with $h^{0,1}(S)=h^{0,2}(S)=0$ and $|L|\neq\varnothing$ of unexpected dimension. Hint: use the rational elliptic surface.

We can always embed H_β in a smooth Hilbert scheme as follows. Fix a sufficiently ample divisor $A\subset S$ such that $h^1(L)=h^2(L)=0$ for all $L\in\mathrm{Pic}^\gamma(S)$, where $\gamma:=[A]+\beta$. Then H_γ is a projective bundle over $\mathrm{Pic}^\gamma(S)$. Hence H_γ is smooth. Scheme theoretic union gives a closed embedding

$$H_\beta\hookrightarrow H_\gamma,\ C\mapsto A+C.$$

We identify H_β with its image inside H_γ and write $H_\beta\subset H_\gamma$. We claim H_β is the zero locus of a section of a tautological sheaf on H_γ. An element $D\in H_\gamma$ lies in $H_\beta\subset H_\gamma$ if and only if

$$s_D|_A=0\in H^0(\mathcal{O}_A(D)),$$

where $s_D:\mathcal{O}\to\mathcal{O}(D)$ is the canonical section (which is defined up to non-zero complex scalars). This argument can be repeated "in families". Let $\mathcal{D}\subset H_\gamma\times S$ be the universal curve and let $\pi: H_\gamma\times A\to H_\gamma$ be projection. The sheaf $F:=\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma\times A})$ has a tautological section s coming from $\mathcal{O}\to\mathcal{O}(\mathcal{D})$ and it is not hard to see that

$$H_\beta=s^{-1}(0)\subset H_\gamma.$$

A priori F need not be a vector bundle. We therefore assume (AS1)

$$\text{AS1}: h^2(L)=0 \text{ for all effective } L\in\mathrm{Pic}^\beta(S).$$

For any $C\in H_\beta$, we have short exact sequences

$$\begin{aligned}&0\longrightarrow\mathcal{O}(-A)\longrightarrow\mathcal{O}\longrightarrow\mathcal{O}_A\longrightarrow 0,\\&0\longrightarrow\mathcal{O}(C)\longrightarrow\mathcal{O}(A+C)\longrightarrow\mathcal{O}_A(A+C)\longrightarrow 0.\end{aligned}$$

Ampleness of A and (AS1) imply $h^1(\mathcal{O}_A(A+C))=0$. Semi-continuity and base change imply $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma\times A})=0$ in a Zariski open neighbourhood of $H_\beta\subset H_\gamma$. Note that this argument only shows F is a vector bundle on a Zariski open neighbourhood of H_β in H_γ. Nevertheless, diagram (D) of the previous section still provides a perfect obstruction theory on H_β which we call the *reduced perfect obstruction theory* on H_β and denote by

$$\mathbb{F}^{\mathrm{red}}\to\mathbb{L}_{H_\beta}.$$

At this stage it is not clear how $\mathbb{F}^{\mathrm{red}}$ is dependent on A. Let us consider what $\mathbb{F}^{\mathrm{red}}$ looks like over a point $p=C\in H_\beta$. Consider the short exact sequence defining A inside $A+C$

$$\begin{aligned}&0\longrightarrow I_A\longrightarrow\mathcal{O}_{A+C}\longrightarrow\mathcal{O}_A\longrightarrow 0,\ I_A=\mathcal{O}_C(-A)\\&0\longrightarrow\mathcal{O}_C(C)\longrightarrow\mathcal{O}_{A+C}(A+C)\longrightarrow\mathcal{O}_A(A)\longrightarrow 0.\end{aligned}\tag{6}$$

Exercise 2.5. Let S be a smooth projective surface and $C \in |K_S|$ an effective divisor. (i) Show that the deformation and obstruction spaces of $C \subset S$ (keeping S fixed) are dual to each other. (ii) Does this also hold for $2C \subset S$, where $2C$ is defined by the square of the ideal defining S? Prove your answer. Hint: use (6).

The associated long exact sequence is

$$\begin{aligned} 0 &\longrightarrow H^0(\mathcal{O}_C(C)) \longrightarrow H^0(\mathcal{O}_{A+C}(A+C)) \longrightarrow H^0(\mathcal{O}_A(A+C)) \\ &\longrightarrow \ker\{H^1(\mathcal{O}_C(C)) \to H^1(\mathcal{O}_{A+C}(A+C))\} \longrightarrow 0. \end{aligned} \tag{7}$$

Recall from Exercise 1.6 that

$$\begin{aligned} H^0(\mathcal{O}_C(C)) &\cong \mathrm{Hom}(I_C, \mathcal{O}_C) \cong T_{H_\beta}|_p \\ H^0(\mathcal{O}_{A+C}(A+C)) &\cong \mathrm{Hom}(I_{A+C}, \mathcal{O}_{A+C}) \cong T_{H_\gamma}|_p. \end{aligned}$$

One can show the map $T_{H_\beta}|_p \to T_{H_\gamma}|_p$ obtained from the above exact sequence is the map on tangent spaces induced by inclusion $H_\beta \subset H_\gamma$. Also note that

$$H^0(\mathcal{O}_A(A+C)) \cong F|_p.$$

The second line of the long exact sequence (7) can be rewritten as well. From the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(A+C) \longrightarrow \mathcal{O}_{A+C}(A+C) \longrightarrow 0 \tag{8}$$

and positivity of A we deduce $H^1(\mathcal{O}_{A+C}(A+C)) \cong H^2(\mathcal{O}_S)$. It turns out that the combined map

$$H^1(\mathcal{O}_C(C)) \to H^1(\mathcal{O}_{A+C}(A+C)) \cong H^2(\mathcal{O}_S) \tag{9}$$

is independent of A. More specifically it comes from the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0, \text{ where } \mathcal{O}_C(C) \cong N_{C/S}. \tag{10}$$

as the connecting morphism $\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)$. This map is known as the *semi-regularity map.*

Exercise 2.6. Use the short exact sequences (8) and (10) to prove that (9) is the semi-regularity map ϕ.

In conclusion, we rewrote (7) as

$$0 \longrightarrow T_{H_\beta}|_p \longrightarrow T_{H_\gamma}|_p \longrightarrow F|_p \longrightarrow \ker\{\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)\} \longrightarrow 0.$$

We deduce that the deformation and obstruction spaces for p are independent of A and given by

$$\begin{aligned} h^0(\mathbb{F}^{\mathrm{red}\vee}|_p) &\cong H^0(N_{C/S}), \\ h^1(\mathbb{F}^{\mathrm{red}\vee}|_p) &\cong \ker\{\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)\}. \end{aligned}$$

We indeed get the familiar deformation space for deforming an effective divisor $C \subset S$ keeping S fixed (see previous lecture). However, it might come as a surprise that the obstruction class lies in $\ker \phi$. This is actually well-known (in the much more general context of lci subvarieties of smooth projective varieties) by work of S. Bloch [Blo], Z. Ran [Ran2], D. Iacono and M. Manetti [IM], and others. We come back to this topic in the next lecture. This perfect obstruction theory was first discovered by M. Dürr, A. Kabanov, and C. Okonek [DKO].

Theorem 2.2 (Dürr-Kabanov-Okonek)**.** *For all S, β satisfying (AS1), there exists a reduced perfect obstruction theory $\mathbb{F}^{\mathrm{red}} \to \mathbb{L}_{H_\beta}$ on H_β with virtual cycle $[H_\beta]^{\mathrm{red}} \in H_{2\mathrm{vd}}(H_\beta)$ of virtual dimension*

$$\mathrm{vd} = \frac{\beta(\beta - \mathsf{k})}{2} + h^{0,2}(S).$$

We mention another perfect obstruction theory on H_β, which can be constructed directly without use of an ambient variety and was also first discovered in [DKO]. Let $\mathcal{C} \subset H_\beta \times S$ be the universal divisor and let $\pi : H_\beta \times S \to H_\beta$ be projection.

Theorem 2.3 (Dürr-Kabanov-Okonek)**.** *There exists a perfect obstruction theory $\mathbb{F} := (R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C}))^\vee \to \mathbb{L}_{H_\beta}$ on H_β with virtual cycle $[H_\beta]^{\mathrm{vir}} \in H_{2\mathrm{vd}}(H_\beta)$ of virtual dimension*

$$\mathrm{vd} = \frac{\beta(\beta - \mathsf{k})}{2}.$$

One can show that these two perfect obstruction theories, when defined, are compatible with the semi-regularity map.

Proposition 2.4. *The semi-regularity map induces an exact triangle*

$$\begin{tikzcd} H^2(\mathcal{O}_S)^* \otimes \mathcal{O}_{H_\beta}[1] \arrow[r] & \mathbb{F} \arrow[r] \arrow[dr] & \mathbb{F}^{\mathrm{red}} \arrow[d] \\ & & \mathbb{L}_{H_\beta} \end{tikzcd}$$

and the diagram commutes.

2.3 Poincaré invariants

We now briefly review the Poincaré invariants introduced by Dürr, Kabanov, and Okonek [DKO]. Let $\mathcal{C} \to H_\beta$ be the universal curve as before (first lecture). Consider the two Abel-Jacobi maps

$$\mathrm{AJ}^+ : H_\beta \to \mathrm{Pic}^\beta(S),\ C \mapsto [\mathcal{O}(C)]$$
$$\mathrm{AJ}^- : H_{\mathsf{k}-\beta} \to \mathrm{Pic}^\beta(S),\ C \mapsto [\mathcal{O}(K_S - C)].$$

Define invariants

$$P_\beta^+(S) := \mathrm{AJ}_*^+\Big(\sum_i c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{pt\}})^i \cap [H_\beta]^{\mathrm{vir}}\Big),$$
$$P_\beta^-(S) := (-1)^{\chi(\beta)} \mathrm{AJ}_*^-\Big(\sum_i (-1)^i c_1(\mathcal{O}(\mathcal{C})|_{H_{\mathsf{k}-\beta} \times \{pt\}})^i \cap [H_{\mathsf{k}-\beta}]^{\mathrm{vir}}\Big),$$
$$\chi(\beta) := \frac{\beta(\beta - \mathsf{k})}{2} + \chi(\mathcal{O}_S).$$

In the first line, $\mathcal{C}$ denotes the universal divisor over H_β and in the second line, the universal divisor over $H_{\mathsf{k}-\beta}$. These are known as *Poincaré invariants* of S, β. Note that

$$P_\beta^\pm(S) \in H_*(\mathrm{Pic}^\beta(S)) \cong \Lambda^* H^1(S, \mathbb{Z}).$$

Here $*$ means we sum over all degrees. The isomorphism arises as follows: $H_*(\mathrm{Pic}^\beta(S)) \cong H^*(\mathrm{Pic}^\beta(S), \mathbb{Z})^*$ by Poincaré duality and

$$H^*(\mathrm{Pic}^\beta(S), \mathbb{Z}) \cong \Lambda^* H^1(S, \mathbb{Z})^*,$$

since $\mathrm{Pic}^\beta(S) \cong H^1(S, \mathbb{R})/H^1(S, \mathbb{Z})$ as a real torus. The Poincaré invariants satisfy a blow-up and wall-crossing formula.

Theorem 2.5 (Blow-up formula, Dürr-Kabanov-Okonek)**.** *Let $\pi : \tilde{S} \to S$ be the blow-up of S in one point and let e denote the exceptional divisor. Using the identification $\pi^* : H^1(S, \mathbb{Z}) \xrightarrow{\cong} H^1(\tilde{S}, \mathbb{Z})$, we have*

$$P^\pm_{\pi^*\beta + le}(\tilde{S}) = \tau_{\leq \beta(\beta-\mathsf{k}) - l(l-1)} P_\beta^\pm(S),$$

for any $\beta \in H_2(S)$ and $l \in \mathbb{Z}$. Here $\tau_{\leq \beta(\beta-\mathsf{k}) - l(l-1)}$ denotes truncation, keeping only cycles of real dimension $\leq \beta(\beta - \mathsf{k}) - l(l-1)$.

Before formulating the wall-crossing formula, we introduce some notation. For any class $\alpha \in H^2(S, \mathbb{Z})$, define

$$[\alpha] := \int_S \alpha \wedge \cdot \in \Lambda^2 H^1(S, \mathbb{Z})^*.$$

Theorem 2.6 (Wall-crossing formula, Dürr-Kabanov-Okonek)**.** *Fix any S, β such that $h^{0,2}(S) = 0$. Then*

$$P_\beta^+(S) - P_\beta^-(S) = \sum_{i=0}^{\min\left\{h^{0,1}(S), \frac{\beta(\beta-\mathsf{k})}{2}\right\}} \frac{\left(\frac{[2\beta-\mathsf{k}]}{2}\right)^{h^{0,1}(S)-i}}{(h^{0,1}(S)-i)!} \cap [\mathrm{Pic}^\beta(S)].$$

Exercise 2.7. Suppose S, β satisfy $h^{0,1}(S) = h^{0,2}(S) = 0$ and $\beta(\beta - \mathsf{k}) \geq 0$. Use the wall-crossing formula to show that $(P_\beta^+(S), P_\beta^-(S)) = (1, 0)$ if $H_\beta \neq \varnothing$ and $(P_\beta^+(S), P_\beta^-(S)) = (0, -1)$ if $H_\beta = \varnothing$. This [DKO, Prop. 4.1].

The formula of Theorem 2.6 is reminiscent of Poincaré's formula for the cycle class of the Brill-Noether locus (this is the reason for the name Poincaré invariants). Let C be a smooth projective curve of genus g, and consider the symmetric product (Hilbert scheme) $C^{[d]}$ and the Abel-Jacobi map

$$\mathrm{AJ} : C^{[d]} \longrightarrow \mathrm{Pic}^d(C), \;\; D \mapsto [\mathcal{O}(D)].$$

The *Brill-Noether locus* is the image $\mathrm{AJ}(C^{[d]})$. For $d \geq g$, the Abel-Jacobi map is surjective.

Exercise 2.8. Why?

Define

$$\theta := \int_C \cdot \in \Lambda^2 H^1(C, \mathbb{Z})^*.$$

Then for $d \leq g$ *Poincaré's formula* computes the class of the Brill-Noether locus

$$[\mathrm{AJ}(C^{[d]})] = \frac{\theta^{g-d}}{(g-d)!} \cap [\mathrm{Pic}^d(C)].$$

The blow-up formula follows from a fairly straight-forward comparison of virtual cycles. The wall-crossing formula is much harder. One of the key ingredients of the proof is the surprising fact that the sheaf $\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ turns out to be a vector bundle on the whole of H_γ *even though* $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is in general non-zero.

Dürr, Kabanov, and Okonek designed these invariants to agree with the Seiberg-Witten invariants of S, β from symplectic geometry. I will not say much about Seiberg-Witten invariants, because I know almost nothing about them. For compact symplectic 4-manifolds with $b_+ > 0$, one can define Seiberg-Witten invariants (originating from the work of N. Seiberg and E. Witten [Wit])

$$\mathrm{SW}_\bullet(S) : H^2(S, \mathbb{Z}) \longrightarrow \Lambda^* H^1(S, \mathbb{Z}).$$

In the case $b_+ = 0$, the invariants depend on a chamber structure and are maps

$$\mathrm{SW}^\pm_\bullet(S) : H^2(S, \mathbb{Z}) \longrightarrow \Lambda^* H^1(S, \mathbb{Z}).$$

Conjecture 2.7 (Poincaré/SW correspondence, Dürr-Kabanov-Okonek)**.** *For any smooth projective surface S and $\beta \in H_2(S)$*

$$P^+_\beta(S) = P^-_\beta(S) = \mathrm{SW}_\beta(S), \text{ if } h^{0,2}(S) > 0$$
$$P^\pm_\beta(S) = \mathrm{SW}^\pm_\beta(S), \text{ if } h^{0,2}(S) = 0.$$

Using the blow-up and wall-crossing formula, Dürr, Kabanov, and Okonek reduce the conjecture to a conjecture about the degree of $[H_\mathsf{k}]^{\mathrm{vir}}$. The latter was proved by H.-l. Chang and Y.-H. Kiem [CK] using a beautiful application of cosection localization.

Theorem 2.8 (Chang-Kiem)**.** *Conjecture 2.7 is true.*

2.4 Stable pairs on surfaces

Recall that the moduli space of stable pairs on S is isomorphic to the relative Hilbert scheme (Theorem 1.1)

$$P_\chi(S, \beta) \cong \mathrm{Hilb}^n(\mathcal{C}/H_\beta),$$

where $\chi = 1 - h + n$ and h is the arithmetic genus of curves in class β. Recall that $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ is naturally an incidence scheme inside $S^{[n]} \times H_\beta$

$$\iota : \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta.$$

Again, we can realize $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ as the zero locus of a section of a vector bundle on $S^{[n]} \times H_\beta$. This works as follows. For any pair (Z, C), we have $Z \subset C$ if and only if

$$s_C|_Z = 0 \in H^0(\mathcal{O}_Z(C)),$$

where $s_C : \mathcal{O} \to \mathcal{O}(C)$ is the canonical section. The family version goes as follows. Let $\mathcal{Z} \subset S^{[n]} \times S$ be the universal subscheme. Consider

$$\mathcal{O}_{\mathcal{Z} \times H_\beta}(S^{[n]} \times \mathcal{C}) \text{ on } S^{[n]} \times S \times H_\beta.$$

We denote this sheaf somewhat sloppily by $\mathcal{O}_{\mathcal{Z}}(\mathcal{C})$. Pushing forward along projection $S^{[n]} \times S \times H_\beta \to S^{[n]} \times H_\beta$ gives a rank n vector bundle $\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})$ with fibre $H^0(\mathcal{O}_Z(C))$ over (Z, C).

Exercise 2.9. Why is $\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})$ a vector bundle?

It is not hard to construct a tautological section of $\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})$ cutting out $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$.

However, this does *not* automatically give a perfect obstruction theory on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$. The Hilbert scheme of points $S^{[n]}$ is smooth of expected dimension, but this does not need to be the case for H_β! Therefore, this construction only gives a *relative* perfect obstruction theory on $P_S := P_\chi(S, \beta)$ over H_β

$$\mathcal{E}^\bullet := \{(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C}))^*|_{P_S} \to \pi^*_{S^{[n]}}(\Omega_{S^{[n]}})|_{P_S}\} \to \mathbb{L}_{P_S/H_\beta},$$

where $\pi_{S^{[n]}} : S^{[n]} \times H_\beta \to S^{[n]}$ is projection. All these perfect obstruction theories can be put together:

Theorem 2.9 (K-Panov-Thomas)**.** *Consider $\pi : P_S \to H_\beta$. There exist natural commutative diagrams*

$$\begin{tikzcd} \mathcal{E}^\bullet[-1] \arrow[r] \arrow[d] & \pi^*\mathbb{F} \arrow[d] \\ \mathbb{L}_{P_S/H_\beta}[-1] \arrow[r] & \pi^*\mathbb{L}_{H_\beta} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}^\bullet[-1] \arrow[r] \arrow[d] & \pi^*\mathbb{F}^{\mathrm{red}} \arrow[d] \\ \mathbb{L}_{P_S/H_\beta}[-1] \arrow[r] & \pi^*\mathbb{L}_{H_\beta}. \end{tikzcd}$$

Consequently, the mapping cones $\mathbb{E}$, $\mathbb{E}^{\mathrm{red}}$ define perfect obstrucion theories on P_S

$$\begin{tikzcd} \pi^*\mathbb{F} \arrow[r, dashed] \arrow[d] & \mathbb{E} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{E}^\bullet \arrow[d] \\ \pi^*\mathbb{L}_{H_\beta} \arrow[r] & \mathbb{L}_{P_S} \arrow[r] & \mathbb{L}_{P_S/H_\beta} \end{tikzcd} \qquad \begin{tikzcd} \pi^*\mathbb{F}^{\mathrm{red}} \arrow[r, dashed] \arrow[d] & \mathbb{E}^{\mathrm{red}} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{E}^\bullet \arrow[d] \\ \pi^*\mathbb{L}_{H_\beta} \arrow[r] & \mathbb{L}_{P_S} \arrow[r] & \mathbb{L}_{P_S/H_\beta} \end{tikzcd}$$

of virtual dimensions $\frac{\beta(\beta-\mathsf{k})}{2} + n$ and $\frac{\beta(\beta-\mathsf{k})}{2} + h^{0,2}(S) + n$. We denote the corresponding virtual cycles by $[P_S]^{\mathrm{vir}}$, $[P_S]^{\mathrm{red}}$.

Finally, denoting the universal stable pair on P_S by $\mathbb{I}^\bullet_S := \{\mathcal{O} \to \mathbb{F}\}$ and projection by $\pi : P_S \times S \to P_S$, then there exists an isomorphisms

$$\mathbb{E}^\vee \cong R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet_S, \mathbb{F}).$$

The last isomorphism of the theorem states that $\mathbb{E}^\vee$ is the natural deformation obstruction complex of stable pairs on S (i.e. the one coming from Le Potier's moduli functor). Recall that we saw in the first lecture that this complex can be related to stable pair theory on the 3-fold $X = \mathrm{Tot}(K_S)$ (the total space of the canonical bundle over S). Denoting the universal object on $P_X := P_\chi(X, \beta)$ by $\mathbb{I}_X^\bullet$, projection by $\pi : P_X \times X \to X$, and recalling that P_S is a component of $P_X^{\mathbb{C}^*}$, we have

$$\mathbb{E}^\vee \overset{\text{Thm. 2.9}}{\cong} R\pi_* R\mathscr{H}om(\mathbb{I}_S^\bullet, \mathbb{F}) \overset{\text{Prop. 1.3}}{\cong} R\pi_* R\mathscr{H}om(\mathbb{I}_X^\bullet, \mathbb{I}_X^\bullet)_0|_{P_S}^{\mathbb{C}^*}.$$

Consequences of this isomorphism will be explored in the final lecture.

One can show that Theorem 2.9 gives a formula for the (non-reduced) virtual cycle [Koo, Prop. 2.2]

$$\iota_*[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{vir}} = (S^{[n]} \times [H_\beta]^{\mathrm{vir}}) \cap c_n(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})) \in H_*(S^{[n]} \times H_\beta). \tag{11}$$

This formula would be a direct consequence of (5) if H_β were smooth of expected dimension. This formula will be used to find a connection with [DKO]'s Poincaré (and hence Seiberg-Witten) invariants in the final lecture.

3 Applications to Severi degrees and Seiberg-Witten invariants

3.1 Reduced theories

Let S be a smooth projective surface, $\beta \in H_2(S)$ an effective curve class and $X = \mathrm{Tot}(K_S)$ the total space of the canonical bundle over S. We want to study Gromov-Witten and stable pair theory of X. We are immediately faced with two problems. The first one is that X is non-compact, therefore the moduli spaces are in general not compact, so we have no virtual cycles. This will be remedied by using Graber-Pandharipande localization in the next section.

There is a second problem. Suppose we have a smooth deformation $(\mathcal{S} \to B, 0)$ of S over a smooth simply connected base B with central fibre $\mathcal{S}_0 \cong S$. The induced family of canonical bundles $(\mathcal{X} \to B, 0)$ has central fibre $\mathcal{X}_0 \cong X$. The Poincaré dual of β lies in $H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ and can be transported to any fibre $\mathcal{S}_t$ to give a class $\beta_t \in H^2(\mathcal{S}_t, \mathbb{Z})$.

Exercise 3.1. Give a precise definition of β_b using Ehresmann's theorem [Voi1, Thm. 9.3].

However, there is no reason for β_t to still be of Hodge type $(1,1)$. The *Noether-Lefschetz locus* of β is defined as

$$\mathrm{NL}_\beta := \{t \in B \mid \beta_t \text{ is of type } (1,1)\} \subset B.$$

The locus NL_β is a complex analytic closed subset. Whenever $\mathrm{NL}_\beta \subset B$ is positive codimensional, we can deform S to a surface where β is not of type $(1,1)$ anymore. If this is the case, then by deformation invariance all Gromov-Witten and stable pair invariants of X and S with class β are zero!

What is the expected codimension of NL_β? Let T_S and Ω_S be the tangent and cotangent bundle of S. There exists a natural map

$$H^1(T_S) \longrightarrow H^2(T_S \otimes \Omega_S) \longrightarrow H^2(\mathcal{O}_S)$$

induced by

$$\begin{aligned} \cup : H^{0,1}(T_S) \otimes H^{1,1}(S) &\longrightarrow H^{1,2}(T_S), \\ T_S \otimes \Omega_S &\longrightarrow \mathcal{O}_S. \end{aligned}$$

We call this composition somewhat sloppily $\cup\beta$. For S as a complex manifold, one can show that $H^1(T_S)$ naturally parametrizes all first order deformations (i.e. deformations over $\mathbb{C}[\epsilon]/(\epsilon^2)$ [Voi1, Sec. 9.1.2]).

Each tangent vector of the base B gives a direction in which we can deform the central fibre S. More precisely, there exists a linear map

$$\mathrm{KS}_0 : T_B|_0 \longrightarrow H^1(T_S)$$

known as the *Kodaira-Spencer map.*

Exercise 3.2. Consider the short exact sequence $0 \to T_S \to T_{\mathcal{S}}|_S \to N_{S/\mathcal{S}} \to 0$, where $N_{S/\mathcal{S}}$ is the normal bundle to S in $\mathcal{S}$. By flatness of $\mathcal{S} \to B$, $N_{S/\mathcal{S}}$ is the pull-back of $N_{0/B} \cong T_B|_0$ to S. Construct KS_0 from these facts.

Consider the composition

$$T_B|_0 \xrightarrow{\mathrm{KS}_0} H^1(T_S) \xrightarrow{\cup\beta} H^2(\mathcal{O}_S).$$

It is a fact from Hodge theory [Voi2] that the kernel of this composition is the Zariski tangent space to the Noether-Lefschetz locus at 0, i.e. $T_{\mathrm{NL}_\beta}|_0$. So the *expected* codimension of $\mathrm{NL}_\beta \subset B$ is $h^{0,2}(S)$. Therefore, when $h^{0,2}(S) > 0$ ordinary Gromov-Witten and stable pair invariants of X and S should be expected to be zero (unless there is reason not to such as in Exercise 3.3 below).

In order to get interesting Gromov-Witten and stable pair invariants, one should remove part of the obstruction bundle corresponding to deformations of S outside the Noether-Lefschetz locus. This problem has been studied by many people. I only mention a few key references. For Gromov-Witten theory on Kähler surfaces (differential geometry setting): [Don, Lee]. For Gromov-Witten theory on abelian and K3 surfaces: [BL1, BL2, Li]. For stable pair theory on canonical bundles over K3 surfaces: [MPT]. I will now present a quite general approach to this problem (joint work with R. P. Thomas [KT1]). I will discuss the case of Gromov-Witten theory of a surface S. The same method works for Gromov-Witten and stable pair theory of $X = \mathrm{Tot}(K_S)$. I need the following assumption (AS2).

$$\mathrm{AS2} : H^1(T_S) \xrightarrow{\cup\beta} H^2(\mathcal{O}_S) \text{ is surjective.}$$

One can show that for any effective divisor $C \subset S$, the map $\cup\beta$ factors through the semi-regularity map

$$H^1(T_S) \longrightarrow H^1(T_S|_C) \longrightarrow H^1(N_{C/S}) \longrightarrow H^2(\mathcal{O}_S).$$

Therefore (AS2) implies our earlier (AS1) (Section 2.2).

Exercise 3.3. Suppose $\beta = \mathsf{k}$ (recall $\mathsf{k} := c_1(\omega_S)$) and $|K_S| \neq \varnothing$. Show that in this case (AS1) and hence (AS2) is not satisfied. In this case, the ordinary Gromov-Witten/stable pair theory of X and S are non-trivial.

The deformation $(\mathcal{S} \to B, 0)$ is called *versal* if KS_0 is surjective and any other smooth deformation $(\mathcal{S}' \to B', 0)$ over a smooth base is the pull-back from $(\mathcal{S} \to B, 0)$ along a pointed morphism $f : (B', 0) \to (B, 0)$. Although we only need surjectivity of KS_0, we simply assume our deformation $(\mathcal{S} \to B, 0)$ is versal. (Technical note: a versal family over a smooth base need not exist. However, a versal family for first order deformations does always exist and this turns out to be enough for the constructions of this section [KT1]. So all we need for this section is (AS2).)

Our assumptions imply

$$T_B|_0 \xrightarrow{\mathrm{KS}_0} H^1(T_S) \xrightarrow{\cup\beta} H^2(\mathcal{O}_S)$$

is surjective, so $T_{\mathrm{NL}_\beta}|_0 \subset T_B|_0$ has codimension $h^{0,2}(S)$. In fact, one can show this also implies $\mathrm{NL}_\beta \subset B$ is smooth of codimension $h^{0,2}(S)$ at 0 [Voi2]. Therefore, we can take a small polydisc $\Delta \subset B$ of dimension $h^{0,2}(S)$ intersecting NL_β transversally at 0. Let $(\mathcal{T} \to \Delta, 0)$ be the induced family.

Let $\overline{M}_{g,m}(\mathcal{T}/\Delta, \beta)$ be the moduli stack of stable maps mapping to the fibres of $\mathcal{T} \to \Delta$. One can show that the natural inclusion

$$\overline{M}_{g,m}(S, \beta) \hookrightarrow \overline{M}_{g,m}(\mathcal{T}/\Delta, \beta)$$

is an isomorphism of stacks. This is not so surprising: at the level of $\mathbb{C}$-points this map is a bijection since the central fibre of $\mathcal{T} \to \Delta$ is (by construction) the only fibre where β is of type $(1, 1)$. As is the case for any smooth family, $\mathcal{T} \to \Delta$ has a relative Gromov-Witten theory

$$\mathbb{E}^{\mathrm{rel}}_{\mathrm{GW}} \longrightarrow L^\bullet_{\overline{M}_{g,m}(\mathcal{T}/\Delta,\beta)/\Delta}.$$

There is a standard procedure for turning a relative perfect obstruction theory over a smooth base B into an absolute one. Consider the diagram

$$\begin{tikzcd}
\mathbb{E}^{\mathrm{red}}_{\mathrm{GW},S} \arrow[r, dashed] \arrow[d, dashed] & \mathbb{E}^{\mathrm{rel}}_{\mathrm{GW}} \arrow[r] \arrow[d] & \Omega_\Delta[1] \arrow[d, equal] \\
L^\bullet_{\overline{M}_{g,m}(\mathcal{T}/\Delta,\beta)} \arrow[r] \arrow[d, "\cong"] & L^\bullet_{\overline{M}_{g,m}(\mathcal{T}/\Delta,\beta)/\Delta} \arrow[r] & \Omega_\Delta[1] \\
L^\bullet_{\overline{M}_{g,m}(S,\beta)}, & &
\end{tikzcd}$$

where the middle row is an exact triangle and $\mathbb{E}^{\mathrm{red}}_{\mathrm{GW},S} \to L^\bullet_{\overline{M}_{g,m}(S,\beta)}$ is induced from the diagram. This is a perfect obstruction theory which we refer to as *reduced Gromov-Witten theory*.

Exercise 3.4. Use the above diagram to show that $\psi : \mathbb{E}^{\mathrm{red}}_{\mathrm{GW},S} \to L^{\bullet}_{\overline{M}_{g,m}(S,\beta)}$ indeed satisfies $h^0(\psi)$ is an isomorphism and $h^{-1}(\psi)$ is surjective. (To show $\mathbb{E}^{\mathrm{red}}_{\mathrm{GW},S}$ is isomorphic to a 2-term complex of locally free sheaves is more complicated.)

Why is this the right theory? It turns out the corresponding invariants are only invariant under deformations of S *inside* the Noether-Lefschetz locus of β. Moreover, over an embedded curve $C \subset S$ corresponding to a point $p \in \overline{M}_{g,m}(S,\beta)$, the deformation and obstruction space are

$$\begin{aligned} h^0(\mathbb{E}^{\mathrm{red}\vee}_{\mathrm{GW},S}|_p) &\cong H^0(N_{C/S}), \\ h^1(\mathbb{E}^{\mathrm{red}\vee}_{\mathrm{GW},S}|_p) &\cong \ker\{\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)\}, \end{aligned}$$

where $\phi : H^1(N_{C/S}) \to H^2(\mathcal{O}_S)$ is the semi-regularity map we encountered in the previous lecture. These are the same deformation and obstruction space as for the perfect obstruction theory $\mathbb{F}^{\mathrm{red}}$ on H_β in the previous lecture. For ordinary non-reduced Gromov-Witten theory $\mathbb{E}_{\mathrm{GW},S}$ of S one has the following deformation and obstruction space

$$\begin{aligned} h^0(\mathbb{E}^{\vee}_{\mathrm{GW}}|_p) &\cong H^0(N_{C/S}), \\ h^1(\mathbb{E}^{\vee}_{\mathrm{GW}}|_p) &\cong H^1(N_{C/S}). \end{aligned}$$

Theorem 3.1 (K-Thomas)**.** *Let S, β satisfy (AS2) and let $X = \mathrm{Tot}(K_S)$. Then there exist natural reduced obstruction theories*

$$\begin{aligned} \mathbb{E}^{\mathrm{red}}_{\mathrm{GW},S} &\longrightarrow \mathbb{L}_{\overline{M}_{g,m}(S,\beta)}, \\ \mathbb{E}^{\mathrm{red}}_{\mathrm{GW},X} &\longrightarrow \mathbb{L}_{\overline{M}_{g,m}(X,\beta)}, \\ \mathbb{E}^{\mathrm{red}}_{\mathrm{PT},X} &\longrightarrow \mathbb{L}_{P_\chi(X,\beta)}, \end{aligned}$$

and their virtual dimensions are $h^{0,2}(S)$ bigger than the virtual dimensions of their non-reduced counterparts.

Exercise 3.5. Explain why the virtual dimension of the reduced theories is $h^{0,2}(S)$ bigger than the virtual dimensions of their non-reduced counterparts.

3.2 Localizing

The moduli spaces $\overline{M}_{g,m}(X,\beta)$, $P_\chi(X,\beta)$ are in general non-compact. For example when S is a K3 surface and β is irreducible, $X = S \times \mathbb{C}$ and

$$\overline{M}_{g,m}(X,\beta) \cong \overline{M}_{g,m}(S,\beta) \times \mathbb{C}.$$

Therefore, we do not have virtual cycles at our disposal. However, the natural action of $\mathbb{C}^*$ on the fibres of X lifts to $\overline{M}_{g,m}(X,\beta)$, $P_\chi(X,\beta)$ and the fixed loci are compact. In fact

$$\begin{aligned} \overline{M}_{g,m}(X,\beta)^{\mathbb{C}^*} &\cong \overline{M}_{g,m}(S,\beta), \\ P_\chi(X,\beta)^{\mathbb{C}^*} &\cong P_\chi(S,\beta) \sqcup \text{other connected components.} \end{aligned}$$

Here "other components" contain stable pairs thickened in the fibre direction as briefly discussed in the first lecture.

Let me quickly repeat the setup of virtual $\mathbb{C}^*$ localization discussed by Tom. Let M be a proper scheme with $\mathbb{C}^*$-action. Assume M is $\mathbb{C}^*$ equivariantly embeddable into a smooth variety. Let $\mathbb{E} \to \mathbb{L}_M$ be a $\mathbb{C}^*$ equivariant perfect obstruction theory (i.e. $\mathbb{E}$ and the map are $\mathbb{C}^*$ equivariant). T. Graber and Pandharipande [GP] show that on each connected component $M_i^{\mathbb{C}^*} \subset M^{\mathbb{C}^*}$ of the fixed point locus, the fixed part $(\mathbb{E}|_{M_i^{\mathbb{C}^*}})^{\mathbb{C}^*}$ of the complex $\mathbb{E}|_{M_i^{\mathbb{C}^*}}$ induces a perfect obstruction theory

$$(\mathbb{E}|_{M_i^{\mathbb{C}^*}})^{\mathbb{C}^*} \to \mathbb{L}_{M_i^{\mathbb{C}^*}}.$$

It is important to stress that the virtual dimension $\mathbb{E}$ and $(\mathbb{E}|_{M_i^{\mathbb{C}^*}})^{\mathbb{C}^*}$ are in general different. In particular the former can be of virtual dimension zero while the latter need not be. Since we work $\mathbb{C}^*$ equivariantly,

$$[M]^{\mathrm{vir}} \in H_*^{\mathbb{C}^*}(M),$$

where $H_*^{\mathbb{C}^*}(M)$ is $\mathbb{C}^*$ equivariant homology. Let $s := c_1^{\mathbb{C}^*}(\mathcal{O} \otimes \mathfrak{t})$, where $\mathfrak{t}$ is a primitive character of $\mathbb{C}^*$, then we have seen in Tom's lecture that

$$H_*^{\mathbb{C}^*}(M) \otimes_{\mathbb{Q}[s]} \mathbb{Q}[s, s^{-1}] \cong H(M^{\mathbb{C}^*}) \otimes_{\mathbb{Q}} \mathbb{Q}[s, s^{-1}].$$

The *Graber-Pandharipande localization formula* [GP] (discussed in Tom's lectures) is the analog for virtual cycles of the Atiyah-Bott localization formula

$$\int_{[M]^{\mathrm{vir}}} \alpha = \sum_i \int_{[M_i^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N_i^{\mathrm{vir}})}\ \alpha|_{M_i^{\mathbb{C}^*}},$$

where $N_i^{\mathrm{vir}} := (\mathbb{E}^\vee|_{M_i^{\mathbb{C}^*}})^m$ is the moving part of the complex and $e(\cdot) := c_{\mathrm{top}}^{\mathbb{C}^*}(\cdot)$ is the $\mathbb{C}^*$ equivariant top Chern class.

Back to the moduli spaces $\overline{M}_{g,m}(X, \beta)$, $P_\chi(X, \beta)$. Let $\mathbb{E}_{\mathrm{GW},X}$ be Gromov-Witten theory of X. Then

$$(\mathbb{E}_{\mathrm{GW},X})^{\mathbb{C}^*}$$

provides a perfect obstruction theory on the fixed locus $\overline{M}_{g,m}(X, \beta)^{\mathbb{C}^*} \cong \overline{M}_{g,m}(S, \beta)$. It is well-known that this is isomorphic to the Gromov-Witten theory $\mathbb{E}_{\mathrm{GW},S}$ of S (e.g. see [KT1]). Next let $\mathbb{E}_{\mathrm{PT},X}$ be stable pair theory of X. Then $(\mathbb{E}_{\mathrm{PT},X})^{\mathbb{C}^*}$ provides a perfect obstruction theory on $P_\chi(X, \beta)^{\mathbb{C}^*}$ and therefore also on $P_\chi(S, \beta)$. Moreover, we saw that on $P_\chi(S, \beta) \cong \mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ this is just (essentially by Proposition 1.3)

$$(\mathbb{E}_{\mathrm{PT},X}^\vee)^{\mathbb{C}^*} \cong R\pi_* R\mathscr{H}om(\mathbb{I}_X^\bullet, \mathbb{I}_X^\bullet)_0^{\mathbb{C}^*} \cong R\pi_* R\mathscr{H}om(\mathbb{I}_S^\bullet, \mathbb{F}).$$

In addition, we also saw in Theorem 2.9 that

$$R\pi_* R\mathscr{H}om(\mathbb{I}_S^\bullet, \mathbb{F}) \cong \mathbb{E},$$

where $\mathbb{E}$ is the perfect obstruction theory on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ coming from embedding it inside $S^{[n]} \times H_\beta$. So the perfect obstruction theories we constructed directly on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ *are* localized stable pair theories on X!

The same turns out to be true for the reduced theories: Suppose (AS2) is satisfied. Then $(\mathbb{E}^{\mathrm{red}}_{\mathrm{GW},X})^{\mathbb{C}^*}$ is isomorphic to the perfect obstruction theory $(\mathbb{E}^{\mathrm{red}}_{\mathrm{GW},S})^{\mathbb{C}^*}$. Similarly on $P_\chi(S,\beta)$, $(\mathbb{E}^{\mathrm{red}}_{\mathrm{PT},X})^{\mathbb{C}^*}$ is isomorphic to the perfect obstruction theory $\mathbb{E}^{\mathrm{red}}$ constructed directly on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ in Theorem 2.9.

Since the moduli spaces we consider can be non-compact, we *define* the various reduced/non-reduced Gromov-Witten and stable pair invariants via the Graber-Pandharipande localization formula. For any $\sigma_1,\ldots,\sigma_m \in H^*(S,\mathbb{Z})$

$$\begin{aligned}
\mathrm{GW}^{\mathrm{red}}_{g,\beta}(X,\sigma_1\cdots\sigma_m) &:= \int_{[\overline{M}_{g,m}(S,\beta)]^{\mathrm{red}}} \frac{1}{e(N^{\mathrm{vir}})}\prod_{i=1}^m \mathrm{ev}_i^*\sigma_i,\\
\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(X,\sigma_1\cdots\sigma_m) &:= \int_{[P_\chi(X,\beta)^{\mathbb{C}^*}]^{\mathrm{red}}} \frac{1}{e(N^{\mathrm{vir}})}\prod_{i=1}^m \tau_0(\sigma_i),\\
\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S,\sigma_1\cdots\sigma_m) &:= \int_{[P_\chi(S,\beta)]^{\mathrm{red}}} \frac{1}{e(N^{\mathrm{vir}})}\prod_{i=1}^m \tau_0(\sigma_i),
\end{aligned}$$

and similarly for non-reduced invariants using non-reduced cycles. Let me explain some of the notation in this definition. Here

$$\mathrm{ev}_i : \overline{M}_{g,m}(X,\beta) \longrightarrow X$$

is evaluation at the ith marked point. Moreover $\tau_0(\cdot)$ are *primary descendent insertions* which are defined as follows. Let

$$\begin{aligned}
\pi_X &: P_\chi(X,\beta)\times X \longrightarrow X\\
\pi_P &: P_\chi(X,\beta)\times X \longrightarrow P_\chi(X,\beta)
\end{aligned}$$

be projections and $\mathbb{I}^\bullet_X = \{\mathcal{O}\to\mathbb{F}\}$ the universal stable pair on $P_\chi(X,\beta)\times X$. Then

$$\tau_0(\cdot) := \pi_{P_X *}(\pi_X^*(\cdot)\cap \mathrm{ch}_2(\mathbb{F})) \in H^*(P_\chi(X,\beta),\mathbb{Q}).$$

The universal sheaf $\mathbb{F}$ has $\mathrm{ch}_2(\mathbb{F})$ as its lowest non-zero Chern character. Higher descendents are defined using higher Chern characters of $\mathbb{F}$ [PP2]. Higher descendents in stable pair theory correspond to ψ-classes in Gromov-Witten theory.

3.3 Universality

In the previous section we defined reduced surface stable pair invariants $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S,\sigma_1\cdots\sigma_m)$ whenever (AS2) is satisfied. However, the perfect obstruction theory $\mathbb{E}^{\mathrm{red}}$ of the previous lecture (and therefore the invariants $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S,\sigma_1\cdots\sigma_m)$) already exists if the weaker assumption (AS1) is satisfied. For the purposes of this lecture, we are interested in two types of insertions. Let $\gamma_1,\ldots,\gamma_{b_1}$ be an integral oriented basis of $H_1(S)/\mathrm{torsion}$ and consider the point class $[pt]\in H_0(S)$. Here $b_1 := b_1(S)$ is the first Betti number of S. Consider invariants

$$\begin{aligned}
&\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S,\gamma_1\cdots\gamma_{b_1}[pt]^m),\\
&\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S,[pt]^m),
\end{aligned}$$

where $\chi = 1 - h + n$ and h is the arithmetic genus of elements of H_β as before. Roughly speaking, the H_1-insertions ensure only stable pairs with support curve in a fixed linear subsystem $|L| \subset H_\beta$ contribute to the invariant. Also, roughly speaking, the point insertions force the support curve to lie in a codimension m linear subsystem $\mathbb{P} \subset |L|$. We come back to this more precisely in the next section. We now want to exploit our hard work by computing these invariants on the ambient smooth projective variety

$$\iota : P_\chi(S, \beta) \cong \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta \hookrightarrow S^{[n]} \times H_\gamma.$$

There is one issue though. Recall that $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ need not vanish. Therefore, for computational purposes, we assume the following stronger version of (AS1)

$$(\mathrm{AS3}) : h^2(L) = 0 \text{ for all } L \in \mathrm{Pic}^\beta(S).$$

Then $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) = 0$ and $\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is a vector bundle on H_γ. The space $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ is cut out by a section of

$$\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) \oplus \pi_*\mathcal{O}_{\mathcal{Z}}(\mathcal{D} - A)$$

on the smooth ambient space $S^{[n]} \times H_\gamma$ (Theorem 2.9). Therefore formula (5) gives

$$\iota_*[P_\chi(S, \beta)]^{\mathrm{red}} = c_{\mathrm{top}}(\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) \oplus \pi_*\mathcal{O}_{\mathcal{Z}}(\mathcal{D} - A)). \tag{12}$$

Therefore the surface stable pair invariants become

$$\int_{S^{[n]} \times H_\gamma} c_{\mathrm{top}}(\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) \oplus \pi_*\mathcal{O}_{\mathcal{Z}}(\mathcal{D} - A)) \cdot \text{some explicit integrand.}$$

We will not discuss the precise shape of the integrand (see [KT2] for details). In the case of the γ_i insertions one can compute the integral over H_γ and obtains

$$\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^m) = \int_{S^{[n]}} \text{some explicit integrand.}$$

Exercise 3.6. Suppose S satisfies $h^{0,1}(S) = 0$. Then $H_\beta = |L|$ is a linear system and $H_\gamma = |L(A)|$. Assume $h^2(L) = 0$ (AS3). Let $\mathcal{O}(1)$ be the tautological line bundle on $|L(A)|$ and $\omega := c_1(\mathcal{O}(1))$. One can show $\mathcal{O}(\mathcal{D}) \cong L(A) \boxtimes \mathcal{O}(1)$. Suppose $n = 0$ so $P_\chi(S, \beta) = |L|$ is the linear system. Prove that $\iota_*[|L|]^{\mathrm{red}} = \omega^r \cap [|L(A)|]$, where $r := \chi(L(A)) - \chi(L)$. Deduce that $[|L|]^{\mathrm{red}} = [|L|]$ when $h^1(L) = 0$.

Exercise 3.7. Suppose S satisfies $h^{0,1}(S) = 0$. Then $H_\beta = |L|$ is a linear system and $H_\gamma = |L(A)|$ as in the previous exercise. Assume $h^2(L) = 0$ (AS3). Let $\mathcal{O}(1)$ be the tautological line bundle on $|L(A)|$ and $\omega := c_1(\mathcal{O}(1))$ as in the previous exercise. Suppose $n = 1$ so $P_\chi(S, \beta) = \mathcal{C}$ is the universal curve. Show that $\pi_*\mathcal{O}_{\mathcal{Z}}(\mathcal{D} - A) \cong L \boxtimes \mathcal{O}(1)$. Use this to show that $\iota_*[\mathcal{C}]^{\mathrm{red}} = c_1(L) \cap \omega^{r+1} \cap [S \times |L(A)|]$, where we suppressed pull-backs along $S \times |L(A)| \to |L(A)|$ and $S \times |L(A)| \to S$ and where $r := \chi(L(A)) - \chi(L)$.

Intersection theory on $S^{[n]}$ has been well-studied. Consider the projections

$$p : \mathcal{Z} \longrightarrow S,$$
$$q : \mathcal{Z} \longrightarrow S^{[n]}.$$

Then for any vector bundle E on S

$$E^{[n]} := q_* p^* E$$

is a vector bundle on $S^{[n]}$ called a *tautological bundle*. Using correspondences between $S^{[n]}$ and $S^{[n-1]} \times S$, G. Ellingsrud, L. Göttsche, and M. Lehn [EGL] give a method for computing integrals of polynomials in Chern classes of tautological bundles on $S^{[n]}$. I will formulate their result in the case of a single line bundle L on S. Suppose P is some polynomial expression in terms of the Chern classes

$$c_i(L^{[n]}), c_j(T_{S^{[n]}}), \tag{13}$$

where $T_{S^{[n]}}$ is the tangent bundle of $S^{[n]}$. Then there exists a rational polynomial $Q_n(x_1, x_2, x_3, x_4)$, only depending on n and the polynomial P, such that for any S, L

$$\int_{S^{[n]}} P = Q_n(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)).$$

In this context $Q_n(x_1, x_2, x_3, x_4)$ is often referred to as a *universal polynomial*. Since our "explicit integrand" can be expressed as a polynomial expression in (13), [EGL] gives:

Theorem 3.2 (K, Thomas)**.** *There exist a rational polynomials $Q_{n,m}(x_1, x_2, x_3, x_4)$, only depending on m, n, such that for any S, β satisfying (AS3) and for any $\chi = 1 - h + n$ and any m*

$$\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^m) = (-1)^{\chi(L)-1-m+n} s^{m+1-\chi(\mathcal{O}_S)} Q_{n,m}(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)),$$

where s is the $\mathbb{C}^*$ equivariant parameter.

A few remarks on this theorem:

(1) Even though the invariants are also defined when (AS1) is satisfied, universal dependence on $c_1(L)^2$, $c_1(L)c_1(S)$, $c_1(S)^2$, $c_2(S)$ is *false* in this more general case [Koo]. This can already be seen by considering cases where the number of points $n = 0$. Then $[P_\chi(S, \beta)]^{\mathrm{red}} = [H^{\mathrm{red}}_\beta]$ and the surface invariants are the Poincaré invariants and hence Seiberg-Witten invariants of [DKO]. One can find S, L and S', L' for which the four Chern numbers are the same, but the Seiberg-Witten invariants distinct.

(2) A similar theorem holds for the invariants $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S, [pt]^m)$ where the universal polynomials also depend on certain explicit intersection numbers on $\mathrm{Pic}^\beta(S)$.

(3) When β is irreducible $P_\chi(X, \beta)^{\mathbb{C}^*} \cong P_\chi(S, \beta)$, so this theorem computes reduced stable pair invariants of X. By the GW/PT correspondence, we get universality results for the corresponding reduced Gromov-Witten invariants of X.

Exercise 3.8. Suppose S satisfies $h^{0,1}(S) = 0$ and $c_1(L) = \beta$ with $h^2(L) = 0$ (AS3). Consider the invariant $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}([pt]^m)$ for $\chi = 1 - h$, i.e. $n = 0$. One can show that $\tau_0([pt])^m = \omega^m$, where $\omega := c_1(\mathcal{O}(1))$ on $|L(A)|$. Combine this with Exercise 3.6 to show that the invariant is zero unless $\chi(L) - 1 \geq m$. When this is the case, one can prove that

$$\frac{1}{e(N^{\mathrm{vir}})} = (-1)^{\chi(L)-1-m} s^{m+1-\chi(\mathcal{O}_S)} (1 + \omega)^{\chi(L)}.$$

Combine this with Exercise 3.6 to show that

$$\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}([pt]^m) = (-1)^{\chi(L)-1-m} s^{m+1-\chi(\mathcal{O}_S)} \binom{\chi(L)}{m+1}.$$

This shows a special case of Theorem 3.2.

Exercise 3.9. Suppose S satisfies $h^{0,1}(S) = 0$ and $c_1(L) = \beta$ with $h^2(L) = 0$ (AS3). Consider the invariant $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}([pt]^m)$ for $\chi = 2 - h$, i.e. $n = 1$. One can show that $\tau_0([pt])^m = \omega^m$, where $\omega := c_1(\mathcal{O}(1))$ on $|L(A)|$. Combine this with Exercise 3.7 to show that the invariant is zero unless $\chi(L) - 1 \geq m$. Moreover one can show

$$\frac{1}{e(N^{\mathrm{vir}})} = (-1)^{\chi(L)-m} s^{m+1-\chi(\mathcal{O}_S)} \frac{c_\bullet(T_S)(1+\omega)^{\chi(L)}}{1 + c_1(L) + \omega}.$$

Combine this with Exercise 3.7 to show that

$$\begin{aligned}\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}([pt]^m) =& (-1)^{\chi(L)-m} s^{m+1-\chi(\mathcal{O}_S)} \\ &\times \left\{ \binom{\chi(L)-1}{m+1} c_2(S) - \binom{\chi(L)-2}{m-1} (K_S \cdot L) - \binom{\chi(L)-3}{m-2} L^2 \right\},\end{aligned}$$

when $m \geq 2$, $\chi(L) \geq 3$. This shows a special case of Theorem 3.2.

3.4 Application to Severi degrees

We now come back to Severi degrees mentioned in the introduction to these lectures. First we define Severi degree for arbitrary linear systems. Let $|L|$ be a linear system on a smooth projective surface S. In what follows we want L to be *sufficiently ample*. The notion of ampleness we use is due to M. Beltrametti and A. J. Sommese [BS]. A line bundle L on S is called *k-very ample* if for any $Z \in S^{[k+1]}$ the restriction map

$$H^0(L) \longrightarrow H^0(L|_Z)$$

is surjective. An example of such an L is the kth power of a very ample line bundle. For all our applications it is certainly sufficient to take L $(2\delta + 1)$-very ample and $h^1(L) = h^2(L) = 0$, although for many parts one can find better bounds.

Ampleness ensures the δ-nodal curves form a locally closed subset of codimension δ (and are smooth points of this locus). Therefore a general δ-dimensional linear subsystem $\mathbb{P}^\delta \subset |L|$ contains a finite number of δ-nodal curves and this number $N_{L,\delta}(S)$ is known as a *generalized Severi degree*. We should think of $N_{L,\delta}(S)$ as the number of δ-nodal curves in $|L|$ going through

$$\chi(L) - 1 - \delta$$

points in general position.

We want to realise $N_{L,\delta}(S)$ as a Gromov-Witten invariant. Therefore we work on some space $\overline{M}_{g,m}(S, \beta)$ with $c_1(L) = \beta$. We want the space to contain all normalizations of δ-nodal curves in $\mathbb{P}^\delta$ so we choose $g := h - \delta$, where h is the arithmetic genus of elements of $|L|$. A priori a stable map $\{f : C \to S\} \in \overline{M}_{g,m}(S, \beta)$ satisfies

$$f_*[C] \in H_\beta.$$

There is a way to ensure $f_*[C] \in |L|$. I will simply state the result (see e.g. [BL2, KT1] for details). Let $\gamma_1, \ldots, \gamma_{b_1} \in H_1(S)/\text{torsion}$ be an integral oriented basis of 1-cycles. Then the insertions

$$\prod_{i=1}^{b_1} \mathrm{ev}_i^* \gamma_i$$

cuts the moduli space down to the locus of stable maps satisfying $f_*[C] \in |L|$. However we want more, we also want $f_*[C] \in \mathbb{P}^\delta$. This is achieved by adding point insertions. This leads to the following insertions

$$(\cdots) := \prod_{i=1}^{b_1} \mathrm{ev}_i^* \gamma_i \cdot \prod_{i=b_1+1}^{b_1+\chi(L)-1-\delta} \mathrm{ev}_i^*[pt].$$

We conclude $m := b_1 + \chi(L) - 1 - \delta$ is the desired number of markings. A short calculation shows that

$$\begin{aligned} &\text{degree } (\cdots) \\ &\text{dimension } [\overline{M}_{h-\delta,m}(S,\beta)]^{\mathrm{vir}} \end{aligned}$$

differ by $h^{0,2}(S)$ so we cannot pair $(\cdots)$ and $[\overline{M}_{h-\delta,m}(S,\beta)]^{\mathrm{vir}}$ to get a number!

Exercise 3.10. Compute the degree and dimension mentioned above and conclude they differ by $h^{0,2}(S)$.

However, the degree of $(\cdots)$ and the dimension of $[\overline{M}_{h-\delta,m}(S,\beta)]^{\mathrm{red}}$ do match. This shows the reduced virtual cycle is the right one for enumerative questions. We therefore expect the reduced Gromov-Witten invariant

$$\mathrm{GW}^{\mathrm{red}}_{h-\delta,\beta}(S, \gamma_1 \cdots \gamma_{b_1} [pt]^{\chi(L)-1-\delta}) = \int_{[\overline{M}_{h-\delta,m}(S,\beta)]^{\mathrm{red}}} (\cdots)$$

to be related to the Severi degree $N_{L,\delta}(S)$. Certainly the Gromov-Witten invariant counts the normalizations of δ-nodal curves on $\mathbb{P}^\delta$, but why does it not count much more?

This question can be addressed by analyzing the geometric content of the general linear subsystem $\mathbb{P}^\delta$. Sufficient ampleness of L implies [KST, KT1]:

(1) All curves with $> \delta$ singularities (in particular non-reduced curves) appear in codimension $> \delta$. Therefore $\mathbb{P}^\delta$ contains no such curves.

(2) $\mathbb{P}^\delta$ contains a finite number of δ-nodal curves and no other curves with exactly δ singularities.

(3) $\mathbb{P}^\delta$ only contains irreducible curves.

(4) $\mathbb{P}^\delta$ contains curves with $< \delta$ singularities (e.g. smooth curves). Each such curve C has geometric genus $> h - \delta$ (e.g. for a smooth curve C the geometric genus is h).

Parts 1,2 and probably 3 were known. Part 4 was proved with Thomas and V. Shende [KST]. From (1) we deduce no multiple covers occur (or else $\mathbb{P}^\delta$ contains non-reduced curves). In general, one can show that if $f : C \to S$ is a stable map (no marked points) from a connected curve of arithmetic genus $g_a(C)$ to a curve $\Sigma := f(C)$ of geometric genus $g_g(\Sigma)$, then

$$g_a(C) \geq g_g(\Sigma) \tag{14}$$

with equality if and only if there are no contracted components and C is smooth.

Exercise 3.11. Let $f : C \to S$ be a stable map (no marked points) from a connected curve and let $\Sigma := f(C)$. Let $C_1, \ldots, C_k$ be the contracted irreducible components of C and $C_{k+1}, \ldots, C_m$ the other irreducible components. Denoting normalization of a curve D by $\overline{D}$, we get $\overline{C} = \overline{C}_1 \sqcup \cdots \sqcup \overline{C}_m$ and $\overline{\Sigma} = \overline{C}_{k+1} \sqcup \cdots \sqcup \overline{C}_m$. (i) Show that $g_a(C) - g_g(\Sigma) = \sum_{i=1}^k (g(\overline{C}_i) - 1) + d$, where d is the number of nodes of C. (ii) Using (i), reason why contracted components with $g(\overline{C_i}) > 0$ contribute positively to $g_a(C) - g_g(\Sigma)$. (iii) Using (i) and stability, reason why contracted $\mathbb{P}^1$'s contribute positively to $g_a(C) - g_g(\Sigma)$. (iv) Deduce $g_a(C) \geq g_g(\Sigma)$ with equality if and only if there are no contracted components and C is smooth.

From inequality (14) and (2)–(4) we deduce that any $f : C \to S$ with $f_*[C] \in \mathbb{P}^\delta$ cannot have contracted components and must be a normalization of a δ-nodal curve. This essentially shows[3]

$$\mathrm{GW}^{\mathrm{red}}_{h-\delta,\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) = N_{L,\delta}(S).$$

We can also realize the Severi degrees as 3-fold Gromov-Witten invariants of $X = \mathrm{Tot}(K_S)$. In the previous section we saw that for any g

$$[\overline{M}_{g,m}(X, \beta)^{\mathbb{C}^*}]^{\mathrm{red}} = [\overline{M}_{g,m}(S, \beta)]^{\mathrm{red}}.$$

Moreover, for dimension reasons

$$\frac{1}{e(N^{\mathrm{vir}})}$$

only contributes an overall power of the equivariant parameter s, i.e.

$$\begin{aligned} &\mathrm{GW}^{\mathrm{red}}_{h-\delta,\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) = s^r \mathrm{GW}^{\mathrm{red}}_{h-\delta,\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) \\ &r := \frac{1}{2}L(L - K_S) - \delta. \end{aligned} \tag{15}$$

This realizes the Severi degree as a 3-fold reduced Gromov-Witten invariant.

Exercise 3.12. Let $\pi : \mathcal{C} \to \overline{M}_{g,m}(S, \beta)$ be the universal curve and $f : \mathcal{C} \to S$ the universal stable map. Then one can show $N^{\mathrm{vir}} \cong R\pi_* f^* K_S \otimes \mathfrak{t}$, where $\mathfrak{t}$ is a primitive representation of $\mathbb{C}^*$. Prove (15).

[3] One also has to show that the normalizations of δ-nodal curves are reduced isolated points in $\overline{M}_{h-\delta}(S, \beta)$ [KT1, Prop. 5.2].

Next we want to study the GW/PT correspondence in this setting. We therefore consider the full generating function

$$\mathrm{GW}^{\mathrm{red}}_{\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta})) := \sum_g \mathrm{GW}^{\mathrm{red}}_{g,\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) u^{2g-2}.$$

Since $\mathbb{P}^\delta$ only contains reduced curves (by (1)), multiple covers do not contribute and the GW generating function can be written as

$$\mathrm{GW}^{\mathrm{red}}_{\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) = \sum_{g=h-\delta}^{\infty} n^{\mathrm{GW}}_{g,\beta}(2\sin(u/2))^{2g-2}.$$

The lowest order term in u of this generating series is $n^{\mathrm{GW}}_{h-\delta,\beta} = s^r N_{L,\delta}(S)$ by the previous discussion. On the PT side, we consider the generating function

$$\mathrm{PT}^{\mathrm{red}}_{\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) := \sum_{\chi} \mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) q^{\chi}.$$

Consider the map $\pi : X \to S$. Similar to the case of Gromov-Witten invariants, one can show that with these insertions only stable pairs $(F, s) \in P_\chi(X, \beta)$ contribute for which

$$[\pi_* F] \in \mathbb{P}^\delta \subset |L| \subset H_\beta.$$

Since $\mathbb{P}^\delta$ only contains reduced curves, only the surface components $P_\chi(S, \beta)$ contribute. Therefore

$$\mathrm{PT}^{\mathrm{red}}_{\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) = \mathrm{PT}^{\mathrm{red}}_{\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}).$$

Using again that all elements of $\mathbb{P}^\delta$ are reduced, we have can write

$$\mathrm{PT}^{\mathrm{red}}_{\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}) = \sum_{g \in \mathbb{Z}} n^{\mathrm{PT}}_{g,\beta}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2g-2}. \tag{16}$$

There is a direct argument using Serre duality on the fibres of the map

$$P_\chi(S, \beta) \longrightarrow H_\beta,$$

which shows that $n^{\mathrm{PT}}_{g,\beta} = 0$ unless $g \in [0, h]$. The argument is a *variation* on a general result of Pandharipande-Thomas [PT3]:

Theorem 3.3 (Pandharipande-Thomas)**.** *Let X be a compact Calabi-Yau 3-fold and β an irreducible curve class. Then*

$$\mathrm{PT}_\beta(X) = \sum_{g=0}^{h} n^{\mathrm{PT}}_{g,\beta}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2g-2},$$

where h is the maximal arithmetic genus of curves in class β.

Inverting (16) gives an expression of $n^{\mathrm{PT}}_{h-\delta,\beta}$ in terms of surface invariants

$$\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}),$$

for $\chi \in [1-h, 1-h+\delta]$.

Exercise 3.13. Carry this out for $\delta = 0, 1, 2, 3$.

This discussion together with the fact that the invariants $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta})$ are given by universal polynomials (Theorem 3.2) implies:

Theorem 3.4 (K-Thomas)**.** *Let S, β, L, δ be as above. Setting $-q = e^{iu}$, the lowest order coefficients in u of the Laurent series*

$$\mathrm{GW}^{\mathrm{red}}_{\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta}), \mathrm{PT}^{\mathrm{red}}_{\beta}(X, \gamma_1 \cdots \gamma_{b_1}[pt]^{\chi(L)-1-\delta})$$

coincide if and only if $N_{L,\delta}(S)$ equals

$$T_\delta(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S)),$$

where $T_\delta(x_1, x_2, x_3, x_4)$ is a certain explicit rational polynomial depending only on δ.

Exercise 3.14. Use Exercises 3.8, 3.9, 3.13 to show that the GW/PT correspondence as phrased in the previous theorem implies $N_{L,1}(S) = 3c_1(L)^2 - 2c_1(L)c_1(S) + c_2(S)$. Use this to show that there are three 1-nodal curves of degree 2 passing though 4 points in general position on $\mathbb{P}^2$ as we saw in the introduction.

Interestingly, this gives a connection with a famous conjecture formulated by L. Göttsche about Severi degrees in 1997.

Conjecture 3.5 (Göttsche)**.** *For any δ, there exists a rational polynomial $T_\delta(x_1, x_2, x_3, x_4)$ depending only on δ, such that for any S, L with L sufficiently ample one has*

$$N_{L,\delta}(S) = T_\delta(c_1(L)^2, c_1(L).c_1(S), c_1(S)^2, c_2(S)).$$

In this section we deduced that a very special case of the GW/PT correspondence for $X = \mathrm{Tot}(K_S)$ implies Göttsche's conjecture. The fact that $N_{L,\delta}(S)$ is equal to the explicit expression $T_\delta(c_1(L)^2, c_1(L)c_1(S), c_1(S)^2, c_2(S))$ of Theorem 3.4 was proved directly by K, Shende, and Thomas [KST]. The first algebro-geometric proof of Göttsche's conjecture was given by Y.-j. Tzeng [Tze] using cobordism methods.

Exercise 3.15. Let $|L|$ be a sufficiently ample linear system on a surface S. Let $\mathbb{P}^1 \subset |L|$ be a generic pencil and let $\mathcal{C} \to \mathbb{P}^1$ be the universal curve. Then $\mathcal{C}$ contains $N_{L,\delta}(S)$ 1-nodal fibres and all other fibres are smooth. (i) Show that $e(\mathcal{C}) = 4 - 4h + N_{L,\delta}(S)$, where $e(\cdot)$ denotes topological Euler characteristic. Hint: stratify $\mathbb{P}^1$ according to fibre type and compute the Euler characteristics of the fibres. (ii) It is not hard to see that $\mathcal{C}$ is the blow-up of S in the intersection points of two generic elements of the pencil. Use this to compute $e(\mathcal{C})$. Also compute $e(\mathcal{C})$ using (4) of the second lecture and show you get the same answer. (iii) Deduce the formula for $N_{L,\delta}(S)$ of Exercise 3.14. This approach is generalized in [KST] to prove Göttsche's conjecture.

4 Application to Seiberg-Witten invariants

Consider any S, β. The previous section was about reduced surface stable pair invariants $\mathrm{PT}^{\mathrm{red}}_{\chi,\beta}(S, \sigma_1 \cdots \sigma_m)$. We now turn our attention to non-reduced surface stable invariants $\mathrm{PT}_{\chi,\beta}(S, \sigma_1 \cdots \sigma_m)$. Recall that the non-reduced invariants are always defined and their virtual cycle is given by (11)

$$\iota_*[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{vir}} = (S^{[n]} \times [H_\beta]^{\mathrm{vir}}) \cap c_n(\pi_* \mathcal{O}_{\mathcal{Z}}(\mathcal{C})),$$

where $\chi = 1 - h + n$, h is the arithmetic genus of curves on H_β, and

$$\iota : \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow H_\beta \times S^{[n]}$$

denotes inclusion. This formula relates non-reduced surface stable pair invariants and the Poincaré invariants of [DKO]. Recall that $P^+_\beta(S) \in \Lambda^* H^1(S, \mathbb{Z})^*$. From now on we only consider the "numerical part" of $P^+_\beta(S) = \mathrm{SW}^+_\beta(S)$, i.e. the part in $\Lambda^{b_1} H^1(S, \mathbb{Z})^* \cong \mathbb{Z}$ and denote this numbers by the same symbol. In the case of "many point insertions" one can use (11) to show:

Proposition 4.1 (K). *For any S, β, and $m := \frac{\beta(\beta - \mathsf{k})}{2}$*

$$\mathrm{PT}_\beta(S, [pt]^m) = s^m P^+_\beta(S)(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2h-2},$$

where s denotes the $\mathbb{C}^*$ equivariant parameter.

Using this relatively simple proposition and the (much deeper!) Poincaré/SW correspondence of [DKO, CK] we arrive at the following:

Theorem 4.2 (K). *Fix any S, β with β irreducible and let $m := \frac{\beta(\beta - \mathsf{k})}{2}$. Setting $-q = e^{iu}$, the lowest order coefficients in u of the Laurent series*

$$\mathrm{GW}'_\beta(X, [pt]^m), \mathrm{PT}_\beta(X, [pt]^m)$$

coincide if and only if

$$\mathrm{GW}'_{h,\beta}(S, [pt]^m) = \mathrm{SW}^+_\beta(S). \tag{17}$$

This theorem uses *disconnected* Gromov-Witten invariants $\mathrm{GW}'_\beta(S, [pt]^m)$ (see lecture 1). Equation (17) is a very special case of the GW/SW correspondence of C. Taubes [Tau1, Tau2]. It would be nice to extent this theorem to *any* algebraic S, β without the irreducibility condition. This requires dealing with other components of $P_\chi(X, \beta)^{\mathbb{C}^*}$.

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