Numerical bifurcation analysis of delay differential equations

Lecture 2

Dirk Roose
Dept. of Computer Science
K.U.Leuven

Dirk.Roose@cs.kuleuven.be

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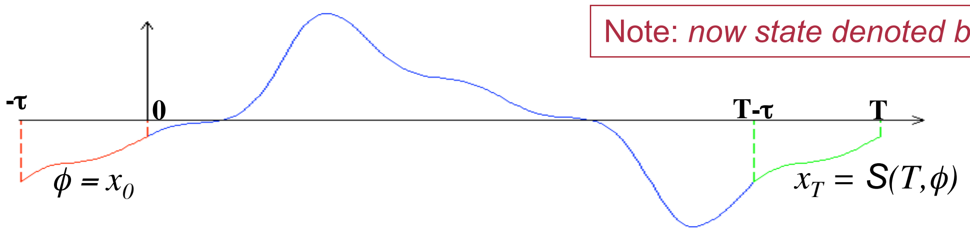
Overview (2)

- *Lecture 2*
  - computation & stability analysis of periodic solutions
  - computation of connecting orbits (homo- & heteroclinic orbits)
  - remark on DDEs with state-dependent delays
  - short introduction to software package PDDE-CONT for continuation and bifurcation analysis of periodic solutions of DDEs
- *Practical session*
  - Demo & hands-on experience with DDE-BIFTOOL and PDDE-CONT

Periodic solutions

Note: *now state denoted by x* !



- autonomous system: period *T* unknown
- periodicity condition *on functions*

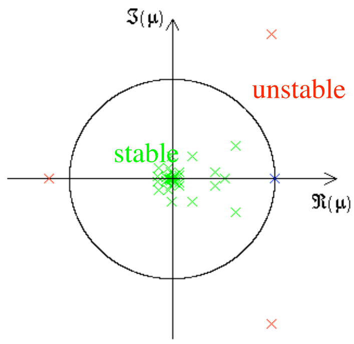
  $$x_T = S(T,\phi) = \phi = x_0$$

  S(.,.): solution operator of original DDE
- continuous derivatives
- constant delays: Monodromy *operator*

  $\mathcal{M} = \partial S(T,\phi) / \partial \phi$

  spectrum of $\mathcal{M}$: Floquet multipliers

  trivial Floquet mult. = 1



- onset of periodic solutions: Hopf bifurcation (imag. eigenval.)

Periodic solutions

DDEs with state-dependent delays

Existence of periodic solutions proven only for particular cases:
e.g. scalar equation, 1 delay, 1 parameter η,
steady state $x^s(\eta) = 0$

if, for $\eta = \eta_m$, $m = 0,1,\ldots$ charact. eq. has solutions $\lambda = \pm i\omega_m$,
then there exists a *slowly oscillating* periodic solution for η_0 with $T=2\pi/\omega_0$

zeros of periodic sol. separated by distances $d > \tau$

Note: *max* $\tau(x(t))$, $t = 0 \ldots T$ is not known in advance !

Numerical experiments indicate that periodic solutions exist for all η_m --> *Hopf-like bifurcation*

Computation of periodic solutions

Computation of periodic solutions

- two point boundary value problem

$$\begin{cases} x'(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_m), \eta), \ t \in [0, T] \\ x_0 = x_T, \\ p(x, T) = 0, \end{cases}$$

x_0 and x_T are function segments on $[-\tau, 0]$ and $[-\tau+T, T]$, respectively, $p(x, T) = 0$ represents a suitable phase condition

- rescale time by factor $1/T$

$$\begin{cases} x'(t) = Tf\left(x(t), x(t-\tau_1/T), \ldots, x(t-\tau_m/T), \eta\right), \quad \text{for } t \in [0, 1], \\ x(\theta+1) - x(\theta) = 0, \quad \text{for } \theta \in [-\tau/T, 0], \\ p(x, T) = 0. \end{cases}$$

Collocation

collocation as used in AUTO, CONTENT, MATCONT, ...
piecewise polynomial (degree *d*) representation

$$L+1 \text{ mesh points } \{0 = t_0 < t_1 < \ldots < t_L = 1\}$$

basis: Lagrange polynomials $P_{i,j}(t)$

representation points in interval $[t_i, t_{i+1}]$

$$t_{i+\frac{j}{d}} = t_i + \frac{j}{d}(t_{i+1} - t_i), \quad j = 0, \ldots, d.$$

$$u(t) = \sum_{j=0}^{d} u(t_{i+\frac{j}{d}}) P_{i,j}(t), \ t \in [t_i, t_{i+1}],$$

u(t) must satisfy the system in the collocation points
(scaled and shifted roots of orthog. polyn.of degree d)

$$c_{i,j} = t_i + c_j(t_{i+1} - t_i), \ i = 0, \ldots, L-1, \ j = 1, \ldots, d,$$

Collocation in case of DDEs

mesh with L+1 mesh points on [0,1]
periodically extended with ℓ points → interval [-τ/T,1]
(approximate) periodic orbit determined by 'variables'

$$u_{i+\frac{j}{d}} := u(t_{i+\frac{j}{d}}),\ i = -\ell, \ldots, L-1,\ j = 0, \ldots, d-1$$

and $u_L := u(t_L)$

$$\textit{starting vector}\quad u_s := [u_{-\ell}, \ldots, u_{i+\frac{j}{d}}, \ldots, u_0]^T$$

$$\textit{final vector}\quad u_f := [u_{L-\ell}, \ldots, u_{i+\frac{j}{d}}, \ldots, u_L]^T$$

periodicity condition: $\boxed{u_f = u_s}$

better: take periodicity into account in system
('folding', 'modulo')

Collocation

- collocation points

$$c_{i,j} = t_i + c_j(t_{i+1} - t_i),\ i = 0, \ldots, L-1,\ j = 1, \ldots, d,$$

- (nonlinear) system of dimension $n(Ld+1)+1$

$$\begin{cases} \dot{u}(c_{i,j}) = Tf(u(c_{i,j}), u((c_{i,j} - \frac{\tau_1}{T}) \bmod 1), \ldots, u((c_{i,j} - \frac{\tau_m}{T}) \bmod 1), \eta) = 0, \\ \qquad i = 0, \ldots, L-1,\ j = 1, \ldots, d \\ u_0 = u_L, \\ p(u) = 0. \end{cases}$$

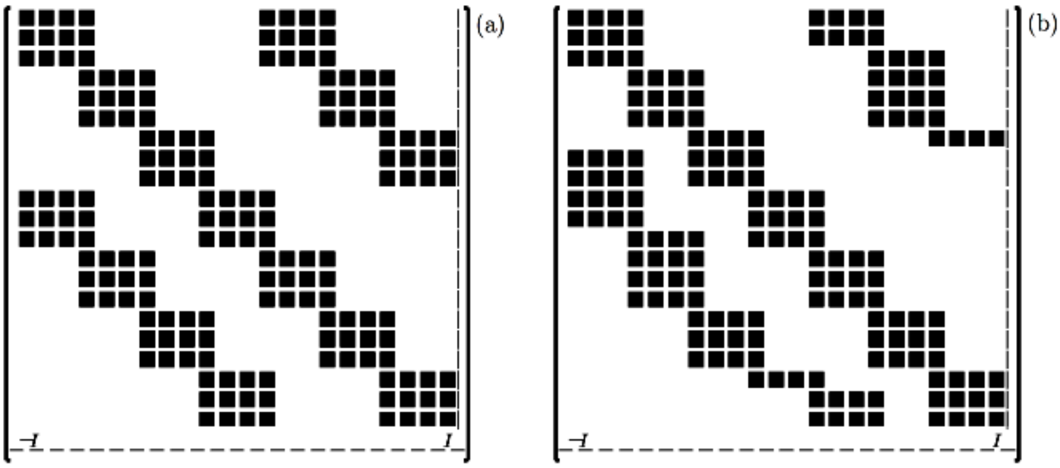
<-- Phase condition

Note: if colloc. point = mesh point: discontinuous derivative

- Solve by Newton iteraton:
  linearization: system matrix: sparse, but not very structured: off-diagonal bands due to delays
  → much more expensive !

Collocation

- structure of collocation matrix
  1 delay τ < period T, L=7, d=3
  left: equidistant mesh; right: nonequidistant mesh



Collocation

Increase efficiency via

– ‘chord-Newton’
– collocation Newton-Picard
– adaptive non-equidistant mesh

Convergence [Engelborgs & Doedel]

$$E = \max_{t\in[0,1]} \|u(t) - u^\star(t)\| \text{ is } \mathcal{O}(h^d)$$

$$\text{and } \mathcal{O}(h^{d+1}) \text{ for Gauss-Legendre collocation}$$

no ‘superconvergence’ at mesh points

Stability of periodic solutions

Monodromy operator

- $x^*(t)$: periodic solution with period T
- (after rescaling → period 1) linearized equation

$$\frac{d}{dt} y(t) = T\left(A_0(t) y(t) + \sum_{j=1}^{m} A_j(t) y(t - \tau_j)\right) \qquad (*)$$

- $U(t,s)$: fundamental solution operator of (*)

$$(U(t,s)\phi_s)(\theta) = y(t+\theta), \quad \theta \in [-1, 0],$$

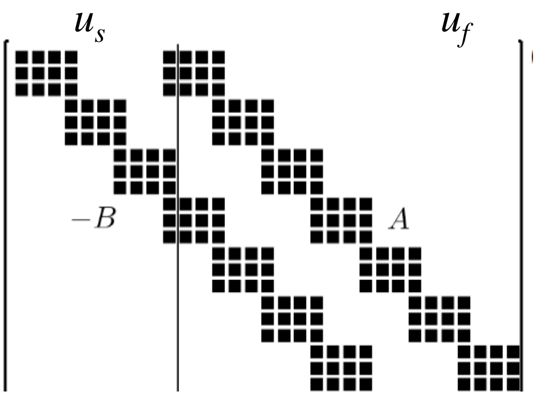
ϕ_s : initial function segment; y : corresponding solution

- Monodromy operator

$$\mathcal{M} : \phi \mapsto y_1 \qquad \mathcal{M} = U(1,0)$$

Floquet multipliers

- discretization of Monodromy operator : $\mathcal{M}_d : u_s \rightarrow u_f$
  matrix representation can be obtained by by collocation



u_s : discretized solution on [- τ/T, 0]

u_f: discretized solution on [1- τ/T, 1]

DDE-BIFTOOL: $u_f = M_d u_s$

if $\tau > T$: dimension of u_s and u_f larger than dimension of discretized orbit; extended matrix: large size → expensive

BUT: u_s and u_f overlap : can be exploited → PDDE-CONT

extended Monodromy operator

- write linearized equation as

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = T \int_0^{\tau/T} \mathrm{d}_\theta \zeta(T\theta, t) y(t-\theta),$$

or taking into account initial function

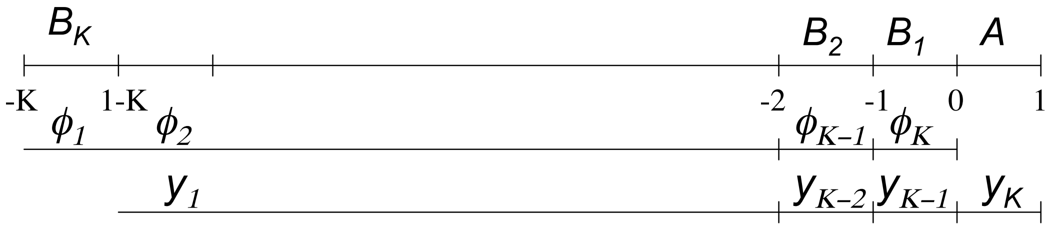
$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - T \int_0^{t} \mathrm{d}_\theta \zeta(T\theta, t) y(t-\theta) - T \int_t^{\tau/T} \mathrm{d}_\theta \zeta(T\theta, t) \phi(t-\theta) = 0.$$

with ζ a matrix function of bounded variation

$$\zeta(T\theta, t) = \begin{cases} 0 & \text{if} \quad \theta \leq 0 \\ A_0(t) & \text{if } 0 < \theta < \tau_1 \\ \vdots & \vdots \\ A_0(t) + \sum_{j=1}^{m} A_j(t) \text{ if} & \tau \leq \theta \end{cases}$$

extended Monodromy operator

- split ϕ and y_1 into K segments $\quad K = \lceil \tau / T \rceil$



$$y_i = \phi_{i+1}. \qquad \phi_i, y_i \in \mathcal{X} := C([-1,0];\mathbb{R}^n)$$

- operators A and B_k acting on unit interval:

  A : differential operator including unknown part of solution

  B_k ,k= 1...K : transforms initial function
- only equation to be solved

$$\mathcal{A}y_K - \sum_{i=1}^{K} \mathcal{B}_i \phi_i = 0, \quad y_K(-1) = \phi_K(0)$$

extended Monodromy operator

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} - T\int_0^t \mathrm{d}_\theta \zeta(T\theta, t) y(t-\theta) - T\int_t^{\tau/T} \mathrm{d}_\theta \zeta(T\theta, t)\phi(t-\theta) = 0.$$

- operators A and B_k acting on unit interval:

  A : differential operator including unknown part of solution

  B_k ,k= 1...K : transforms initial function

$$(\mathcal{A}\phi)(\theta) = \frac{\mathrm{d}\phi(\theta)}{\mathrm{d}t} - T\int_0^{1+\theta} \mathrm{d}_\gamma \zeta(T\gamma, \theta)\phi(\theta-\gamma),$$

$$(\mathcal{B}_i\phi)(\theta) = T\int_{i+\theta}^{i+1+\theta} \mathrm{d}_\gamma \zeta(T\gamma, \theta)\phi(i+\theta-\gamma),$$

extended Monodromy operator

$$\mathcal{A}y_K - \sum_{i=1}^{K} \mathcal{B}_i \phi_i = 0, \quad y_K(-1) = \phi_K(0)$$

- eliminate explicit boundary condition: introduce extended operators on $\hat{\mathcal{X}} = \{(\varphi, c) \in \mathcal{X} \times \mathbb{R}^n \,:\, c = \varphi(0)\}$

$$\hat{\mathcal{A}} = \begin{pmatrix} \mathcal{A} & 0 \\ \mathcal{L} & 0 \end{pmatrix}, \quad \hat{B}_i = \begin{pmatrix} \mathcal{B}_i & 0 \\ 0 & 0 \end{pmatrix} \text{ for } i < N \quad \hat{B}_N = \begin{pmatrix} \mathcal{B}_N & 0 \\ 0 & I \end{pmatrix}$$

$$\mathcal{L}\varphi = \varphi(-1)$$

- Stability information: sufficient to construct extended Monodromy operator on

$$\tilde{X} = \left\{((\phi_1, c_1), \ldots, (\phi_N, c_N)) \in \hat{\mathcal{X}}^N \,:\, \phi_k(0) = c_k = \phi_{k+1}(-1),\, 1 \leqslant k < N\right\}$$

Floquet multipliers

- extended Monodromy operator

$$\tilde{\mathcal{M}} = \begin{pmatrix} 0 & \hat{I} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \hat{I} \\ \hat{\mathcal{A}}^{-1}\hat{\mathcal{B}}_1 & \hat{\mathcal{A}}^{-1}\hat{\mathcal{B}}_2 & \cdots & \hat{\mathcal{A}}^{-1}\hat{\mathcal{B}}_N \end{pmatrix}$$

- discretized version computed by collocation and by inverting discretized $\hat{\mathcal{A}}$ operator
- spectrum computed using the Arnoldi-Lanczos (in ARPACK): iterative process, matrix-vector mult. requires only one solution step with $\hat{\mathcal{A}}$
- if few delays: most matrices $\hat{B}_i$ are zero -> efficient

Floquet multipliers

- operator $\mathcal{M}$: used in DDE-BIFTOOL
- operator $\tilde{\mathcal{M}}$: used in PDDE-CONT
- both methods equivalent, same accuracy for Floquet multipliers

Connecting orbits
(homo- & heteroclinic orbits)

Connecting orbits

- Connecting orbit $x^\star(t)$

$$\lim_{t\to-\infty} x^\star(t) = x^- \quad \text{and} \quad \lim_{t\to+\infty} x^\star(t) = x^+$$

A defining condition for a connecting orbit is that it is contained in both the stable manifold of x^+ and the unstable manifold of x^-. A classical approach in the ODE case is to approximate this condition by truncating the time domain to an interval of length T_c and to apply (so-called) projection boundary conditions [7]: one end point of the connecting orbit is required to lie in the unstable eigenspace of x^- and the other end point in the stable eigenspace of x^+. The projection boundary conditions, therefore, replace the stable and unstable manifolds by their linear approximations near the steady states.

Connecting orbits

Here, the boundary conditions need to be written in terms of solution segments. Furthermore, x^+ has infinitely many eigenvalues with negative real parts (see Sec. 1) and so it is impossible to write the final function segment as a linear combination of all stable eigenfunctions. Instead, it is required that the end function segment is in the orthogonal complement of all unstable left eigenfunctions. We will assume for notational convenience that (1) only contains one delay τ; however, the method is implemented in DDE-BIFTOOL for the general case of m fixed delays.

The condition for the initial function segment $x_0(\theta)$ can be written as

$$x_0(\theta) = x^- + \varepsilon \sum_{k=1}^{s^-} \alpha_k v_k^- e^{\lambda_k^- \theta} \qquad \left(\sum |\alpha_k|^2 = 1\right),$$

where s^- is the number of unstable eigenvalues λ^-, with corresponding eigenvectors v^-. The α_k are unknown coefficients, and ε is a measure for the desired accuracy. An extra condition is added to ensure continuity at $\theta = 0$.

Connecting orbits

- instead of writing final solution segment as linear combination of all stable eigenfunctions:

  require that solution segment lies in orthogonal complement of all unstable left eigenfunctions
- using bilinear form : final solution segment in complement of unstable eigenspace: s^+ conditions*;*

  s^+ *:* # unstable eigenvalues of x^+*;*

  w^+_k*: left eigenvectors*
- Assume only one delay: complementarity condition:

$$w_k^{+*}(x(T_c)-x^+)+\int_{-\tau}^{0} w_k^{+*} e^{-\lambda_k^+(\theta+\tau)} A_1(x^+,\eta)\left(x(T_c+\theta)-x^+\right) d\theta = 0$$

$k = 1, \ldots, s^+$

State-dependent DDEs

State-dependent DDEs

$$\begin{cases} \dfrac{d}{dt}x(t) = f_1(x(t),\, x(t-\tau(x(t)))) \\ \tau(x(t)) = g_1(x(t)), \end{cases} \qquad (1)$$

$$\begin{cases} \dfrac{d}{dt}x(t) = f_2(x(t),\, x(t-\tau(t)),\, \tau(t)) \\ \dfrac{d}{dt}\tau(t) = g_2(x(t),\, x(t-\tau(t)),\, \tau(t)), \end{cases} \qquad (2)$$

$$\begin{cases} \dfrac{d}{dt}x(t) = f_3(x(t),\, x(t-\tau(t)),\, \tau(t)) \\ \displaystyle\int_{t-\tau(t)}^{t} g_3(x(s))ds = 1, \end{cases} \qquad (3)$$

Stability of steady state of sd-DDE

Hence to study the local stability of a steady state of (1), we linearize (1) at x^* by treating $\tau \equiv \tau^*$. The resulting linear equation is a *constant delay* differential equation,

$$\frac{d}{dt}y(t) = D_1 f_1(x^*,\, x^*)y(t) + D_2 f_1(x^*,\, x^*)y(t-\tau^*)\,,$$

Car traffic model

The traffic model in Orosz et *al.* [61, 60] describes the dynamics of N cars on a circular track. Each car has a velocity v_i and an associated headway h_i defined as the distance to the car in front. The headways h_i are calculated from the velocities as

$$\dot{h}_i(t) = v_{i+1}(t) - v_i(t). \tag{44}$$

Because of the circular track, we assume that $v_{N+1} = v_1$ and $h_N = L - \sum_{i=1}^{N-1} h_i$. Each car tries to reach its optimal velocity, which is a function of the headway that can be expressed as

$$\dot{v}_i(t) = \beta(V(h_i(t-1) - v_i(t)), \tag{45}$$

$$V(h) = \begin{cases} 0 & 0 \leq h \leq 1, \\ v^0 \frac{(h-1)^3}{1+(h-1)^3} & h > 1. \end{cases}$$

steady-state solution $\quad h_i^* = L/N, \quad v_i^* = V(h_i^*).$

Traffic model: periodic solutions

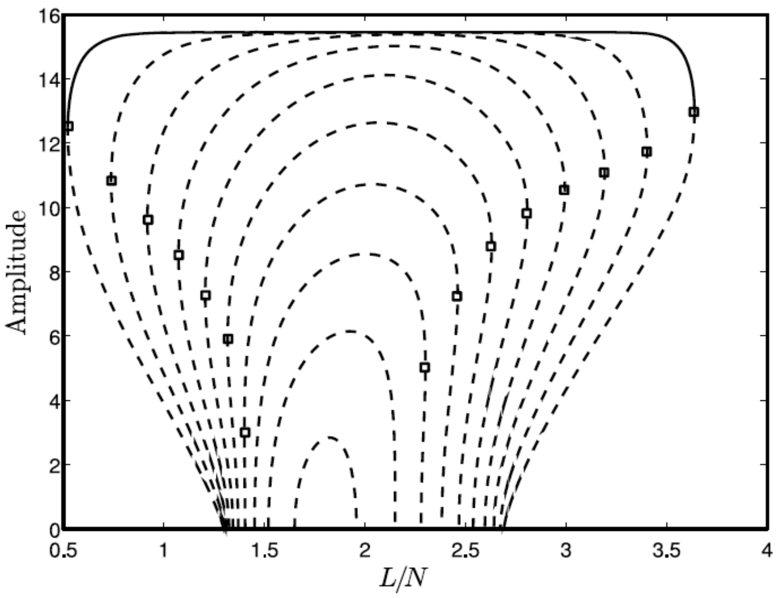


Fig. 8. Periodic solution branches of (44), (45). Unstable solutions are denoted by dashed lines, continous lines refer to stable solutions and boxes denote fold bifurcations.