Towards the analysis of codim 2 bifurcations in planar Filippov systems

Yu.A. Kuznetsov

Department of Mathematics, Utrecht University

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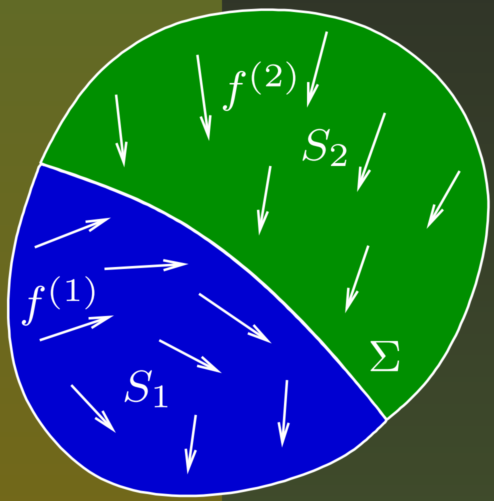
1. Filippov systems



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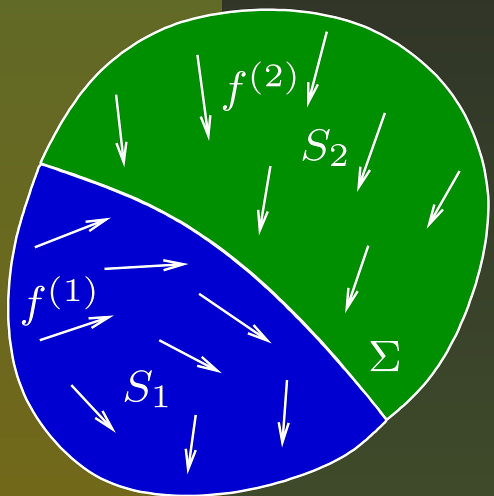

$$\dot{x} = \begin{cases} f^{(1)}(x), & x \in S_1, \\ f^{(2)}(x), & x \in S_2. \end{cases}$$

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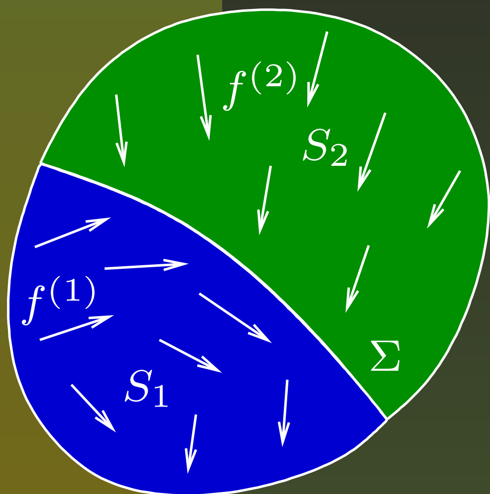


$$\dot{x} = \begin{cases} f^{(1)}(x), & x \in S_1, \\ f^{(2)}(x), & x \in S_2. \end{cases}$$

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^2 : H(x) < 0\}, \\ S_2 &= \{x \in \mathbb{R}^2 : H(x) > 0\}, \end{aligned}$$

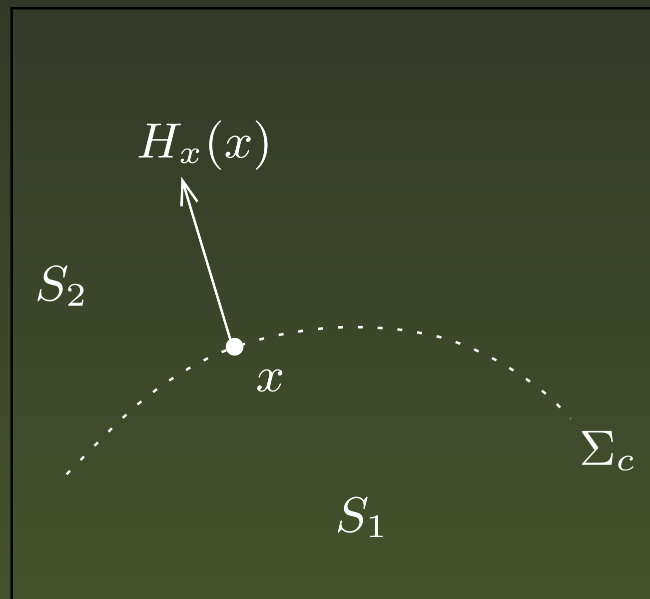
where $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ has nonvanishing gradient $H_x(x)$ on

$$\Sigma = \{x \in \mathbb{R}^2 : H(x) = 0\}.$$

For $x \in \Sigma$, define $\sigma(x) = \langle H_x(x), f^{(1)}(x)\rangle\langle H_x(x), f^{(2)}(x)\rangle$.

Crossing orbits

On $\Sigma_c = \{x \in \Sigma : \sigma(x) > 0\}$:



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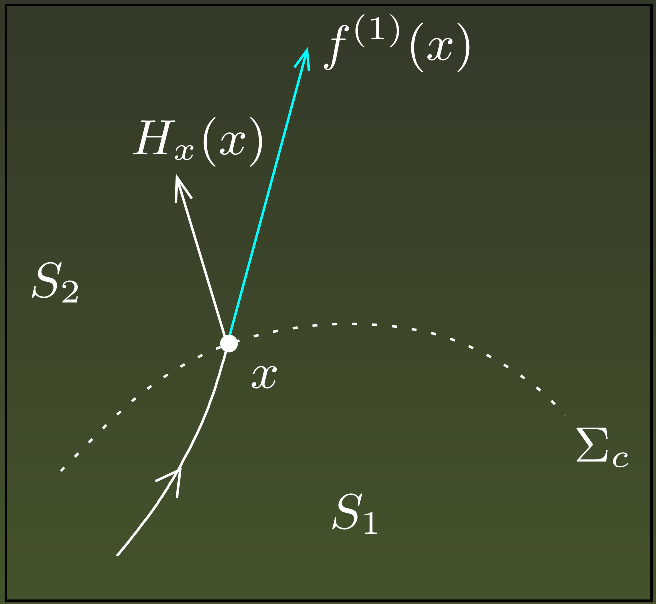

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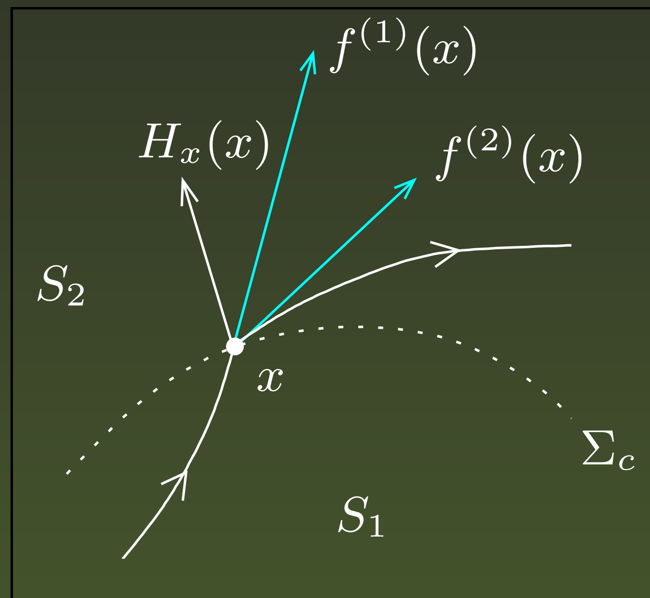

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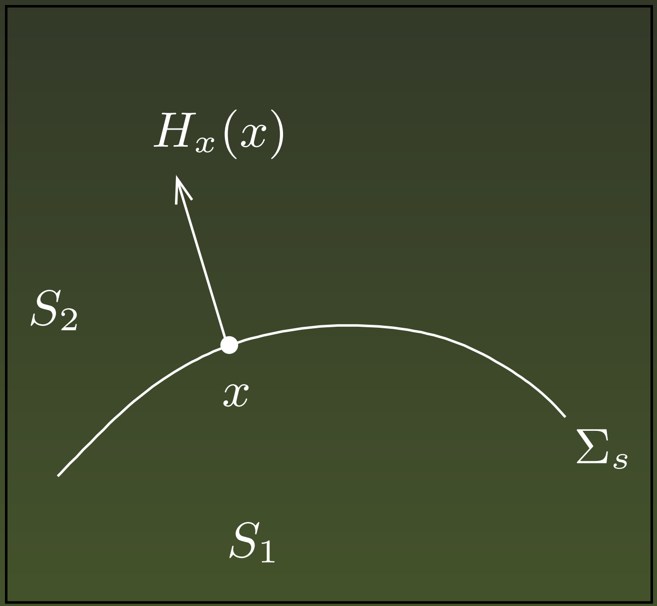
Filippov sliding orbits

On $\Sigma_s = \{x \in \Sigma : \sigma(x) \leq 0\}$:



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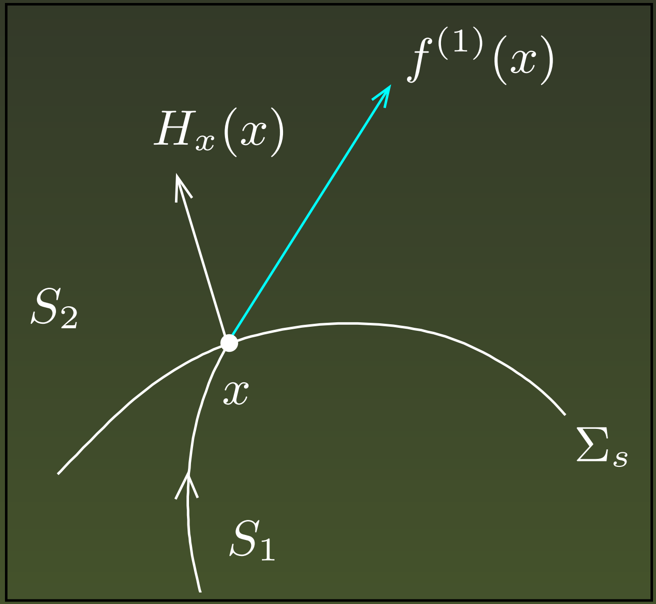

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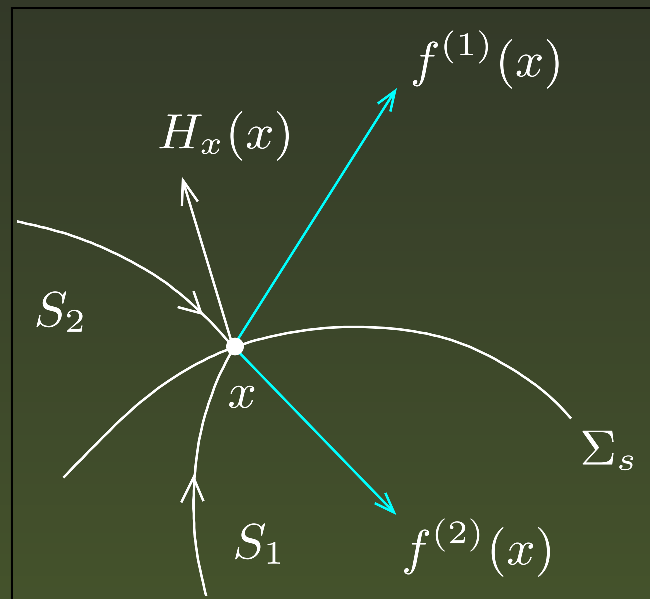

Filippov sliding orbits

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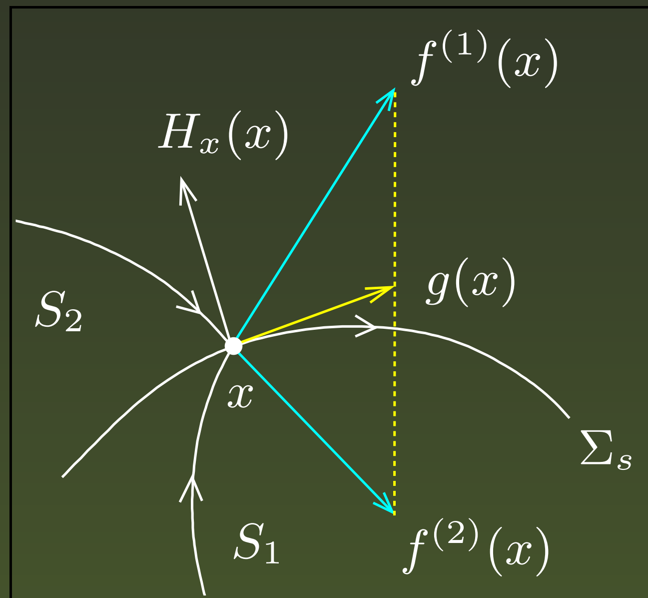

$$\dot{x} = g(x), \quad x \in \Sigma_s,$$

where $g(x) = \lambda f^{(1)}(x) + (1 - \lambda) f^{(2)}(x)$ with

$$\lambda = \frac{\langle H_x(x), f^{(2)}(x) \rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x) \rangle}.$$

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Special sliding points

- **Singular sliding points:**

$$x \in \Sigma_s, \ \langle H_x(x), f^{(2)}(x) - f^{(1)}(x) \rangle = 0, \quad f^{(1,2)}(x) \neq 0.$$

Special sliding points

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$$x \in \Sigma_s,\ \langle H_x(x), f^{(2)}(x) - f^{(1)}(x)\rangle = 0,\quad f^{(1,2)}(x) \neq 0.$$

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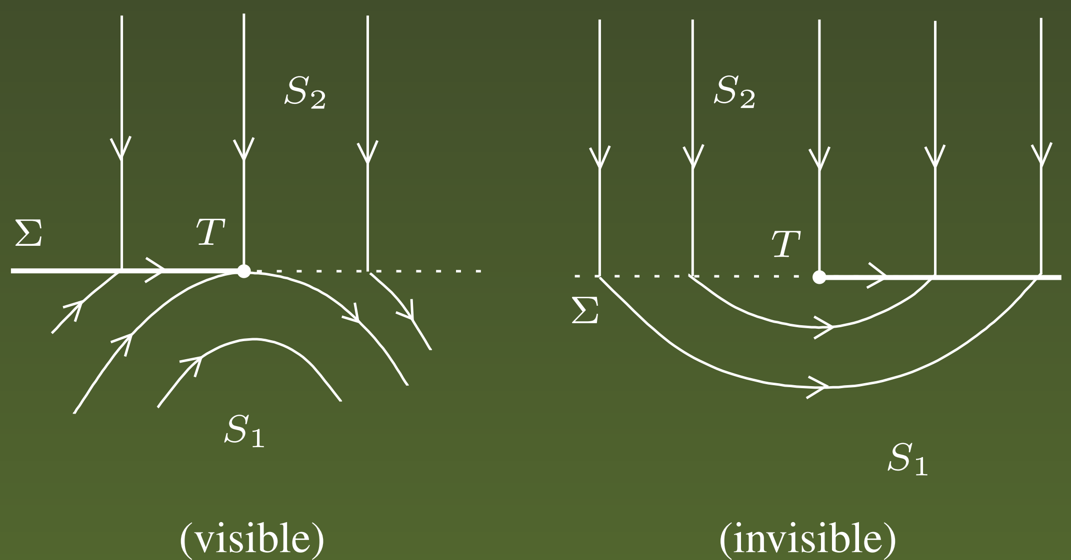

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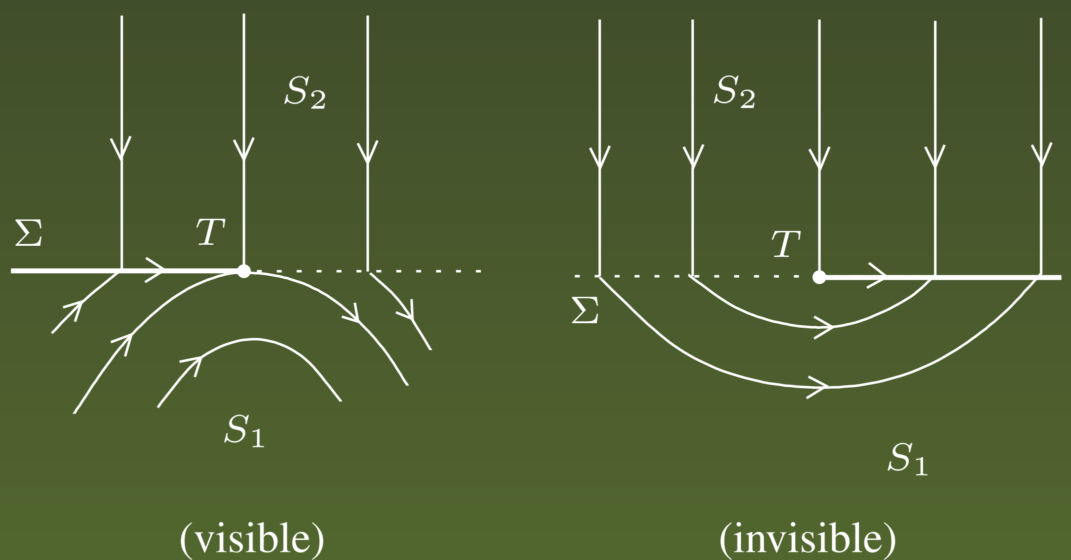

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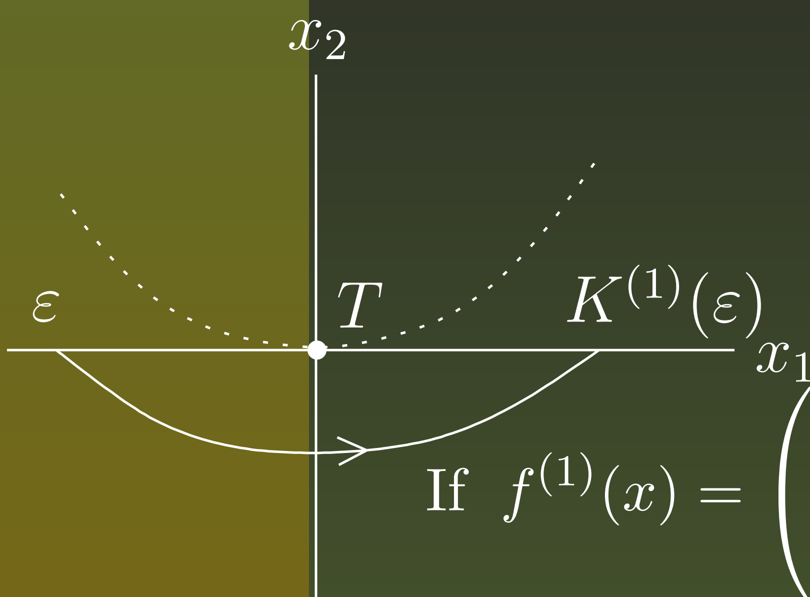
Quadratic tangent point T



$$x_2 = \frac{1}{2}\nu x_1^2 + O(x_1^3)$$

If $$f^{(1)}(x) = \begin{pmatrix} p + ax_1 + bx_2 + \cdots \\ cx_1 + dx_2 + \frac{1}{2}qx_1^2 + rx_1x_2 + \frac{1}{2}sx_2^2 + \cdots \end{pmatrix},$$

then $\nu = \dfrac{c}{p}$ and

$$K^{(1)}(\varepsilon) = -\varepsilon + k_2^{(1)}\varepsilon^2 + O(\varepsilon^3), \quad k_2^{(1)} = \frac{2}{3}\left(\frac{a+c}{p} - \frac{q}{2c}\right).$$

[Filippov, 1988]

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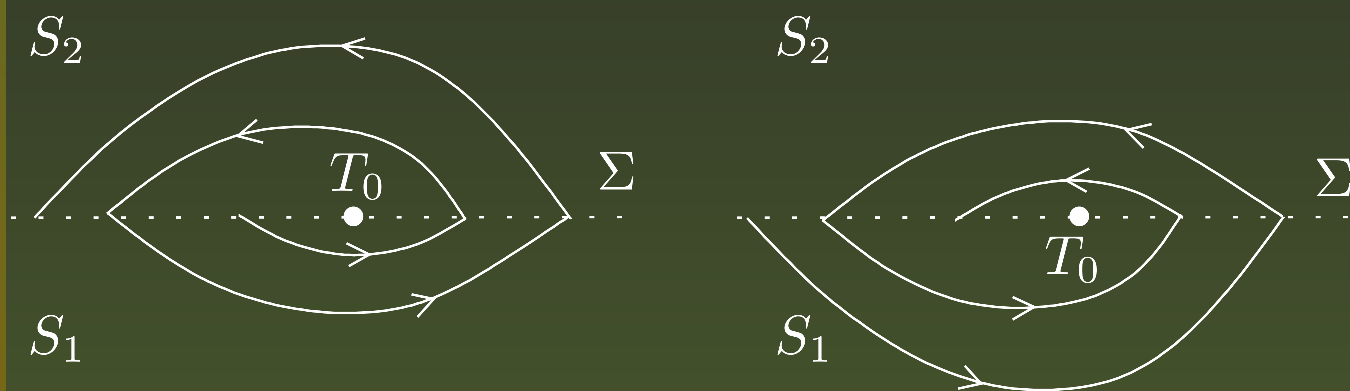
Fused focus (singular sliding point)

When two invisible tangent points coincide, define the Poincaré map:

$$P(\varepsilon) = \varepsilon + k_2\varepsilon^2 + O(\varepsilon^3), \quad k_2 = k_2^{(1)} - k_2^{(2)}.$$



unstable $(k_2 > 0)$ stable $(k_2 < 0)$

2. Codim 1 local bifurcations

Collisions of

- standard equilibria with Σ
- tangent points
- pseudo-equilibria

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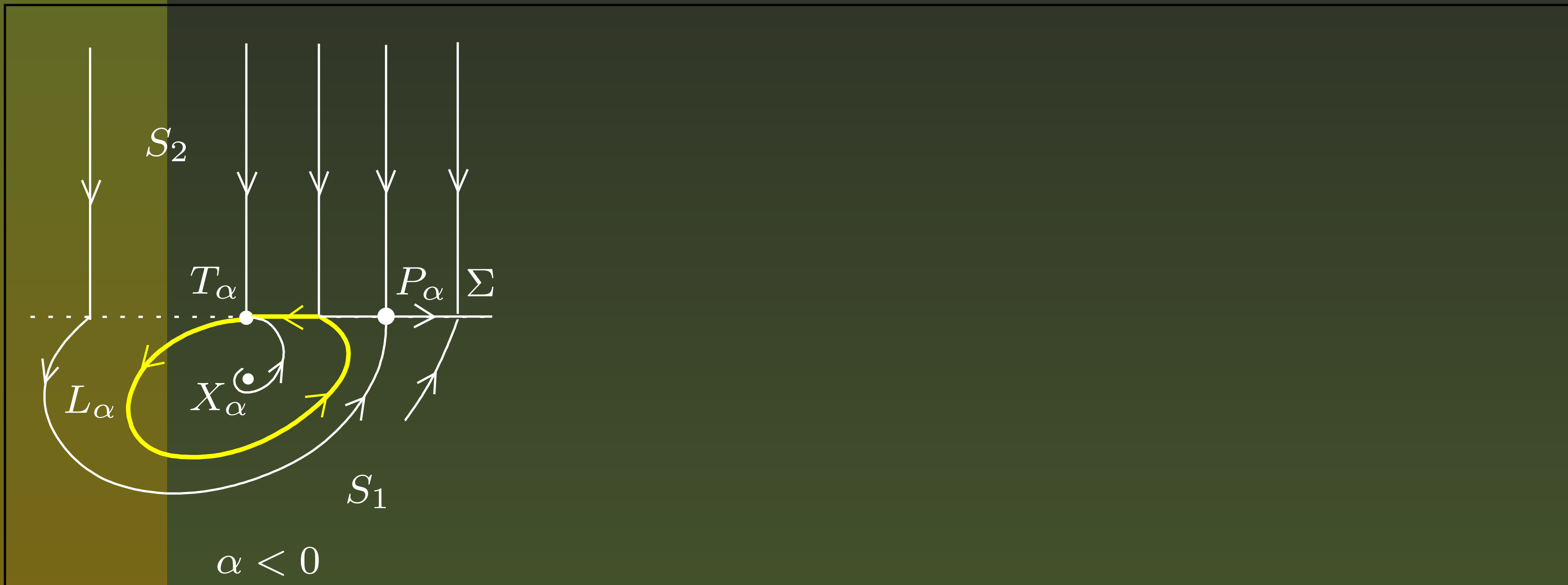
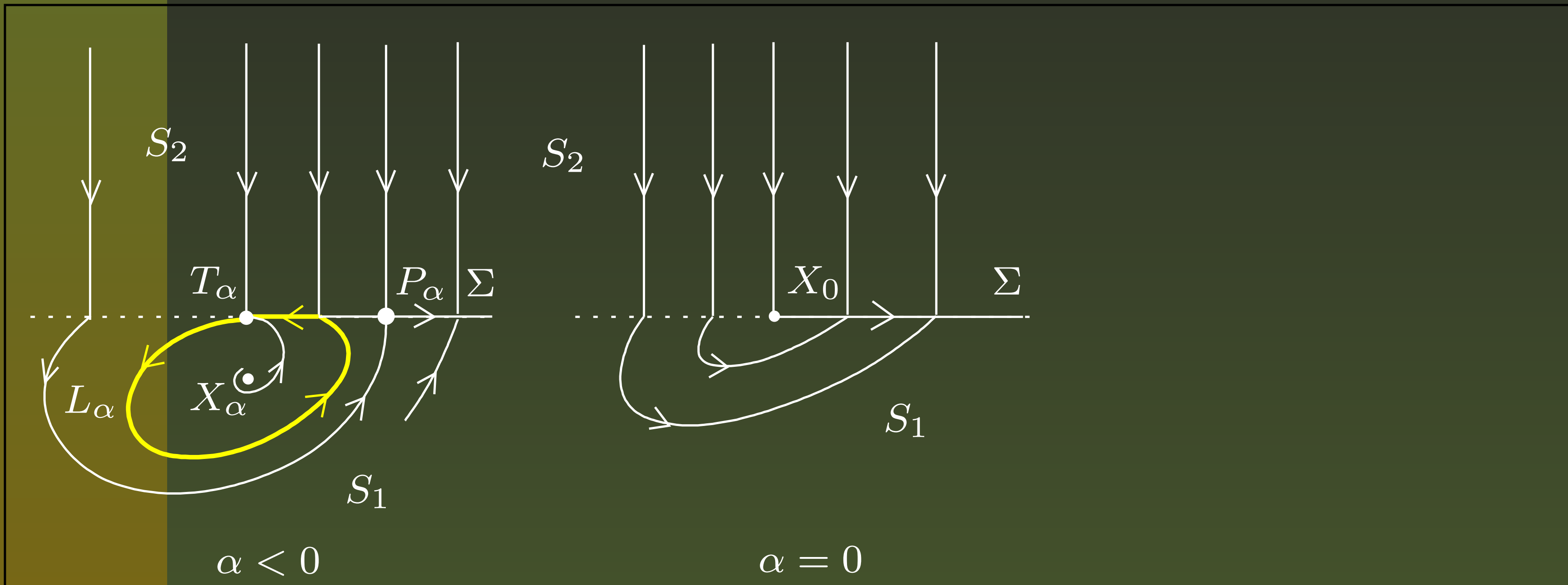
Boundary focus: BF_1



Boundary focus: BF_1



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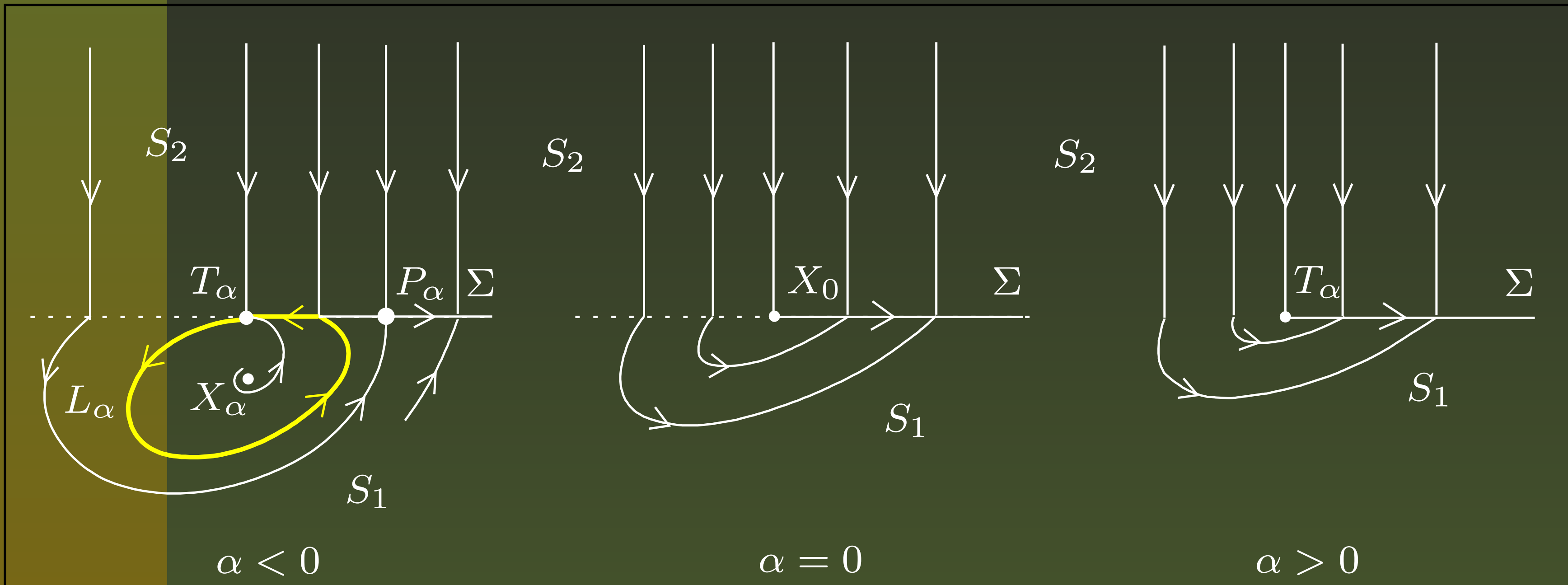
Boundary focus: BF_1



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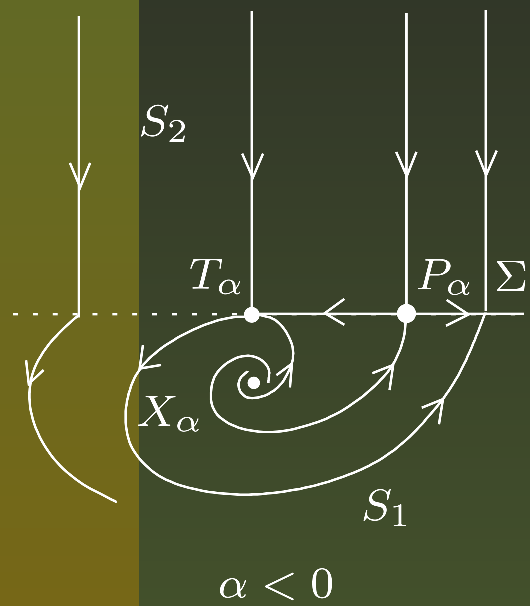
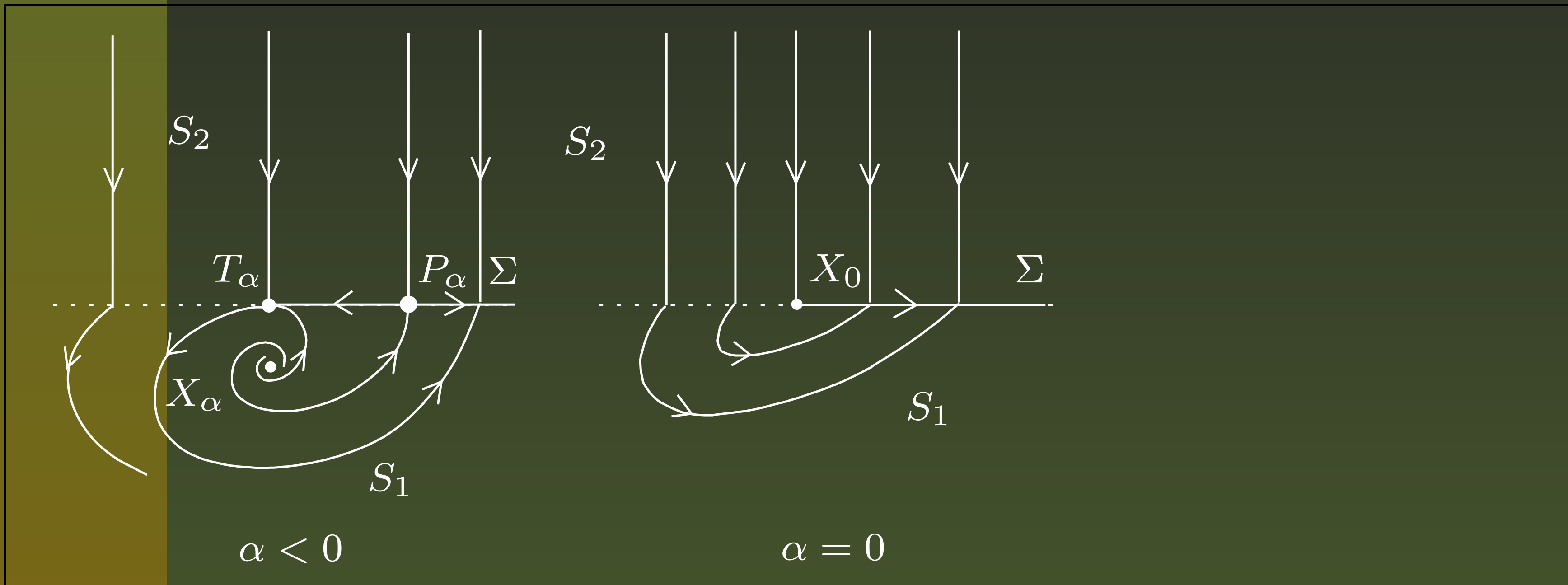
Boundary focus: BF_2



Boundary focus: BF_2



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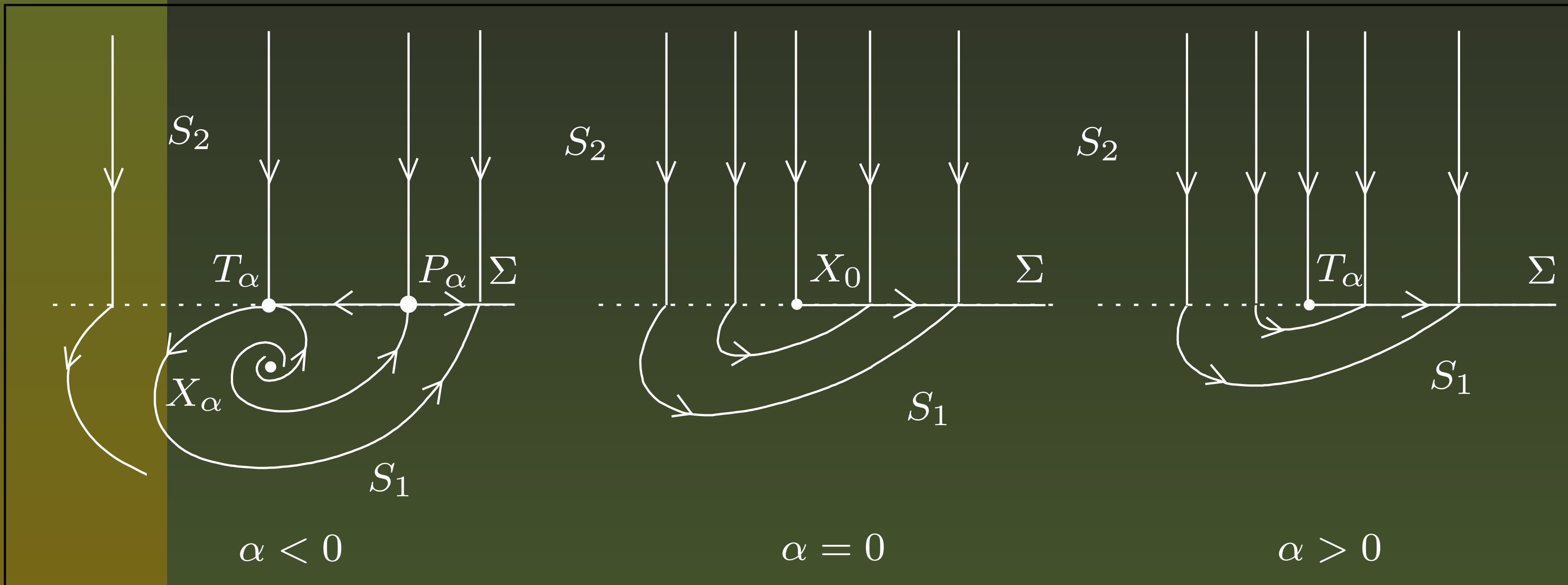
Boundary focus: BF_2



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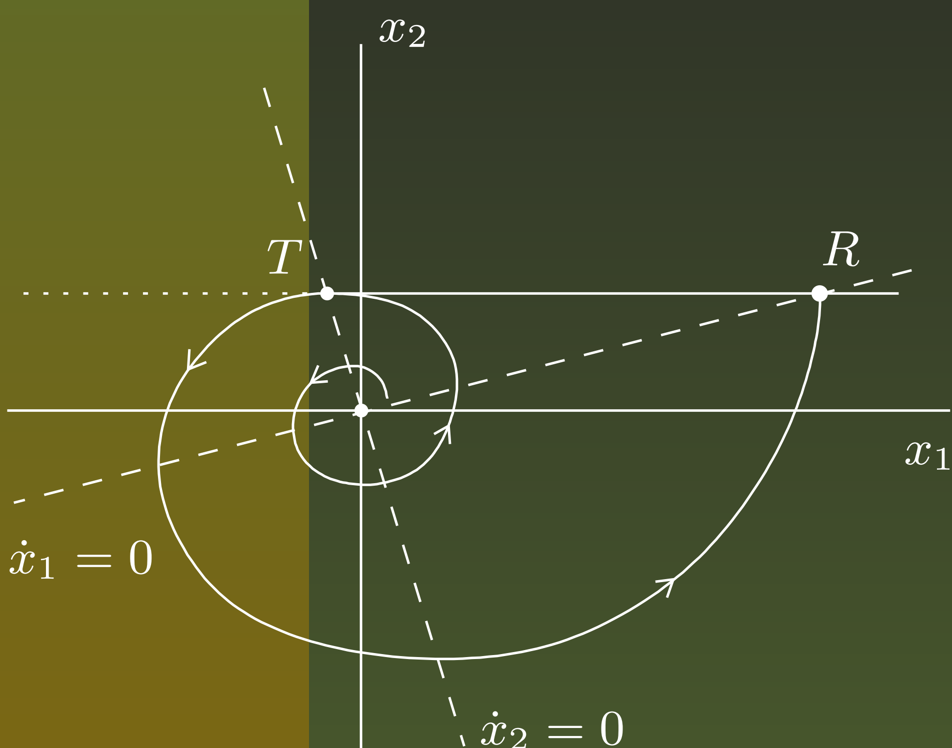
Degenerate boundary focus: BDF



$$\begin{cases} \dot{x}_1 &= ax_1 + bx_2, \\ \dot{x}_2 &= cx_1 + dx_2, \end{cases}$$

$$T = \left(-\frac{d}{c}, 1\right), \quad R = \left(-\frac{b}{a}, 1\right).$$

$$\frac{d-a}{2\omega} \operatorname{tg}\left[\frac{\omega}{a+d} \ln\left(-\frac{bc}{a^2}\right)\right] = 1,$$

where $\omega = \frac{1}{2}\sqrt{-(a-d)^2 - 4bc}$.

Boundary focus: BF_3

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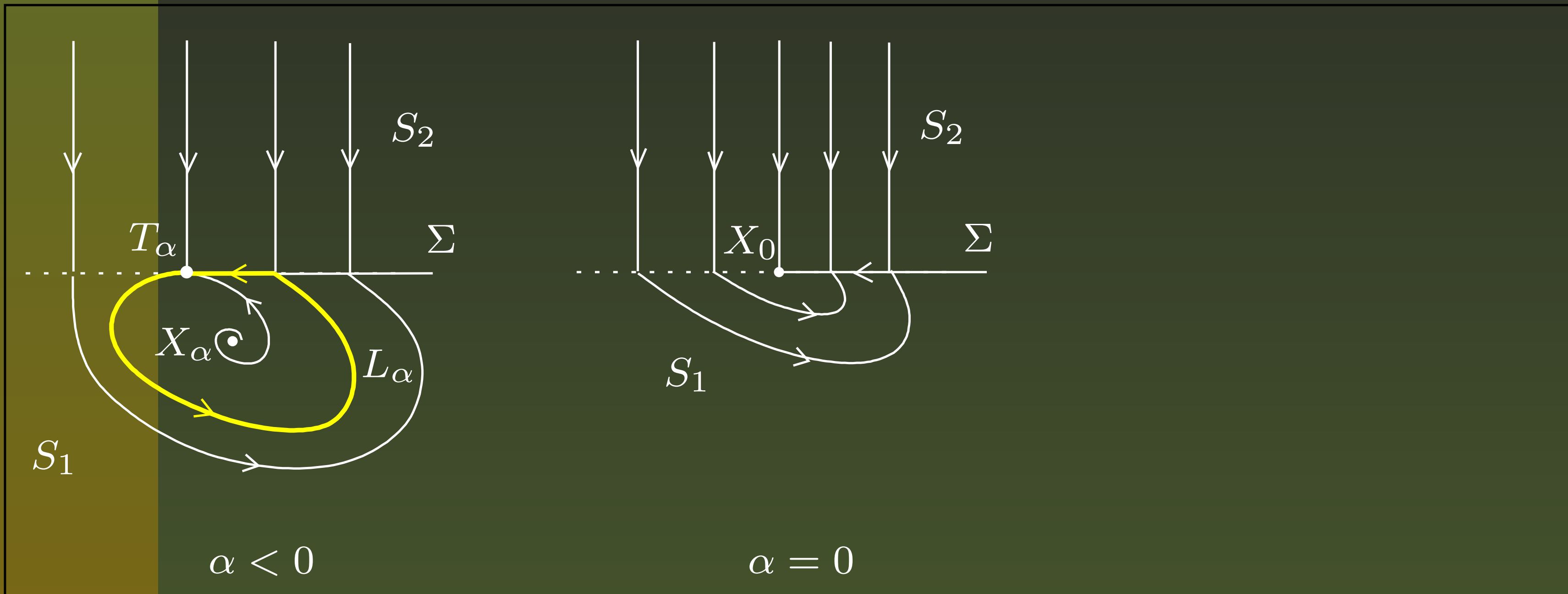
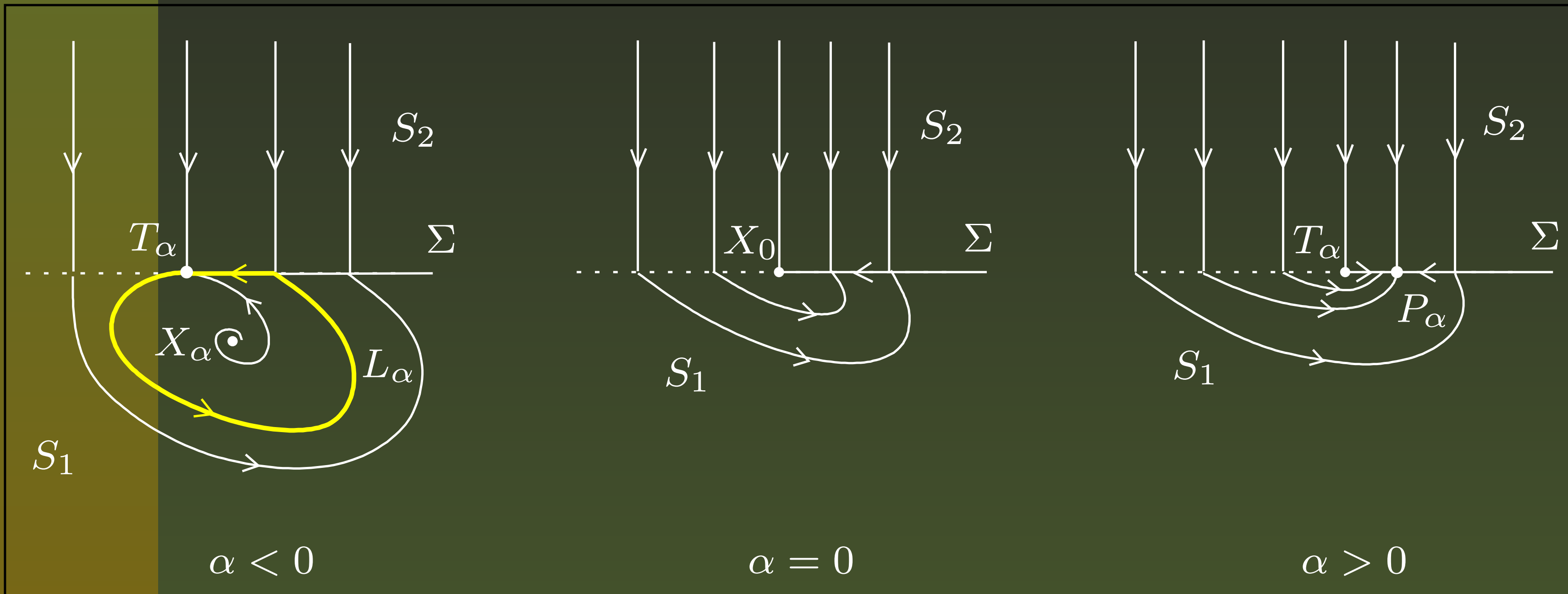


Boundary focus: BF_3



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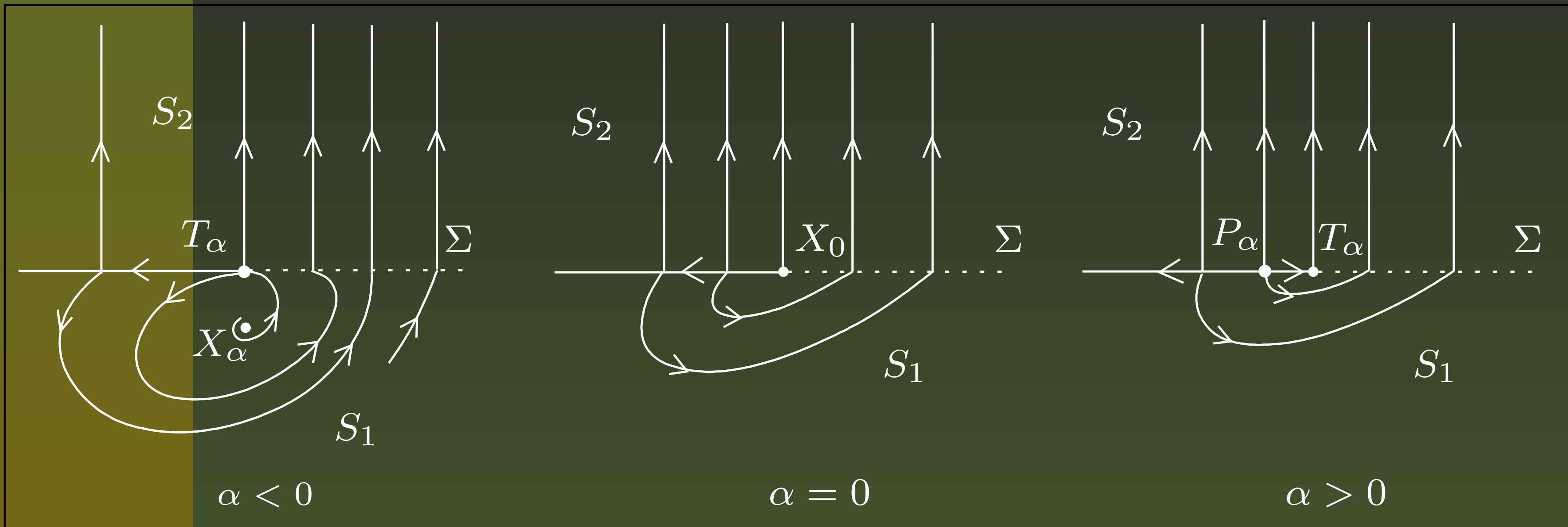
Boundary focus: BF_4



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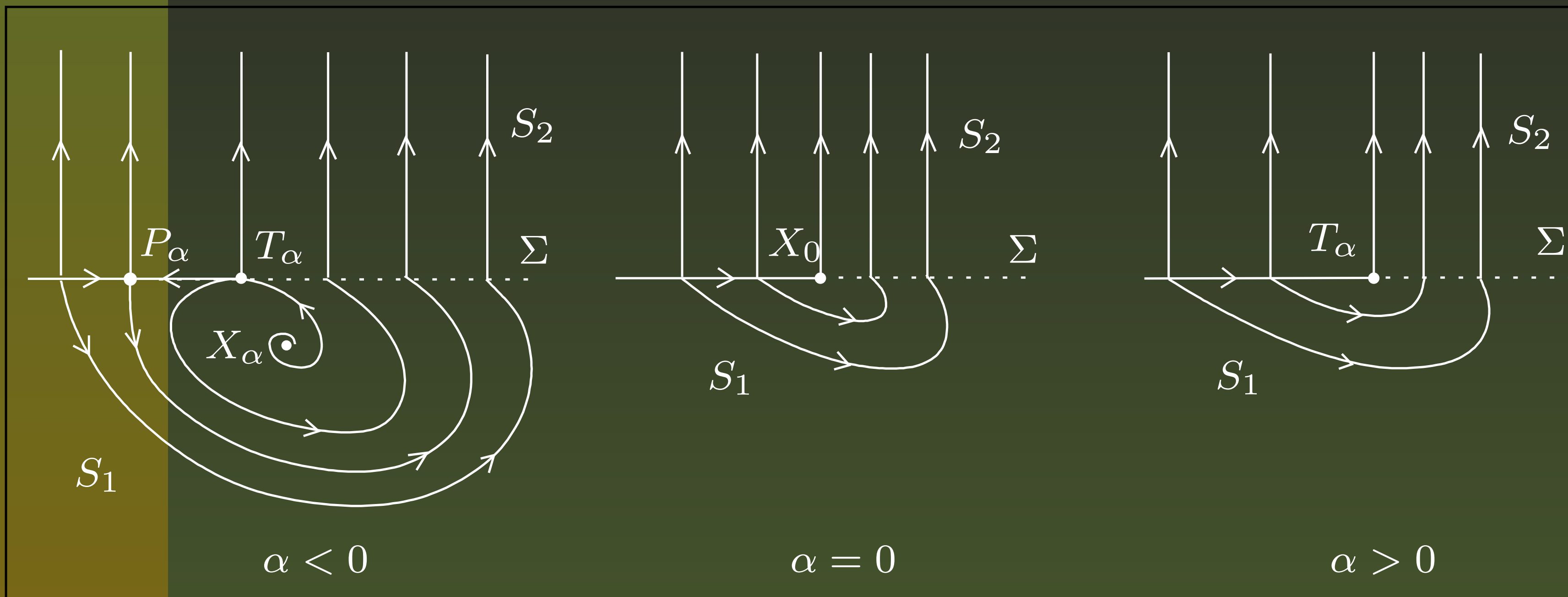
Boundary focus: BF_5



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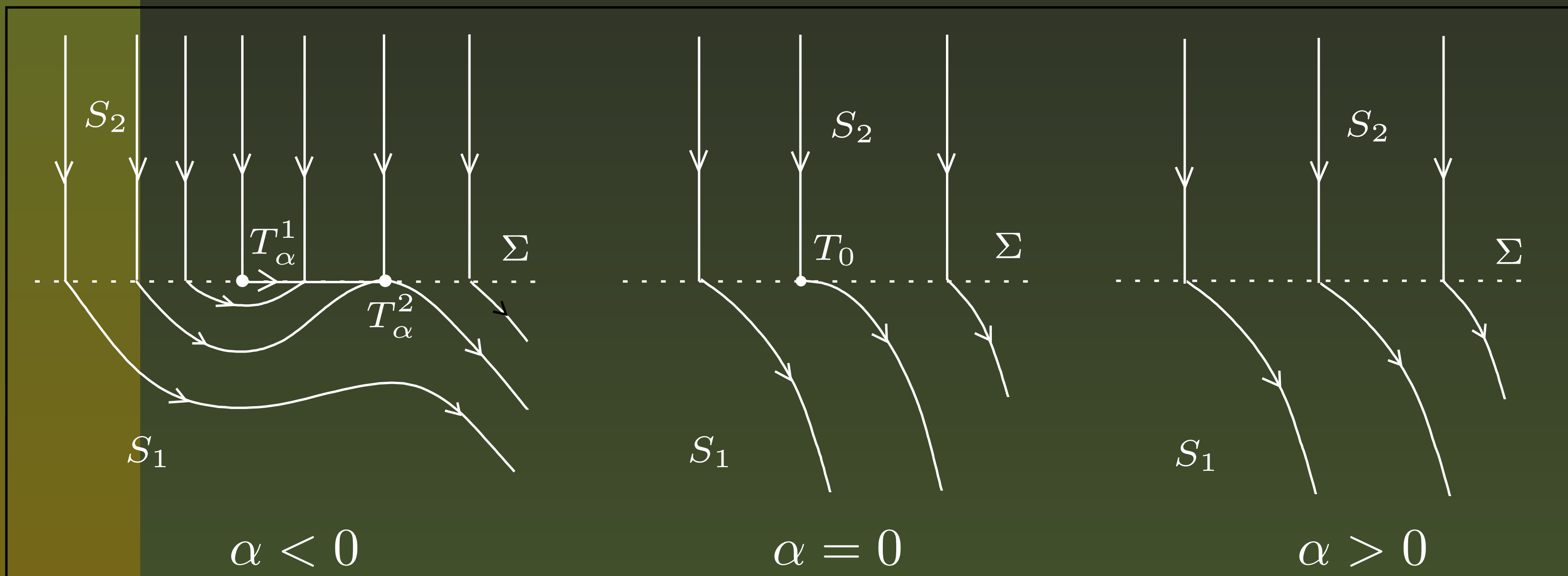
Double tangency: DT_1



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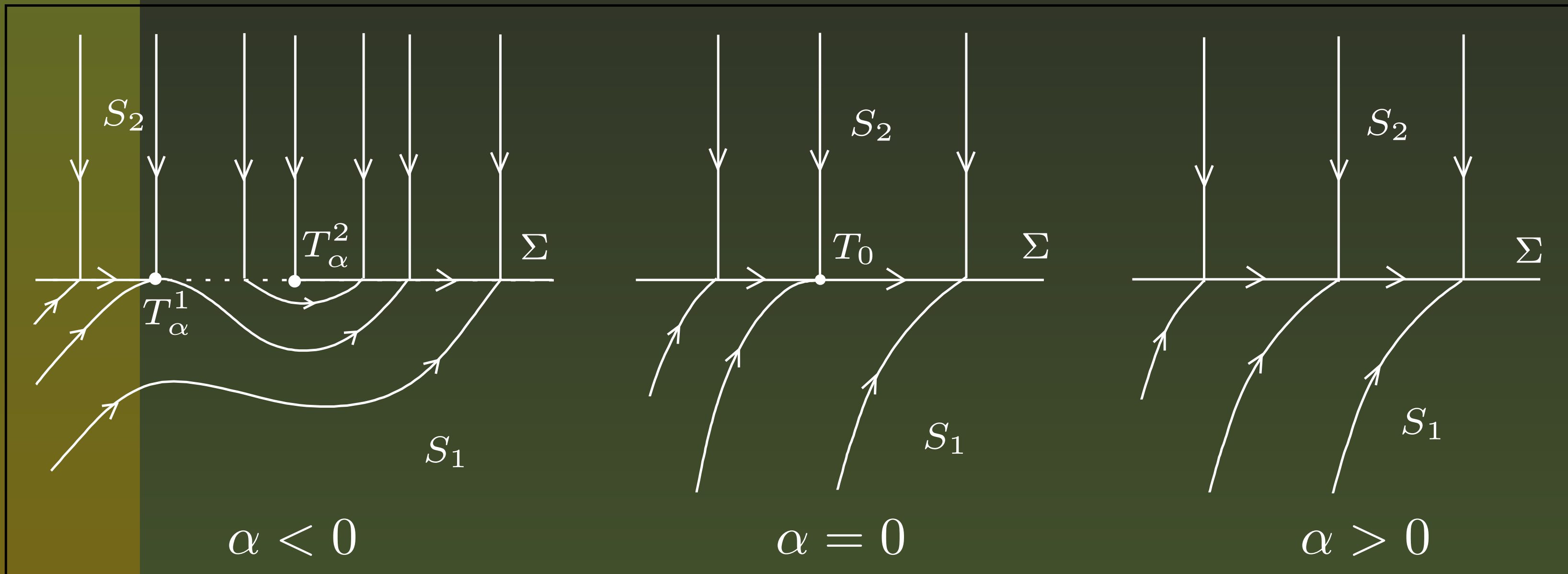
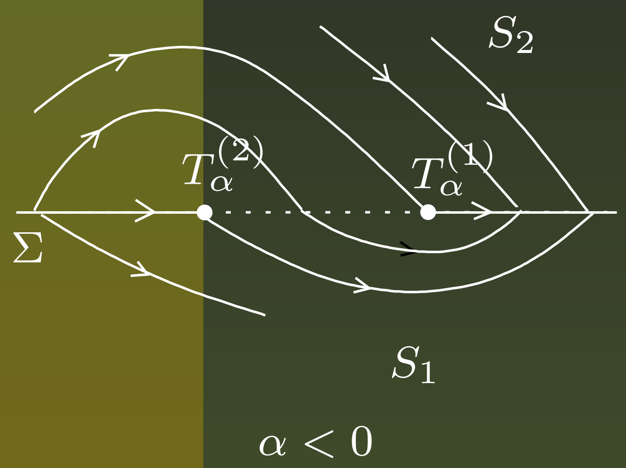
Double tangency: DT_2



Collision of two invisible tangencies: II_1



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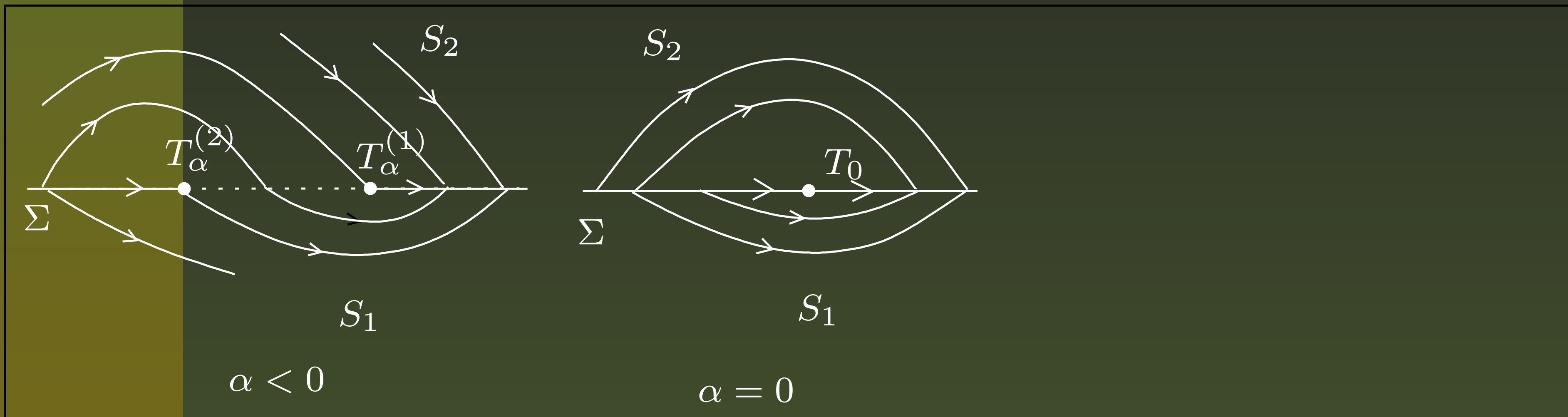

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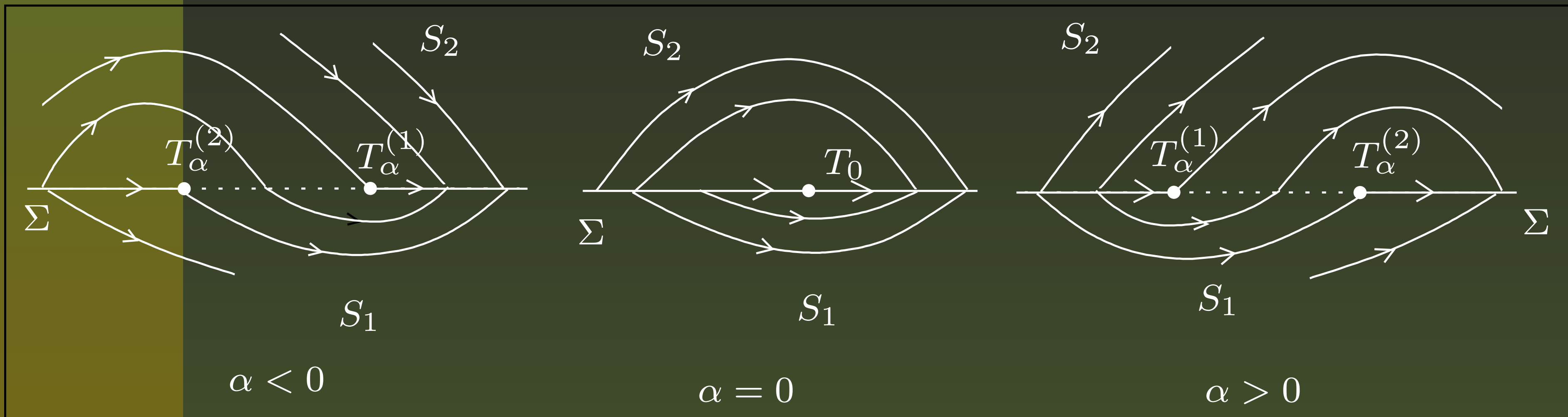

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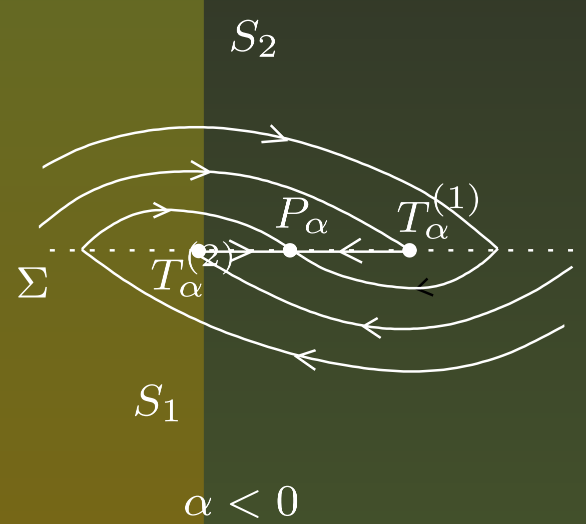
Collision of two invisible tangencies: II_2



[Gubar', 1971; Filippov, 1988]

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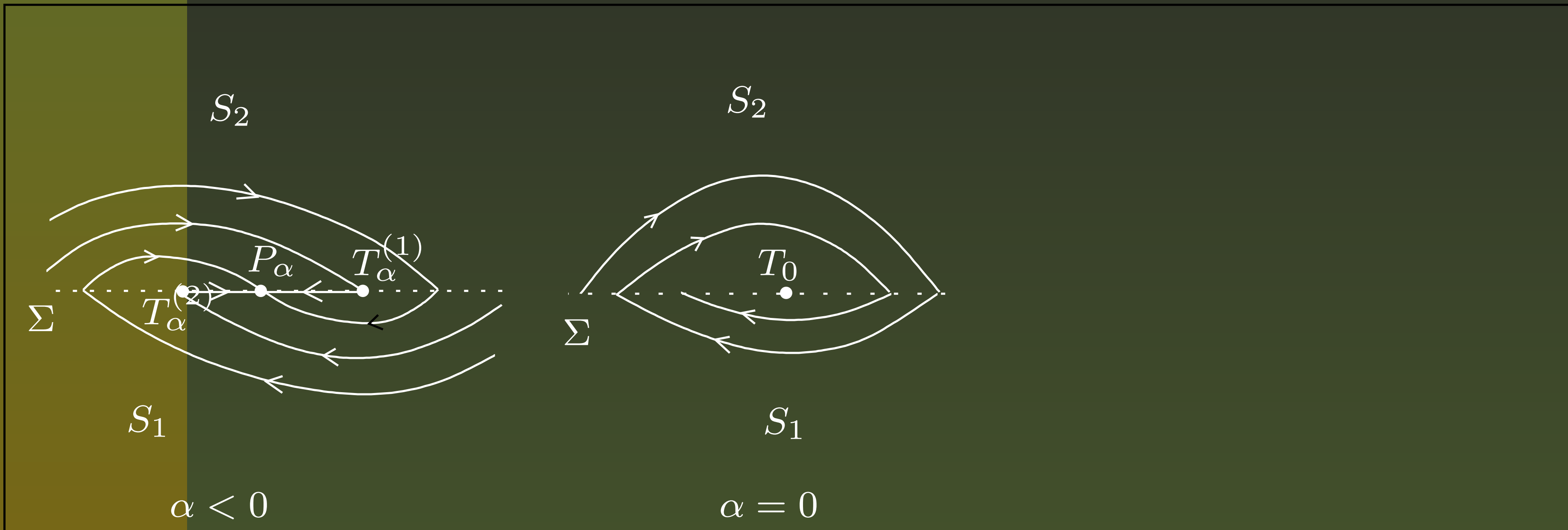

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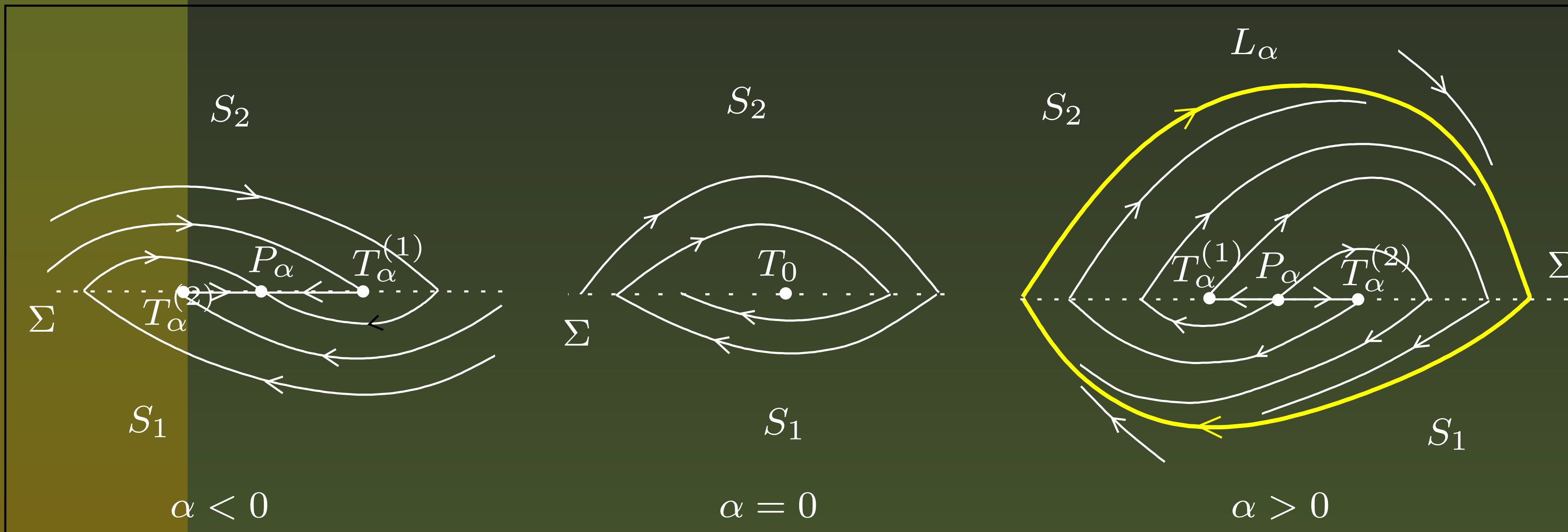

[Gubar', 1971; Filippov, 1988]

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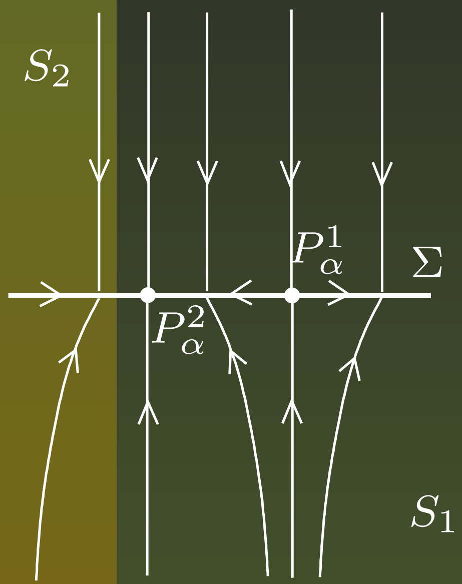
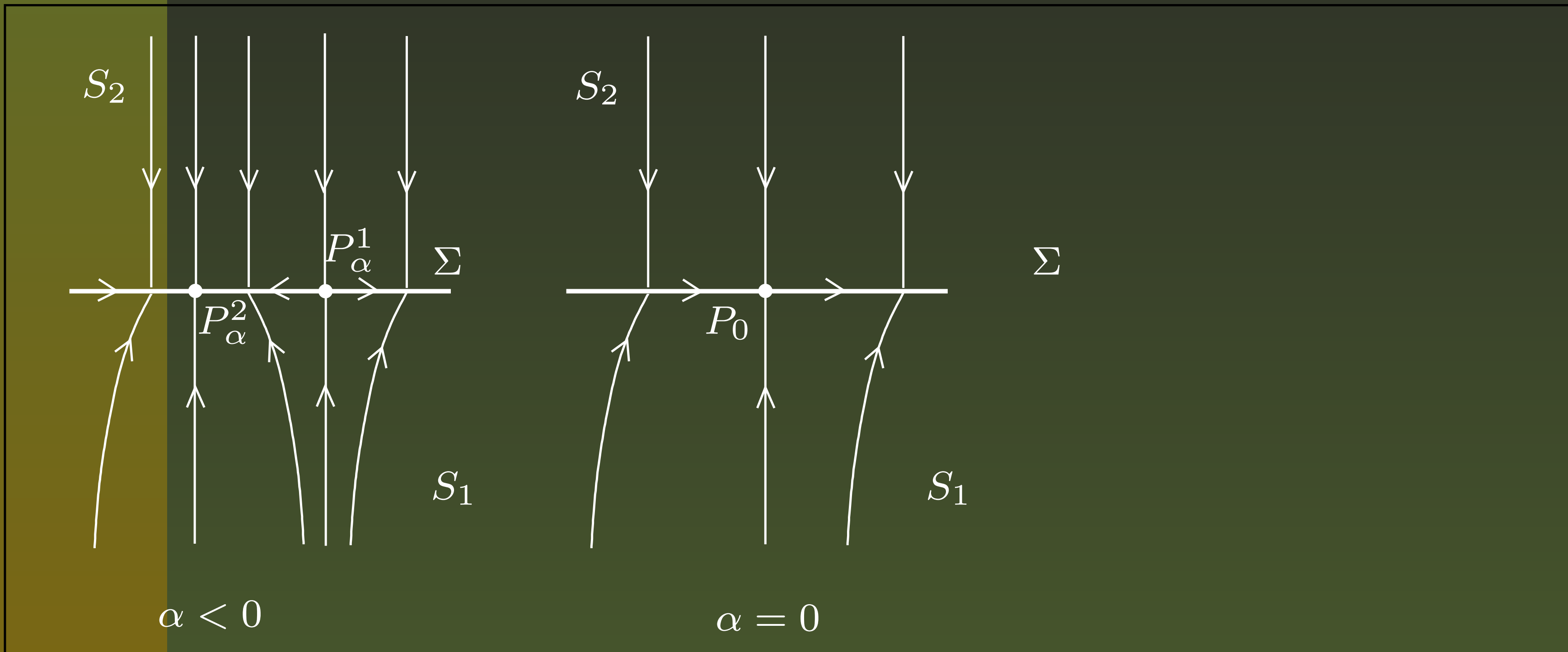
Pseudo-saddle-node: PSN



$\alpha < 0$

Pseudo-saddle-node: PSN



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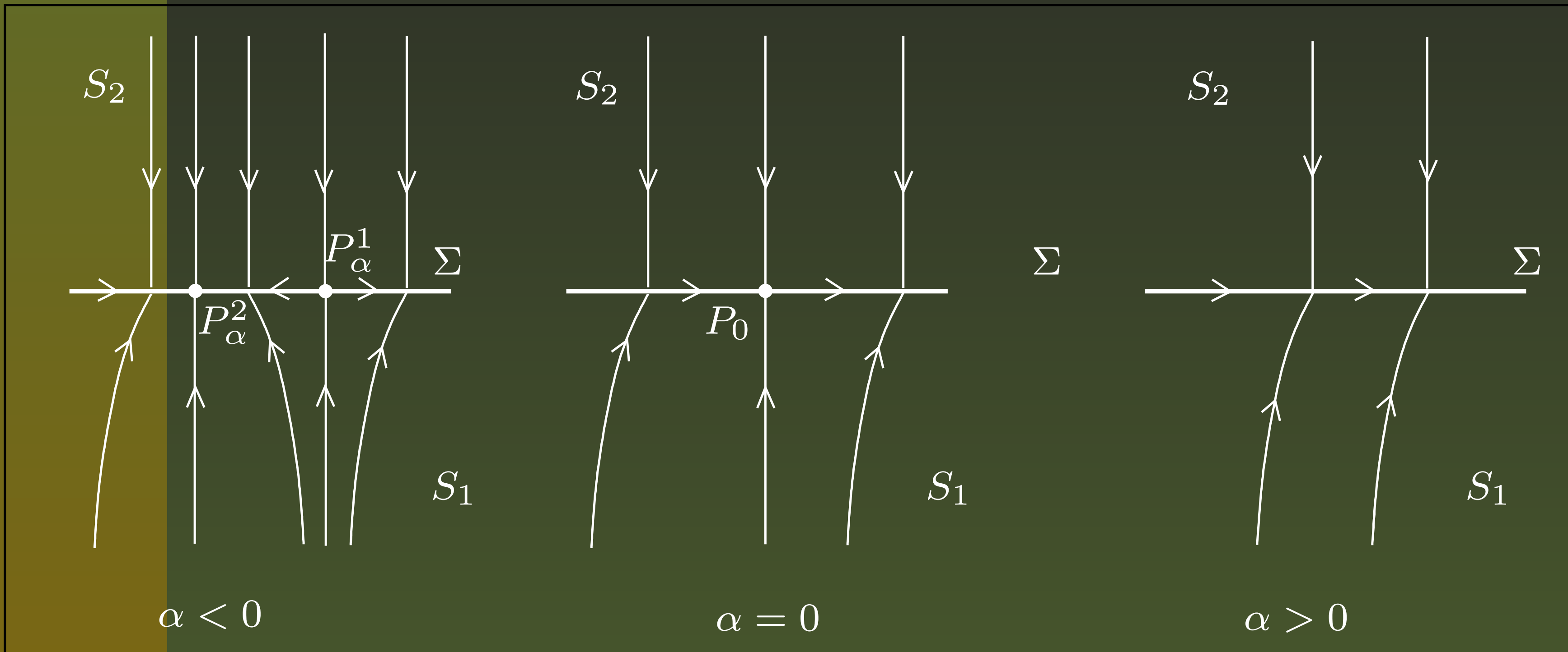

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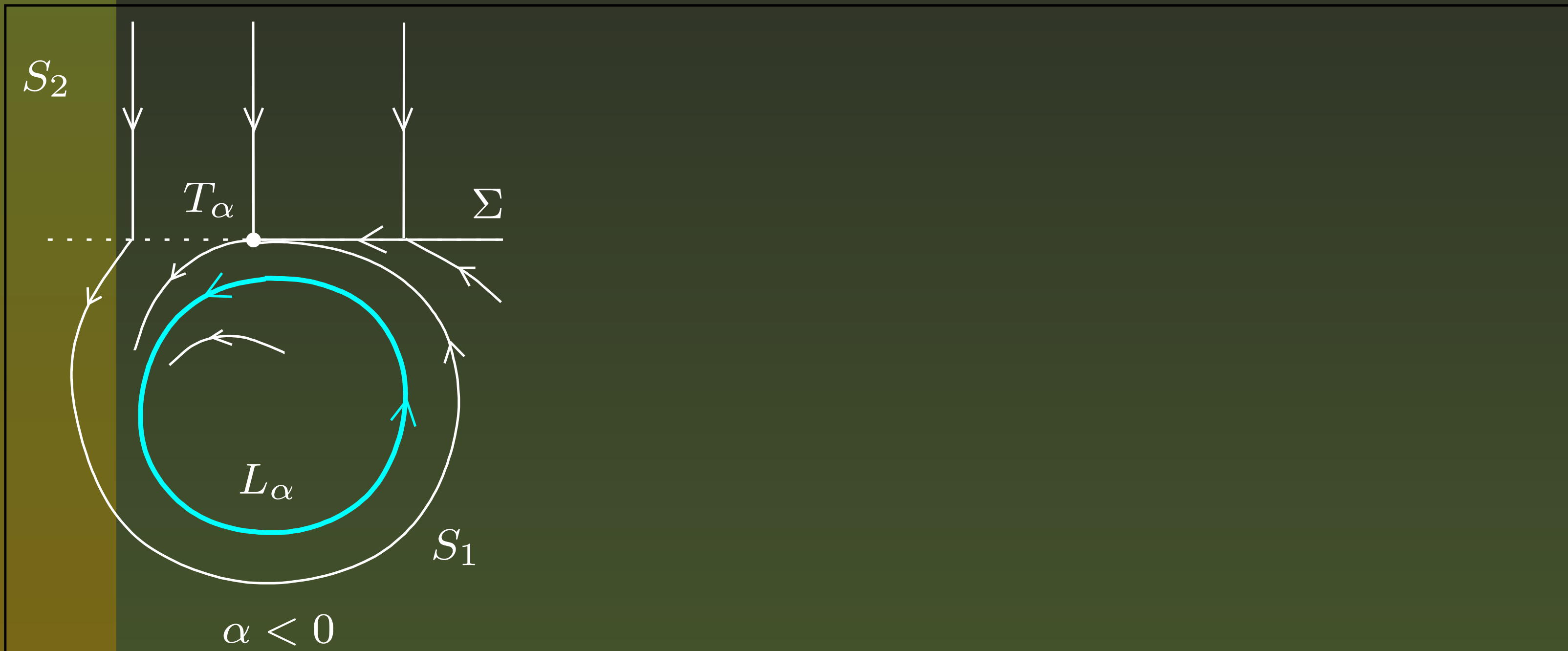
3. Codim 1 global bifurcations

- Bifurcations of sliding cycles:
  - Grazing-sliding
  - Adding-sliding
  - Switching-sliding
  - Crossing-sliding
- Pseudo-homoclinic bifurcations:
  - Homoclinic orbit to a pseudo-saddle-node
  - Homoclinic orbit to a pseudo-saddle
- Sliding homoclinic orbit to a saddle
- Pseudo-heteroclinic bifurcations

Grazing-sliding: TC_1



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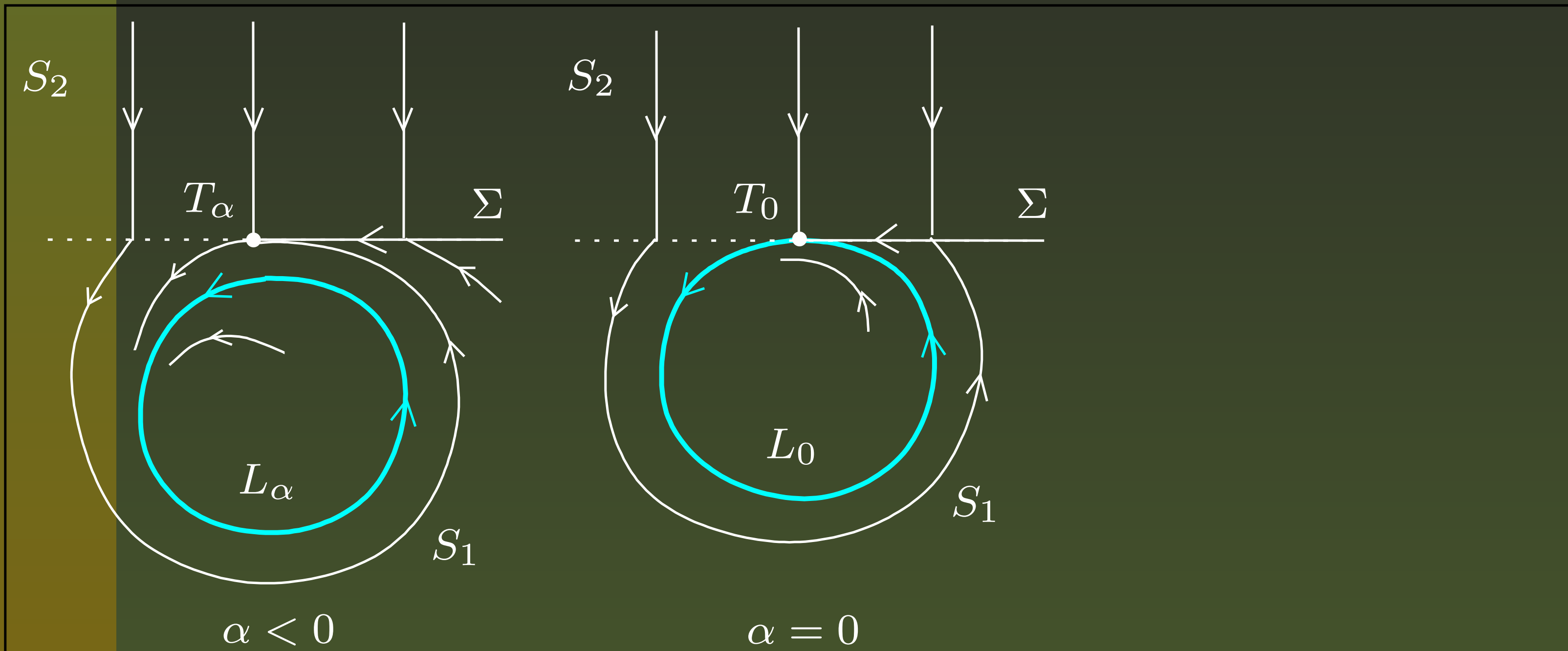
Grazing-sliding: TC_1



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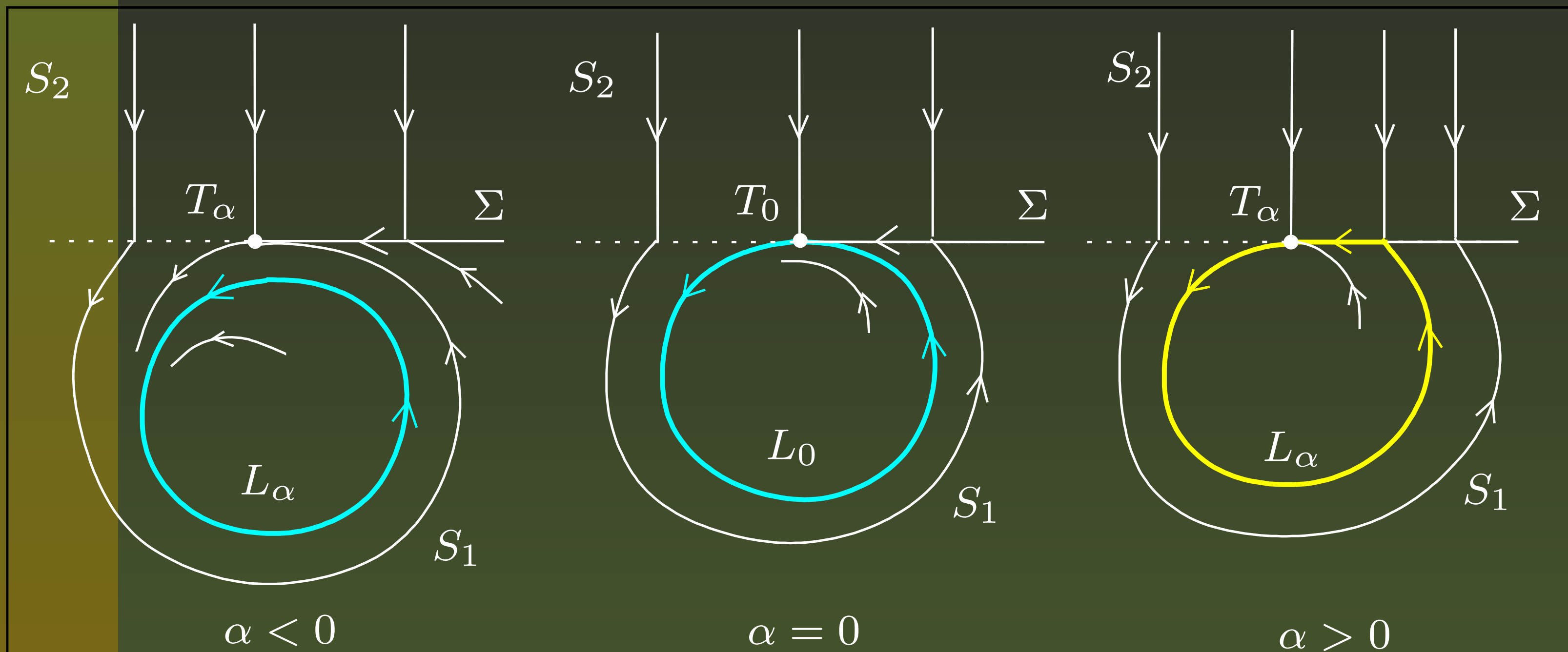

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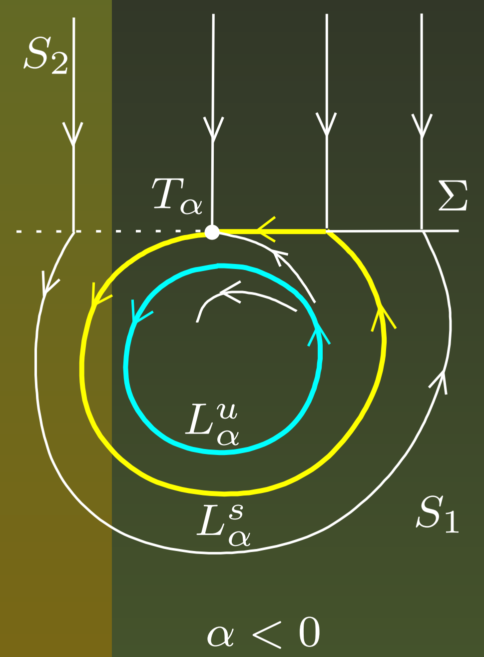
Grazing-sliding: TC_2



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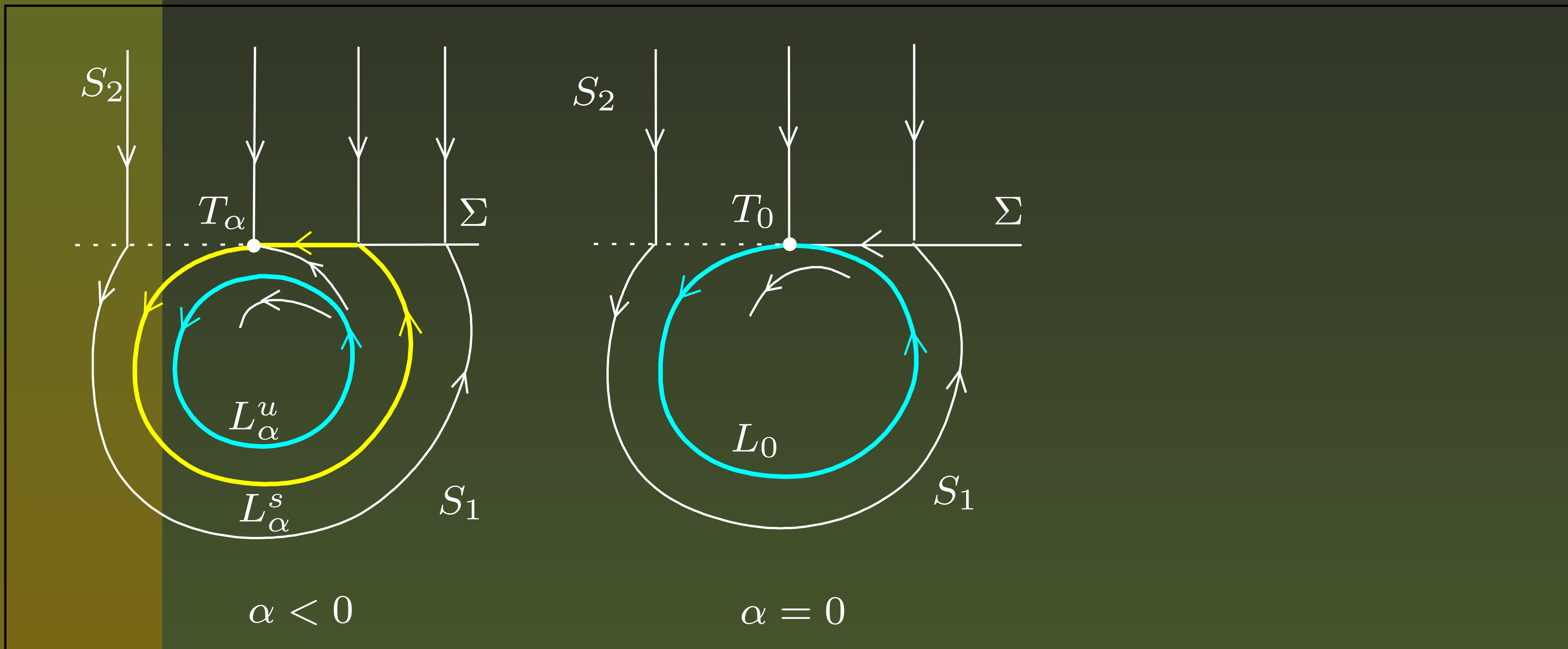

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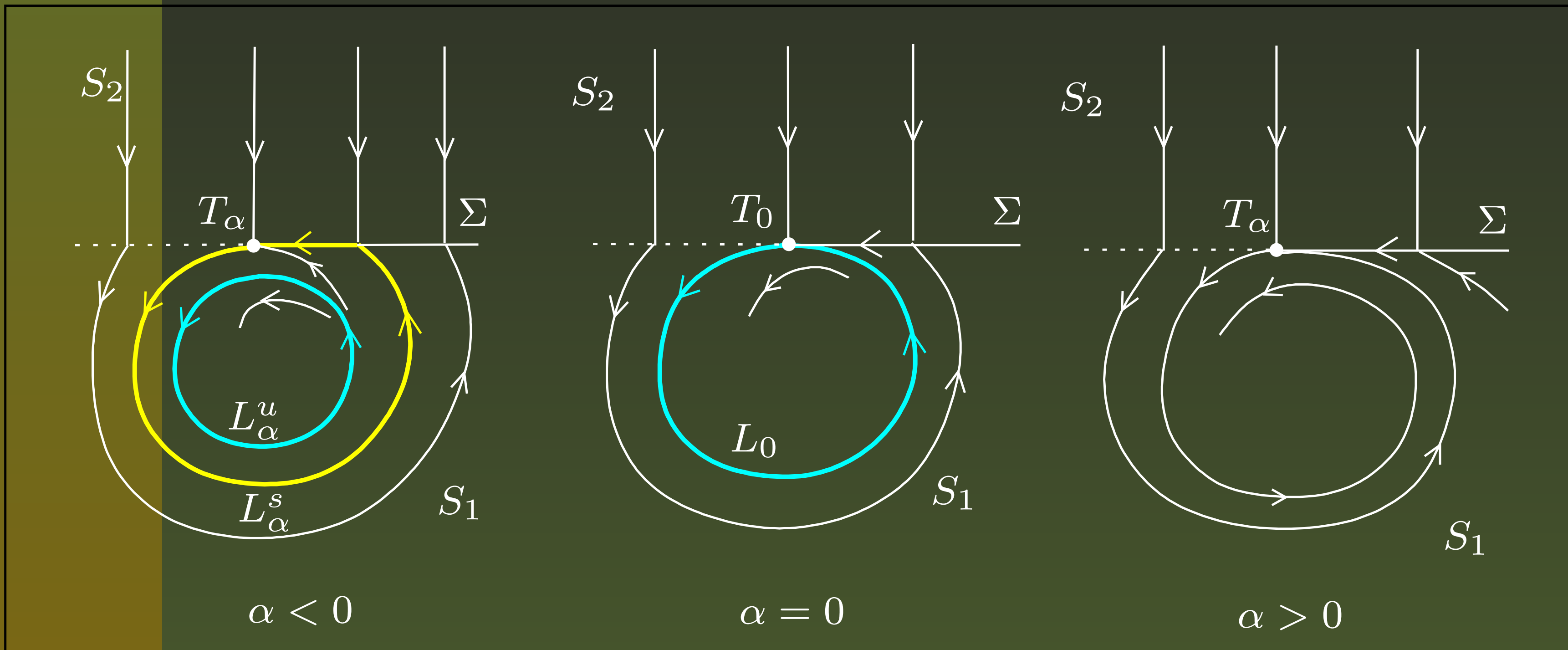

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Adding-sliding: DT_2 with global reinjection

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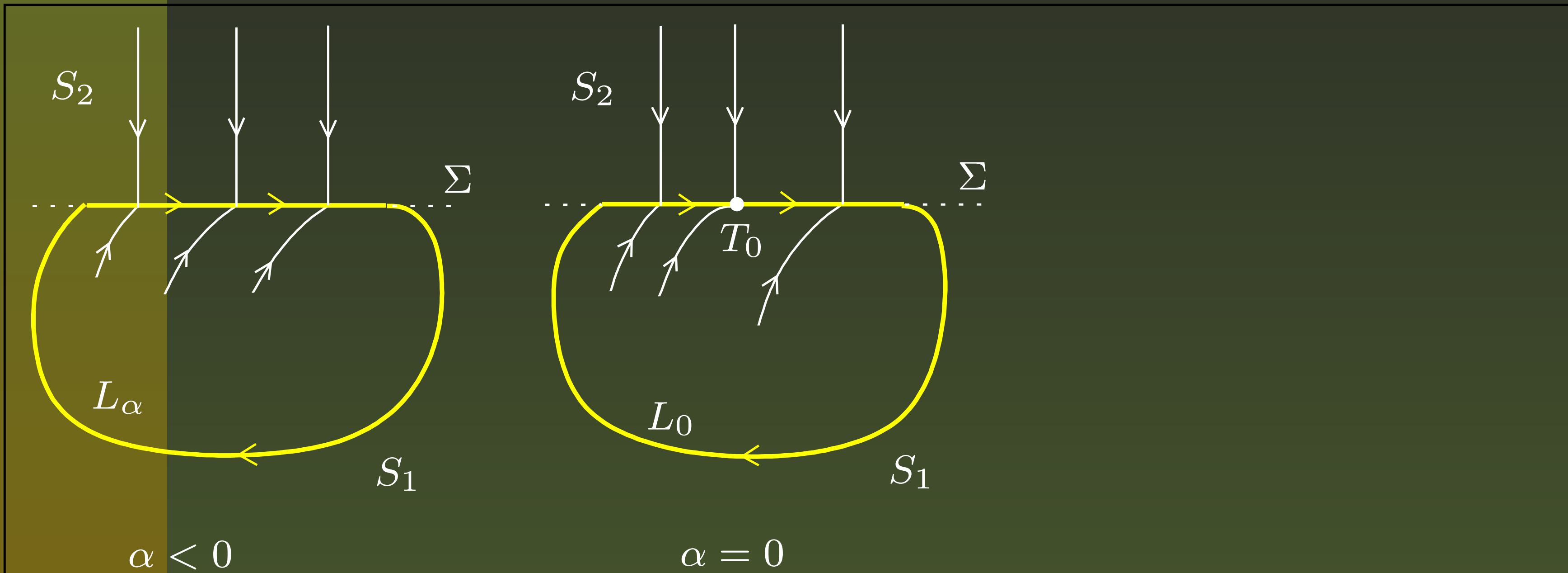

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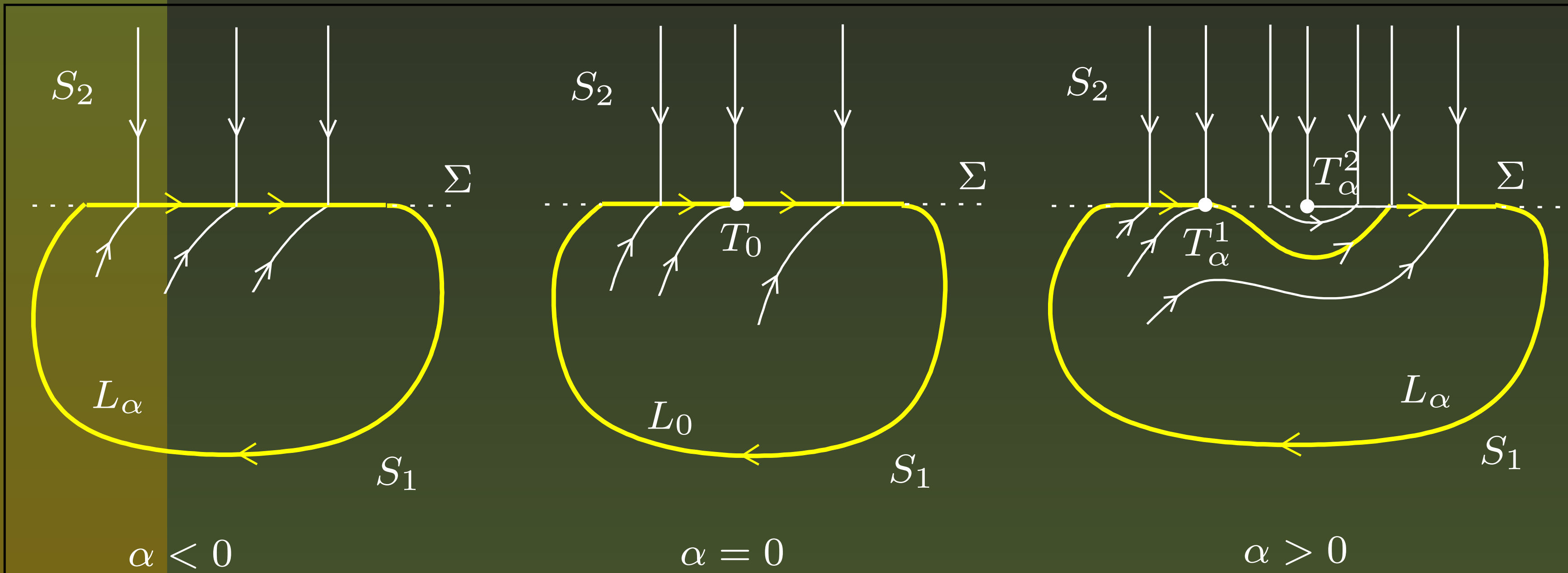
Adding-sliding: DT_2 with global reinjection



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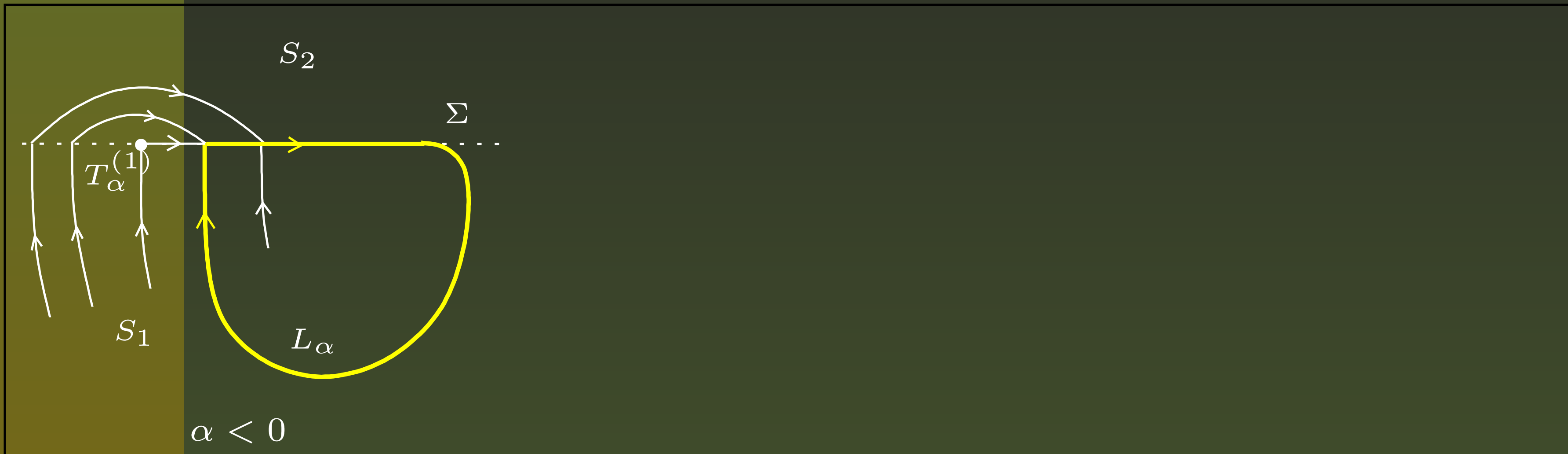
Switching-sliding



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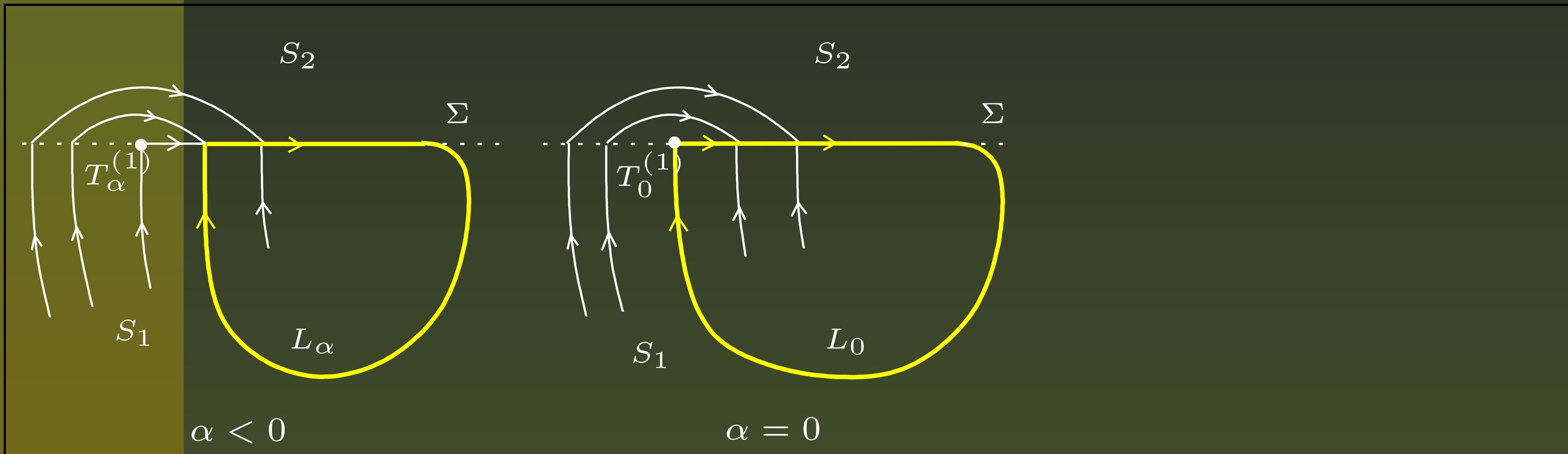

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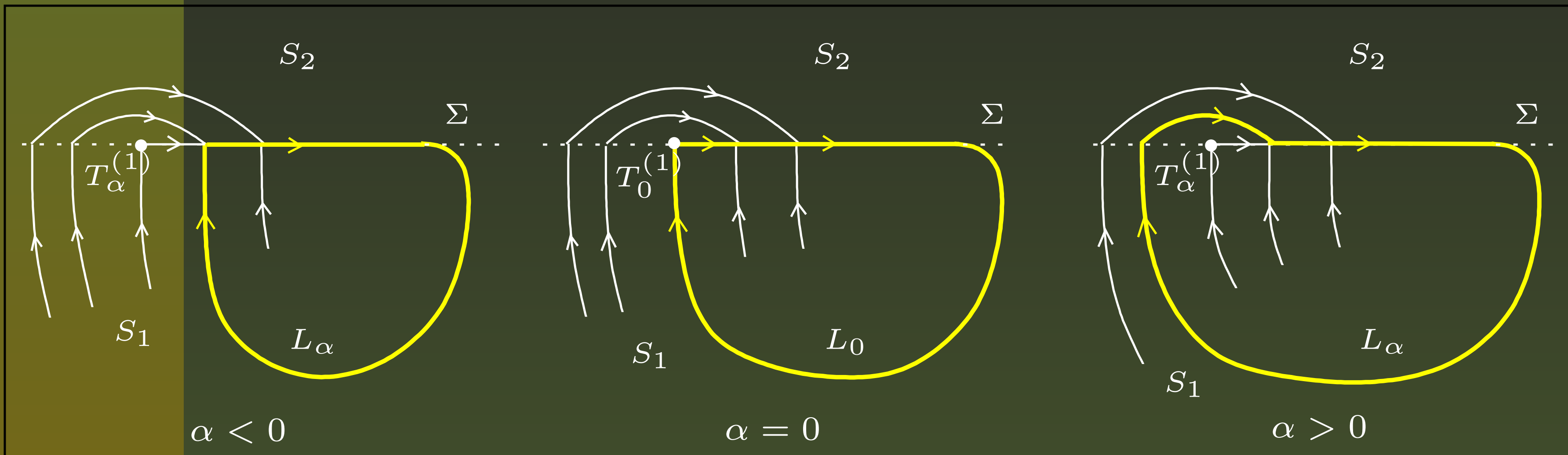

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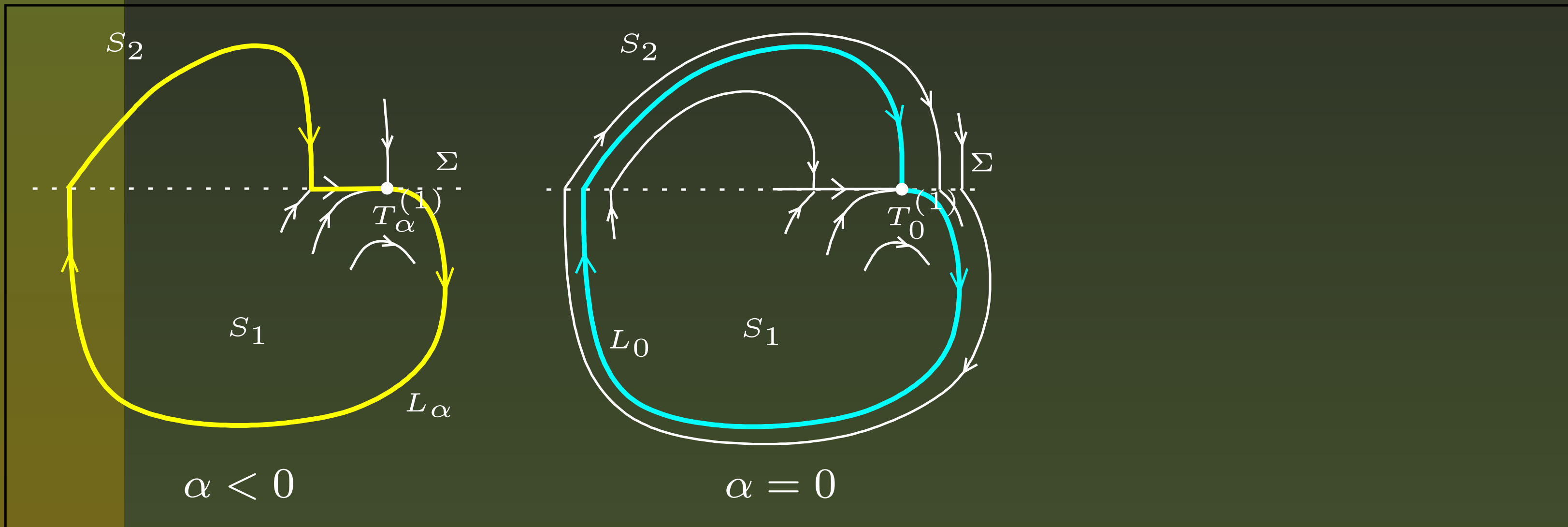
Crossing-sliding

Crossing-sliding



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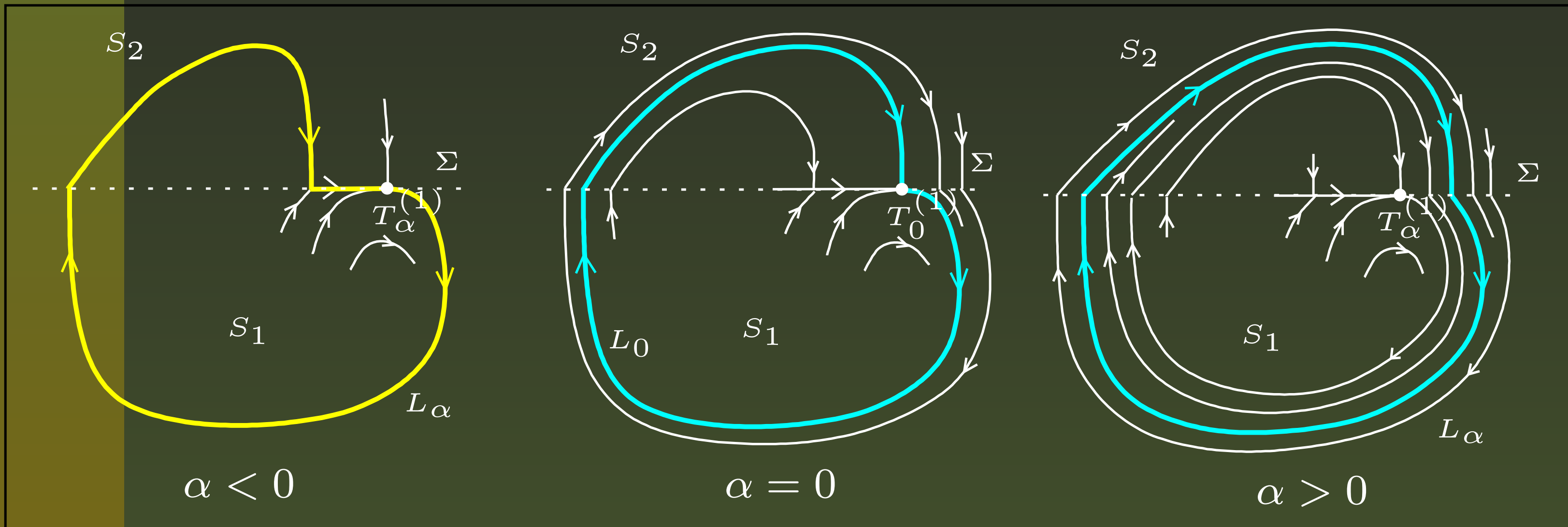

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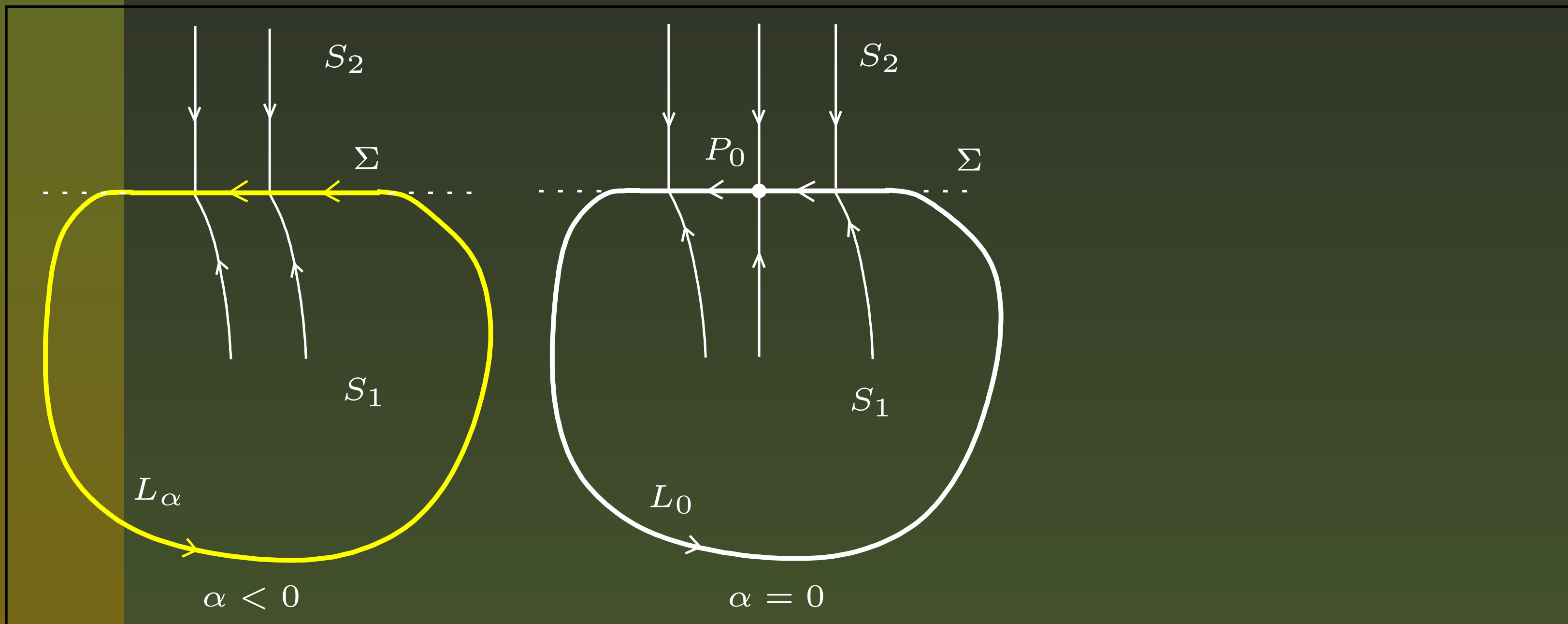
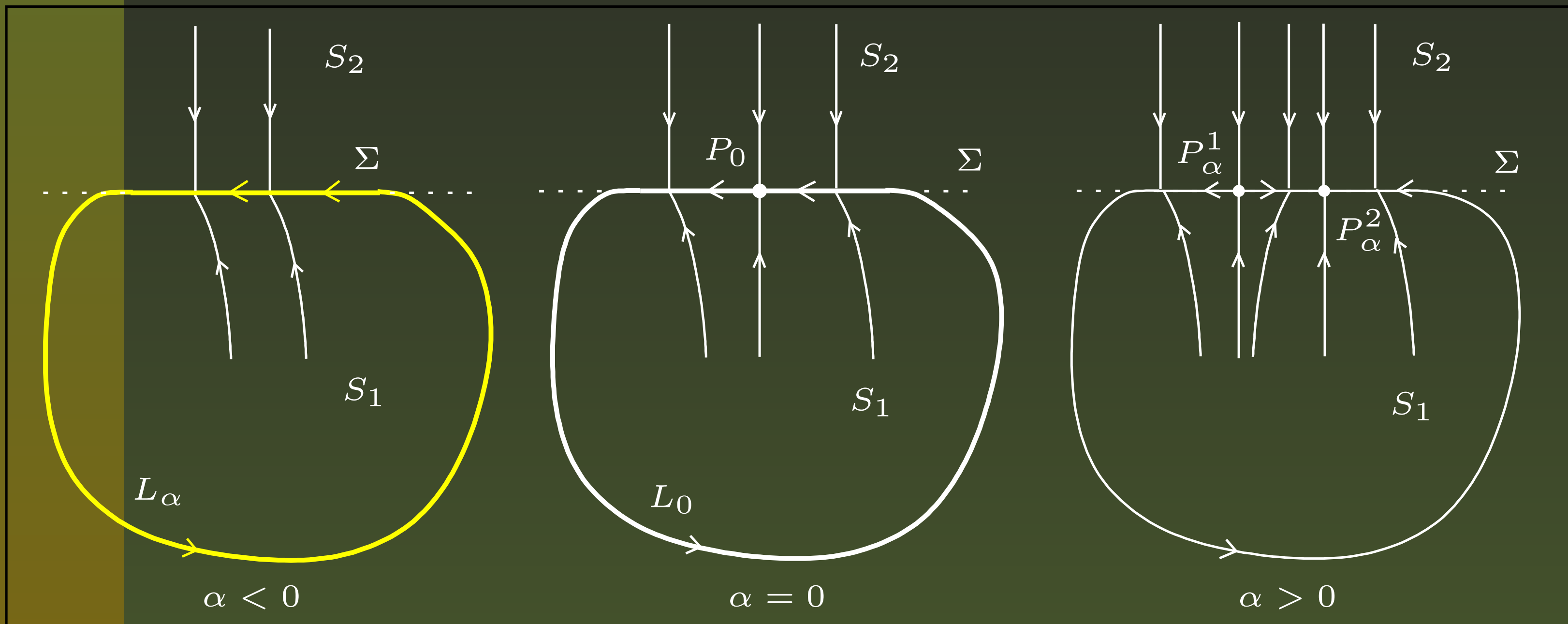
Homoclinic orbit to a pseudo-saddle-node

Homoclinic orbit to a pseudo-saddle-node



Homoclinic orbit to a pseudo-saddle-node



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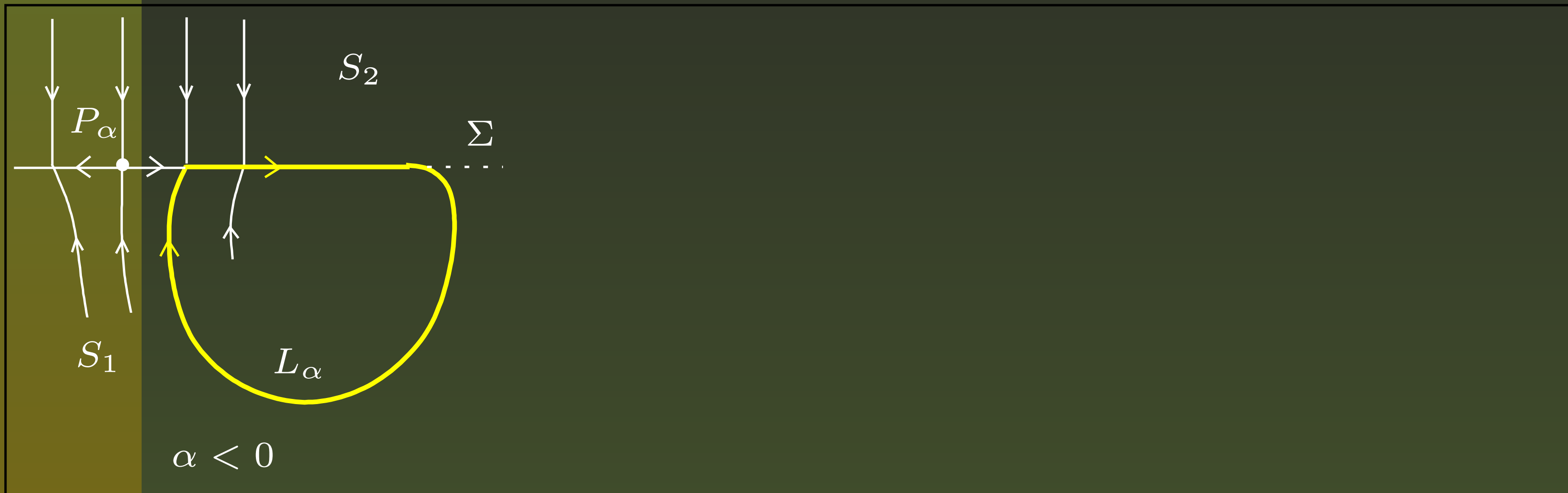
Homoclinic orbit to a pseudo-saddle: TGP



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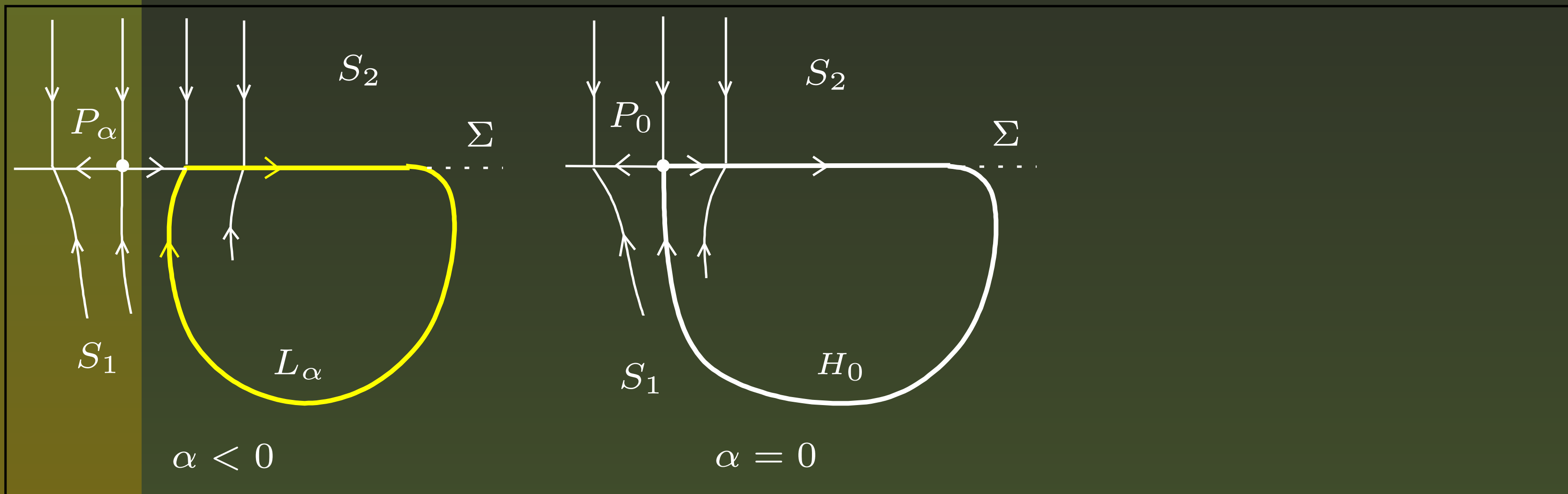
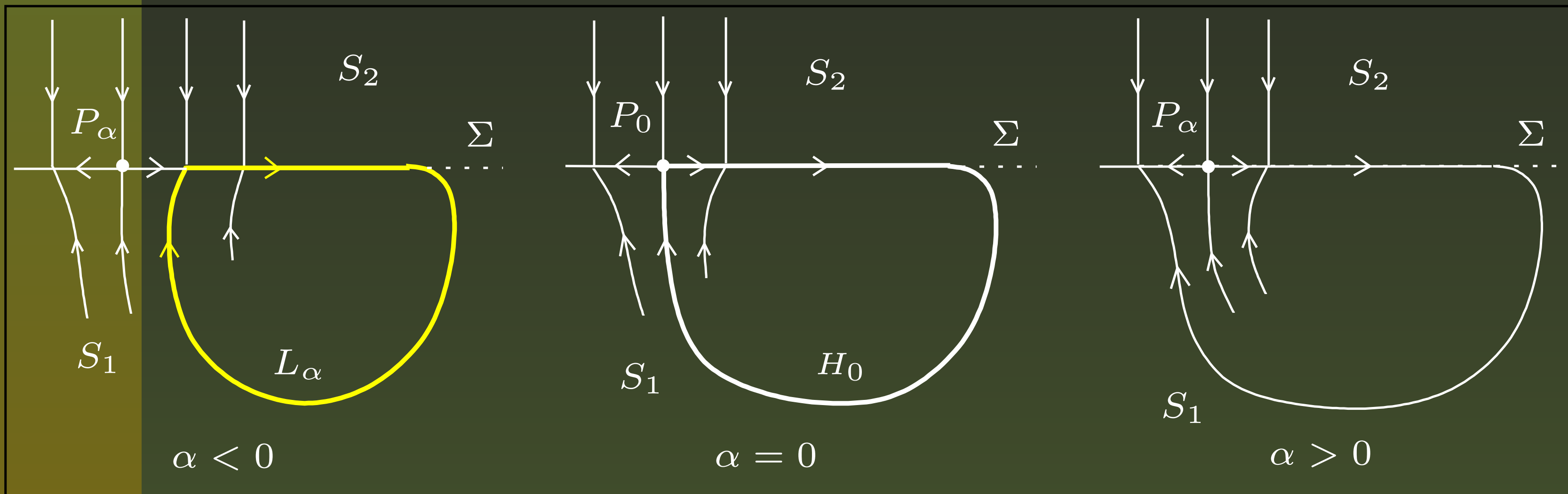
Homoclinic orbit to a pseudo-saddle: *TGP*



Homoclinic orbit to a pseudo-saddle: *TGP*



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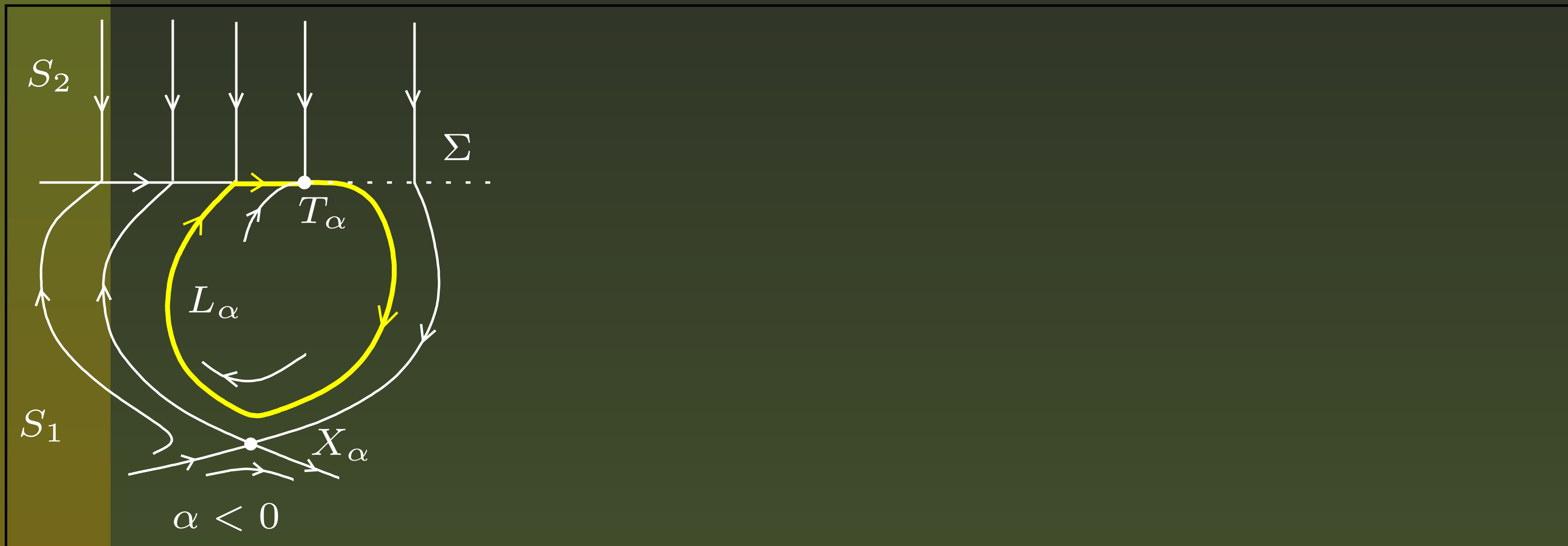
Sliding homoclinic orbit to a saddle



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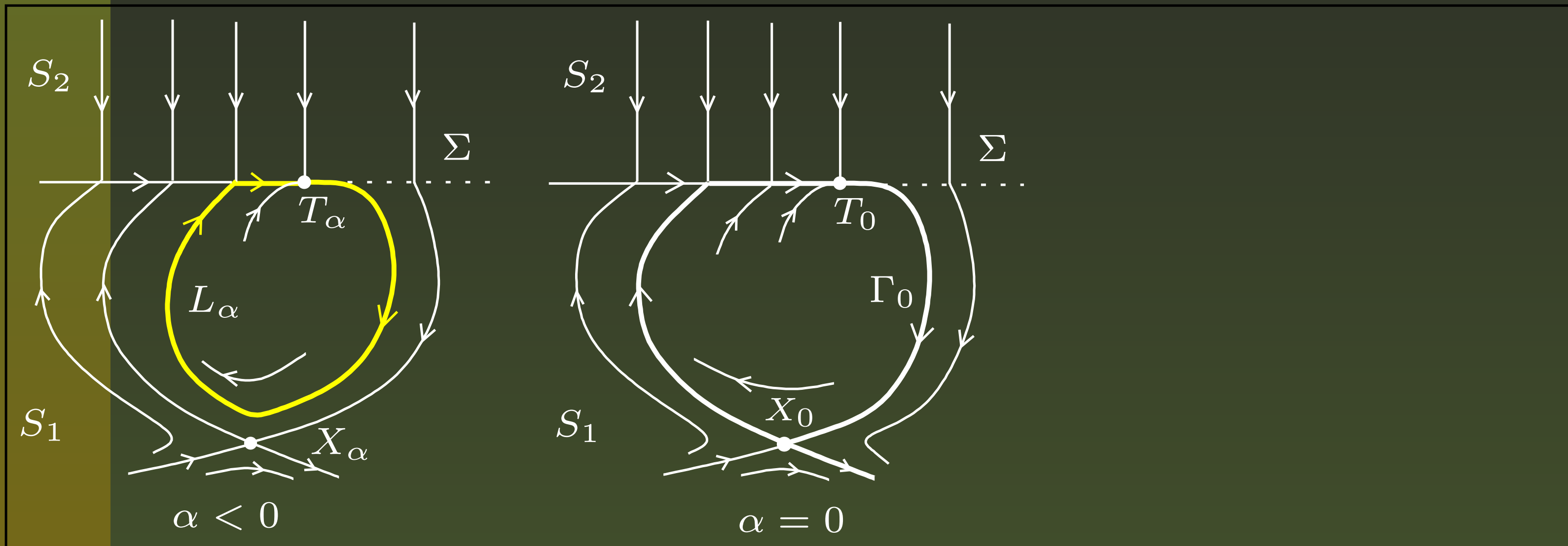
Sliding homoclinic orbit to a saddle



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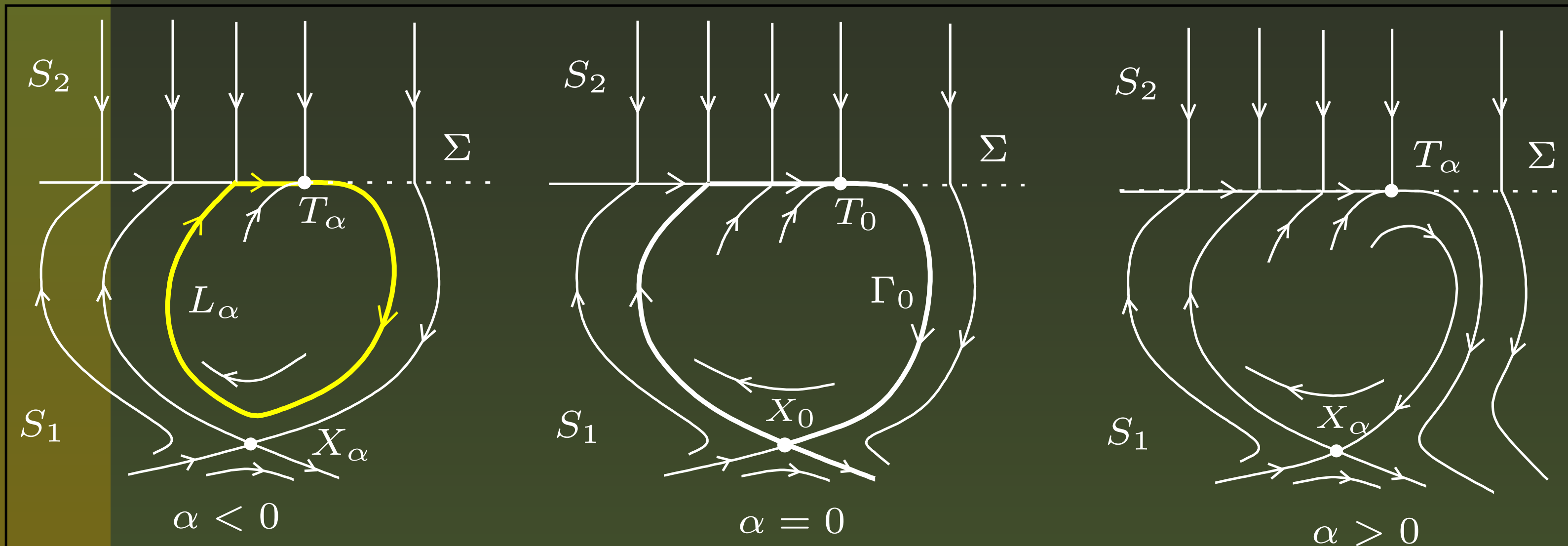
Sliding homoclinic orbit to a saddle



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4. Examples of codim 2 bifurcations

- Local bifurcations:
  - Degenerate boundary focus
  - Boundary Hopf
- Global bifurcations:
  - Sliding-grazing of a nonhyperbolic cycle (fold-grazing)

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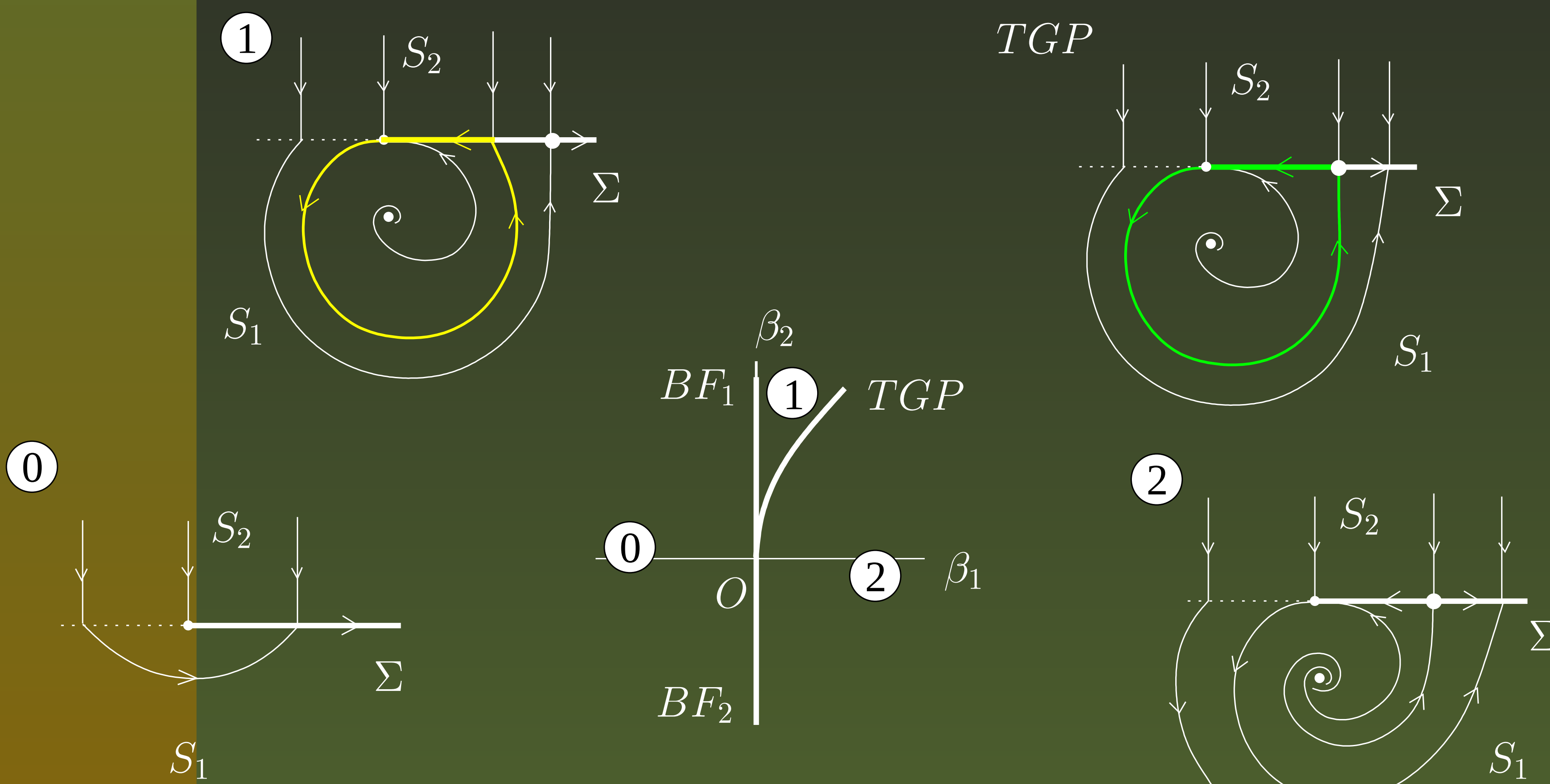
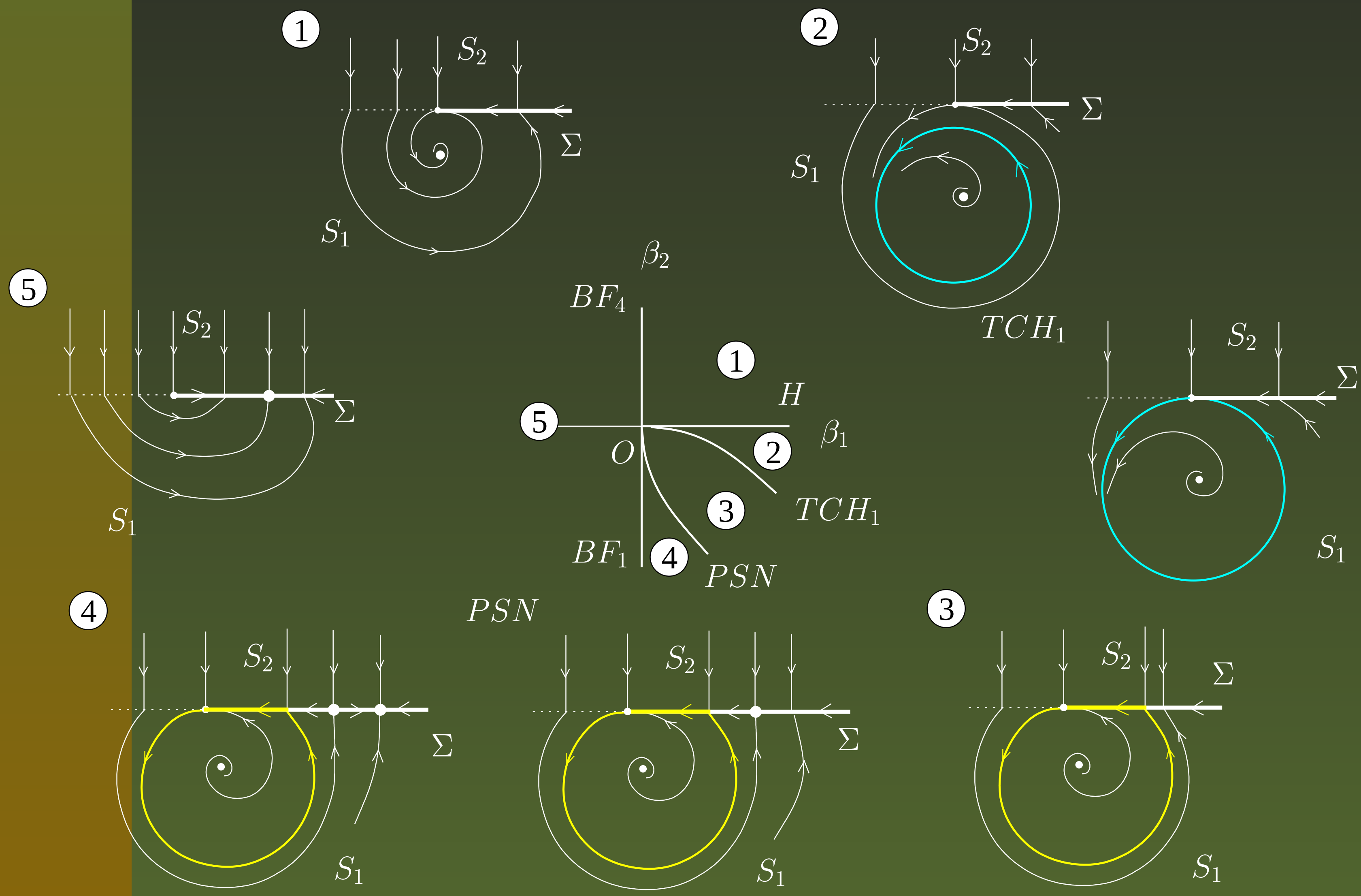
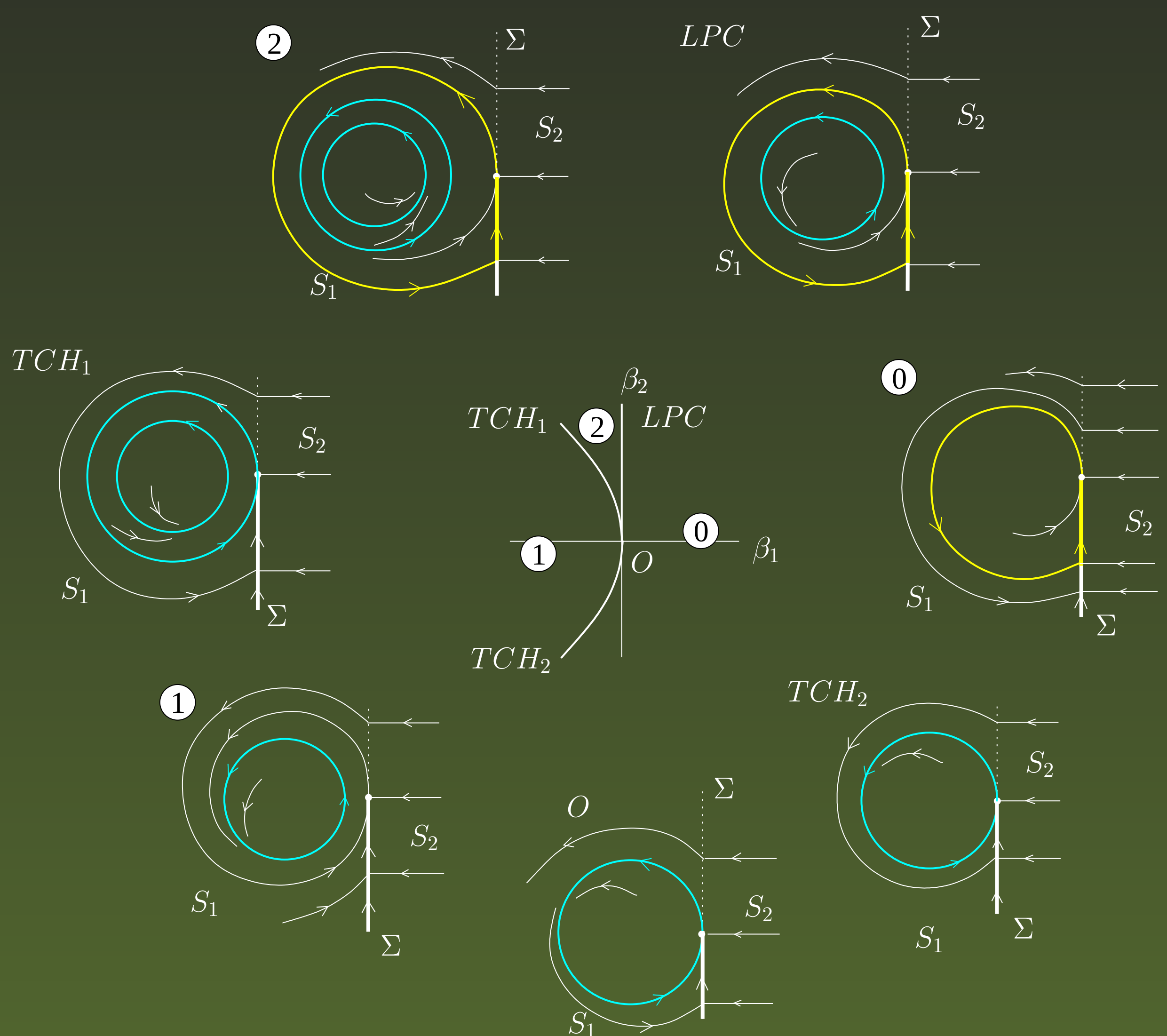
Degenerate boundary focus: DBF



Boundary Hopf: *BHP*



Fold-grazing: FG_1



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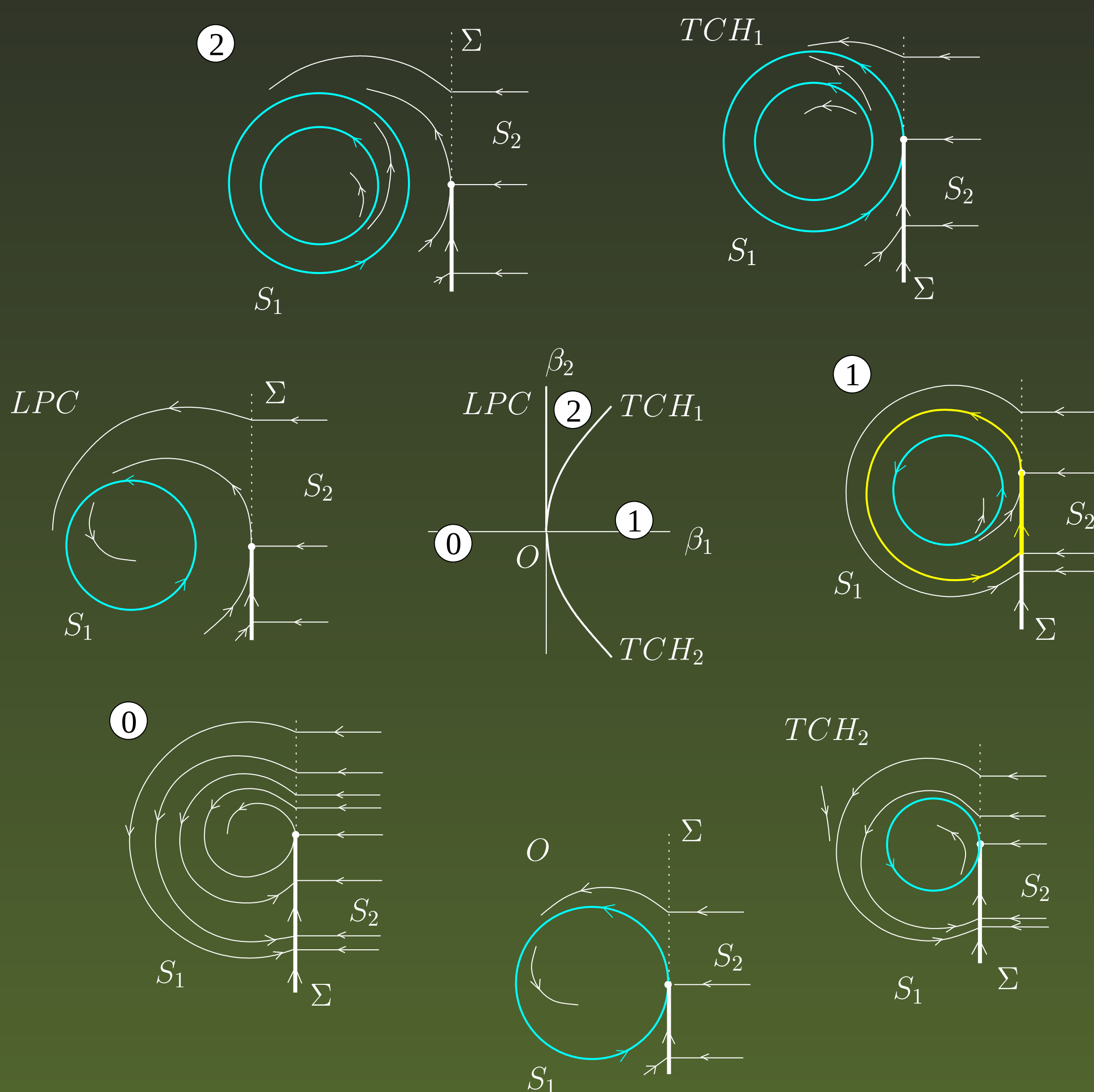
Fold-grazing: FG_2



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5. Example: Harvesting a prey-predator community

- Rosenzweig-MacArthur model
- Harvesing control
- Two-parameter bifurcation diagram

Rosenzweig-MacArthur-Holling model

$$\begin{cases} \dot{x}_1 & = & x_1(1-x_1) - \dfrac{ax_1x_2}{b+x_1} \\ \dot{x}_2 & = & \dfrac{ax_1x_2}{b+x_1} - cx_2 \end{cases}$$

Nontrivial zero-isoclines:

$$x_2 = \frac{1}{a}(b+x_1)(1-x_1), \quad x_1 = \frac{bc}{a-c}.$$

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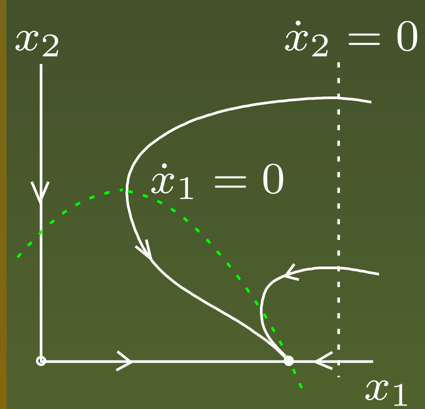
Rosenzweig-MacArthur-Holling model

$$\begin{cases} \dot{x}_1 &= x_1(1-x_1) - \dfrac{ax_1x_2}{b+x_1} \\ \dot{x}_2 &= \dfrac{ax_1x_2}{b+x_1} - cx_2 \end{cases}$$

Nontrivial zero-isoclines:

$$x_2 = \frac{1}{a}(b+x_1)(1-x_1), \quad x_1 = \frac{bc}{a-c}.$$



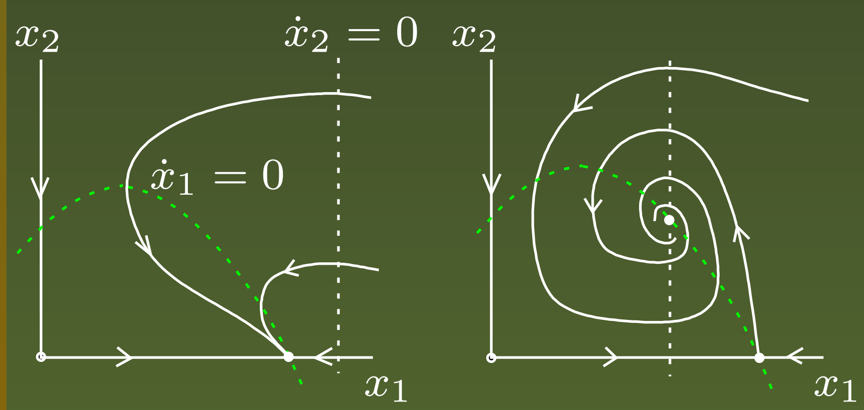
(a)

Rosenzweig-MacArthur-Holling model

$$\begin{cases} \dot{x}_1 = x_1(1-x_1) - \dfrac{a x_1 x_2}{b+x_1} \\ \dot{x}_2 = \dfrac{a x_1 x_2}{b+x_1} - c x_2 \end{cases}$$

Nontrivial zero-isoclines:

$$x_2 = \frac{1}{a}(b+x_1)(1-x_1), \quad x_1 = \frac{bc}{a-c}.$$



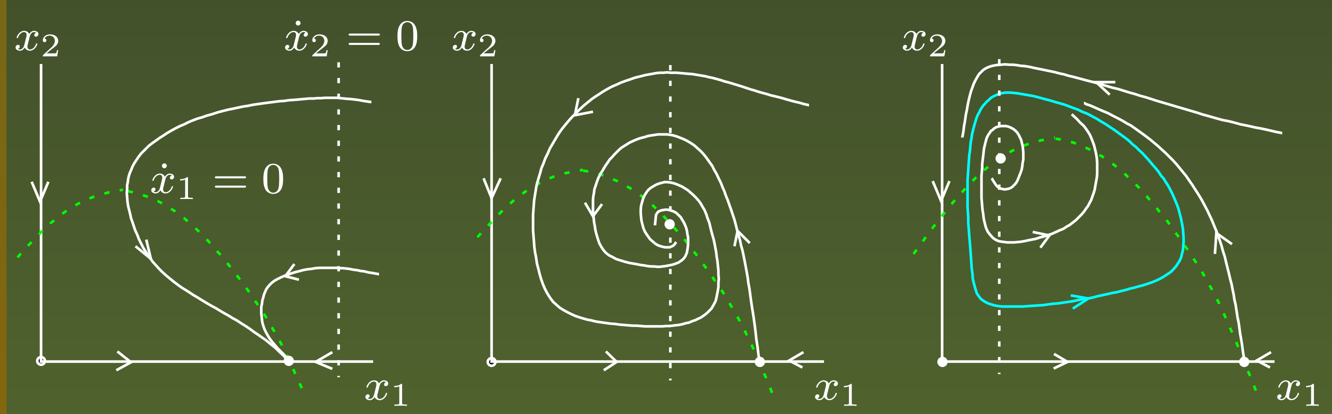
(a) (b)

Rosenzweig-MacArthur-Holling model

$$\begin{cases} \dot{x}_1 & = & x_1(1-x_1) - \dfrac{ax_1x_2}{b+x_1} \\ \dot{x}_2 & = & \dfrac{ax_1x_2}{b+x_1} - cx_2 \end{cases}$$

Nontrivial zero-isoclines:

$$x_2 = \frac{1}{a}(b+x_1)(1-x_1), \quad x_1 = \frac{bc}{a-c}.$$



(a) (b) (c)

Harvesting control

Assume that the predator population is harvested at constant effort $e > 0$ only when abundant ($x_2 > \alpha_5$). This leads to a planar Filippov system:

$$\dot{x} = \begin{cases} f^{(1)}(x), & x_2 > \alpha_5, \\ f^{(2)}(x), & x_2 < \alpha_5, \end{cases}$$

where

$$f^{(1)} = \begin{pmatrix} x_1(1 - x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - dx_2 \end{pmatrix}, \quad f^{(2)} = \begin{pmatrix} x_1(1 - x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - dx_2 - ex_2 \end{pmatrix},$$

$$\psi(x_1) = \frac{ax_1}{\alpha_2 + x_1}.$$

Fix $a = 0.3556, \quad d = 0.0444, \quad e = 0.2067.$

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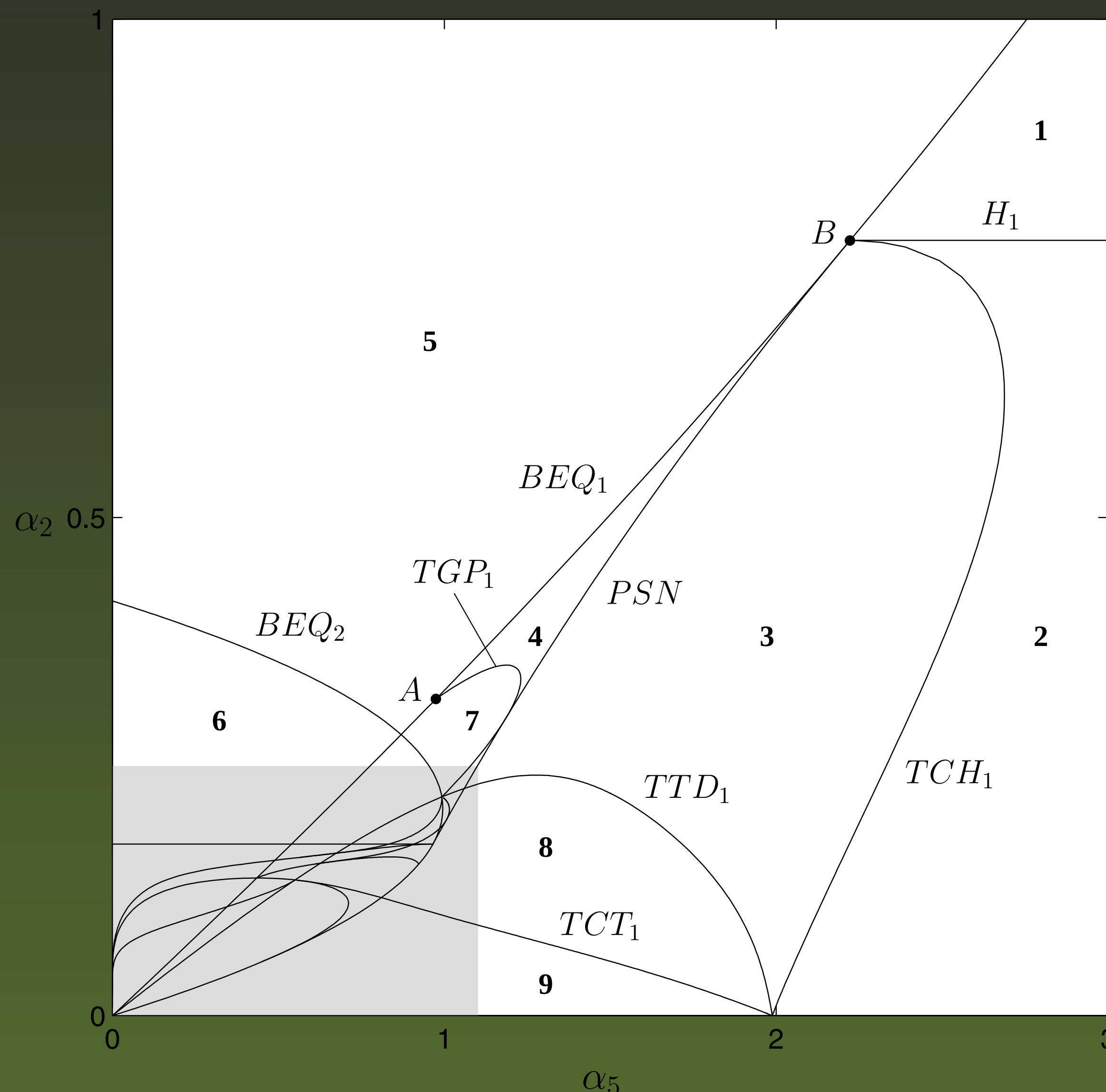
Two-parameter bifurcation diagram



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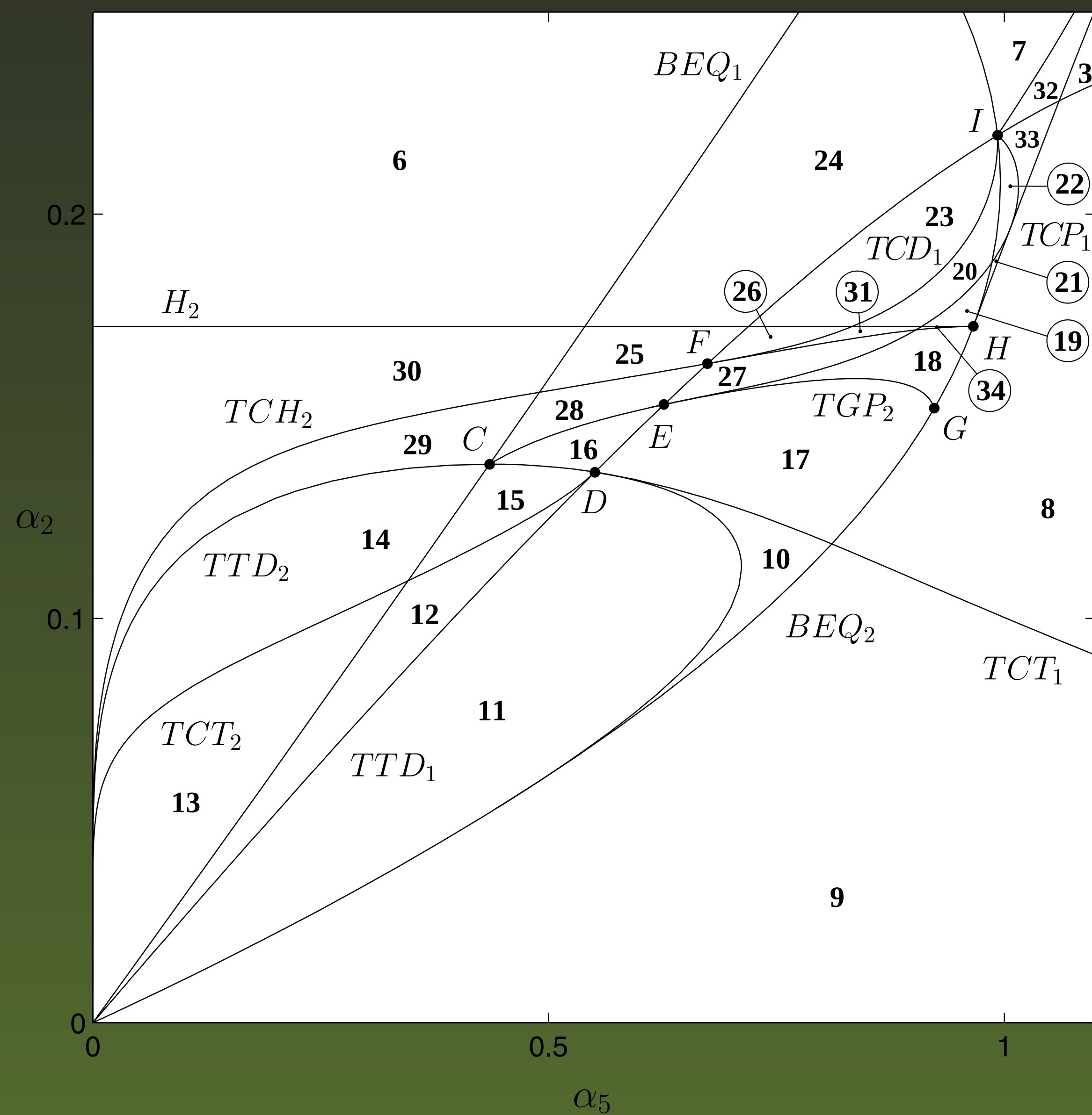
Two-parameter bifurcation diagram (zoom in)



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6. Open problems

1. Complete analysis of codim 2 local bifurcations for $n = 2$.
2. Bifurcation diagrams of smooth approximations of Filippov systems and their limits. A codim 1 bifurcation can
   - disappear: $BF_{3,4}, DT_{1,2}, II_1, TC_1, CC$;
   - become a single smooth bifurcation: $BF_{2,5}, PSN \to \mathsf{LP}, II_2 \to \mathsf{H}, TC_2 \to \mathsf{LPC}, TGP \to \mathsf{HOM}$;
   - split into a several smooth bifurcations: $BF_1 \to \mathsf{H} + \mathsf{LP}$.

   What happens to codim 2 bifurcations?
3. Grazing of nonhyperbolic cycles when $n = 3$ (codim 2).
4. Other codim 1 and 2 local and global bifurcations when $n \geq 3$.