Lecture 2. Filippov systems: Continuation of sliding bifurcations

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Contents

References:

- Dercole, F. and Kuznetsov, Yu.A. User Guide to SclideCont 2.0, 2005. `http://www.math.uu.nl/people/kuznet/cm/slidecont.pdf`
- Dercole, F. and Kuznetsov, Yu.A. SlideCont: An AUTO97 driver for bifurcation analysis of Filippov systems. *ACM TOMS* **31** (2005), no.1, 95-119.
- Kowalczyk, P., di Bernardo, M., Champneys, A. R., Hogan, S. J., Homer, M., Piiroinen, P. T., Kuznetsov, Yu. A., and Nordmark, A. Two-parameter discontinuity-induced bifurcations of limit cycles: Classification and open problems. *Int. J. Bifurcation & Chaos* **16** (2006), 601-629.

1. Reminder on Filippov systems

$$\dot{x} = \begin{cases} f^{(1)}(x,\alpha), & x \in S_1(\alpha), \\ f^{(2)}(x,\alpha), & x \in S_2(\alpha), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$,

$$S_1(\alpha) = \{x \in \mathbb{R}^n : H(x,\alpha) < 0\}, \quad S_2(\alpha) = \{x \in \mathbb{R}^n : H(x,\alpha) > 0\},$$

$H : \mathbb{R}^n \to \mathbb{R}$ is smooth with $H_x(x,\alpha) \neq 0$ on the **discontinuity boundary**

$$\Sigma(\alpha) = \{x \in \mathbb{R}^n : H(x,\alpha) = 0\}.$$

For $x \in \Sigma(\alpha)$, define $\sigma(x,\alpha) = \langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle\langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle$ and introduce the **crossing set:**

$$\Sigma_c(\alpha) = \{x \in \Sigma : \sigma(x,\alpha) > 0\},$$

and the **sliding set:**

$$\Sigma_s(\alpha) = \{x \in \Sigma : \sigma(x,\alpha) \leq 0\}.$$

Points $x \in \Sigma_s(\alpha)$, where

$$\langle H_x(x,\alpha), f^{(2)}(x,\alpha) - f^{(1)}(x,\alpha)\rangle = 0$$

are called **singular sliding points**.

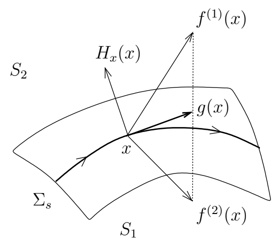
Crossing orbits: At $x \in \Sigma_c(\alpha)$ cross.

Sliding orbits: For a regular sliding point $x \in \Sigma_s(\alpha)$ define the **Filippov vector**

$$g(x,\alpha) = \lambda(x,\alpha) f^{(1)}(x,\alpha) + (1-\lambda(x,\alpha)) f^{(2)}(x,\alpha),$$

where

$$\lambda(x,\alpha) = \frac{\langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle}{\langle H_x(x,\alpha), f^{(2)}(x,\alpha) - f^{(1)}(x,\alpha)\rangle}.$$



This gives the **sliding system**:

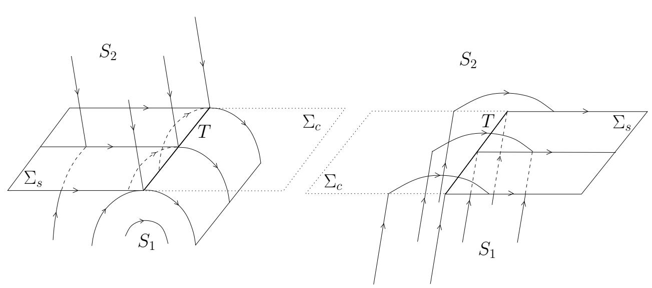
$$\dot{x} = g(x,\alpha), \quad x \in \Sigma_s(\alpha). \qquad (2)$$

- **pseudo-equilibrium**: $g(P,\alpha) = 0$ and $f^{(i)}(P,\alpha)$ are transversal to $\Sigma_s(\alpha)$ and anti-collinear;

- **boundary equilibrium**: $f^{(i)}(X,\alpha) = 0$;

- **tangent point**: Both $f^{(i)}(T,\alpha) \neq 0$ but

$$\langle H_x(T,\alpha), f^{(i)}(T,\alpha)\rangle = 0.$$

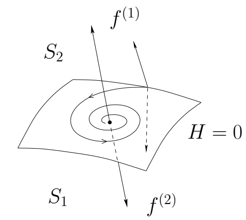


2. Supported n-dimensional cases

2.1. Local objects

Pseudo-equilibrium:

$$\begin{cases} H(x,\alpha) = 0, \\ \lambda_1 f^{(1)}(x,\alpha) + \lambda_2 f^{(2)}(x,\alpha) = 0, \\ \lambda_1 + \lambda_2 - 1 = 0. \end{cases}$$



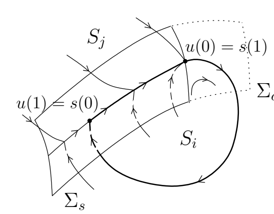
Boundary equilibrium:

$$\begin{cases} f^{(i)}(x,\alpha) = 0, \\ H(x,\alpha) = 0. \end{cases}$$

2.2. Global objects

Sliding cycle with one standard segment:

$$\begin{cases} \dot{u} - T_i f^{(i)}(u,\alpha) = 0, \\ \dot{s} - T_0 g(s,\alpha) = 0, \\ H(u(0),\alpha) = 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle = 0, \\ s(0) - u(1) = 0, \\ s(1) - u(0) = 0. \end{cases}$$



Notice that the condition $H(u(1),\alpha) = 0$ holds automatically due to the invariancy of Σ_s.

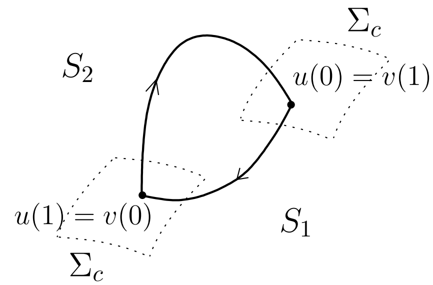
Singilarities and test functions:

$$t(u,\alpha) = \begin{pmatrix} H_{x_2}(u,\alpha) \\ -H_{x_1}(u,\alpha) \end{pmatrix} \qquad (n=2)$$

- boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n=2$): $\langle t(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$
- tangent point of $f^{(j)}$ at $u(0)$: $\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$
- tangent point of $f^{(i)}$ at $u(1)$: $\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$
- tangent point of $f^{(j)}$ at $u(1)$: $\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$
- pseudo-equilibrium at $u(1)$ ($n=2$): $\langle t(u(1),\alpha), g(u(1),\alpha)\rangle$

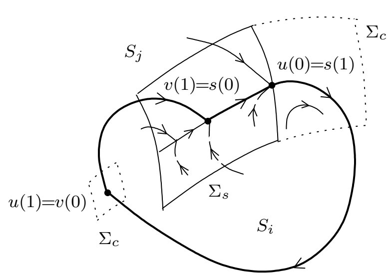
Crossing cycle:

$$\begin{cases} \dot{u} - T_1 f^{(1)}(u,\alpha) &= 0, \\ \dot{v} - T_2 f^{(2)}(v,\alpha) &= 0, \\ H(u(0),\alpha) &= 0, \\ H(u(1),\alpha) &= 0, \\ u(1)-v(0) &= 0, \\ v(1)-u(0) &= 0. \end{cases}$$



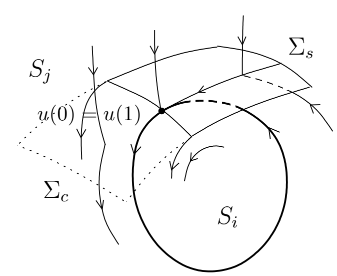
Sliding cycle with two standard segments:

$$\begin{cases} \dot{u} - T_i f^{(i)}(u,\alpha) &= 0, \\ \dot{v} - T_j f^{(j)}(u,\alpha) &= 0, \\ \dot{s} - T_0 g(s,\alpha) &= 0, \\ H(u(0),\alpha) &= 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &= 0, \\ H(u(1),\alpha) &= 0, \\ v(0)-u(1) &= 0, \\ s(0)-v(1) &= 0, \\ s(1)-u(0) &= 0. \end{cases}$$



Grazing cycle:

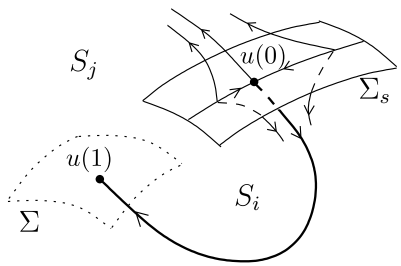
$$\begin{cases} \dot{u} - T f^{(i)}(u,\alpha) &= 0, \\ H(u(0),\alpha) &= 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &= 0, \\ u(0)-u(1) &= 0. \end{cases}$$



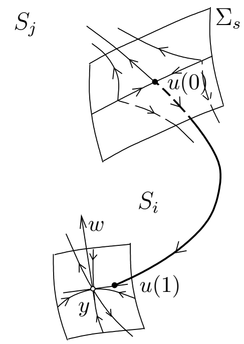
Grazing in multidimensional ($n \geq 3$) Filippov systems can create chaos.

Pseudo-equilibrium to boundary connection:

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0,\\ H(u(0),\alpha) &=& 0,\\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) &=& 0,\\ \lambda_i + \lambda_j - 1 &=& 0,\\ H(u(1),\alpha) &=& 0. \end{array}\right.$$



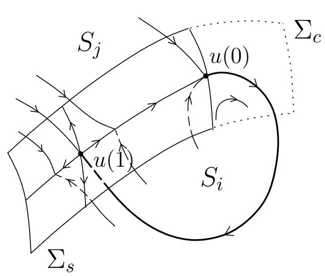
Pseudo-equilibrium to saddle connection:



$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0,\\ H(u(0),\alpha) &=& 0,\\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) &=& 0,\\ \lambda_i + \lambda_j - 1 &=& 0,\\ f^{(i)}(y,\alpha) &=& 0,\\ \left[f_x^{(i)}(y,\alpha)\right]^T w - \nu_k w &=& 0,\\ \langle w,w\rangle - 1 &=& 0,\\ \langle w, y-u(1)\rangle &=& 0. \end{array}\right.$$

Tangent point to pseudo-equilibrium connection:

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0,\\ H(u(0),\alpha) &=& 0,\\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &=& 0,\\ H(u(1),\alpha) &=& 0,\\ \lambda_i f^{(i)}(u(1),\alpha) + \lambda_j f^{(j)}(u(1),\alpha) &=& 0,\\ \lambda_i + \lambda_j - 1 &=& 0. \end{array}\right.$$



3. Introduction to AUTO97

- **Algebraic continuation problems:**

$$F(x,\alpha,\beta) = 0, \quad F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_d} \to \mathbb{R}^{n_d}$$

with $n + m + m_d = n_d + 1$.

- **Boundary-value continuation problems:**

$$\left\{\begin{array}{rcl} \dot{u}(\tau) - F(u(\tau),\alpha,\beta) &=& 0, \quad \tau \in [0,1],\\ B(u(0),u(1),\alpha,\beta) &=& 0, \end{array}\right.$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m_d} \to \mathbb{R}^{n_d}, \quad B : \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \times \mathbb{R}^m \times \mathbb{R}^{m_d} \to \mathbb{R}^{n_b}$$

with $m + m_d = n_b - n_d + 1$.

exp.f

```
C----------------------------------------------------------------------
C----------------------------------------------------------------------
C   exp :    A boundary value problem (Bratu's equation)
C----------------------------------------------------------------------
C----------------------------------------------------------------------
C
      SUBROUTINE FUNC(NDIM,U,ICP,PAR,IJAC,F,DFDU,DFDP)
C     ---------- ----
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION U(NDIM),PAR(*),F(NDIM)
C
       E=EXP(U(1))
       F(1)=U(2)
       F(2)=-PAR(1)*E
C
      RETURN
      END
C
      SUBROUTINE STPNT(NDIM,U,PAR,T)
C     ---------- -----
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION U(NDIM),PAR(*)
C
       PAR(1)=0
       U(1)=0.0
       U(2)=0.0
C

      RETURN
      END
C
      SUBROUTINE BCND(NDIM,PAR,ICP,NBC,U0,U1,FB,IJAC,DBC)
C     ---------- ----
C
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION PAR(*),ICP(*),U0(NDIM),U1(NDIM),FB(NBC)
C
       FB(1)=U0(1)
       FB(2)=U1(1)
C
      RETURN
      END
C
      SUBROUTINE ICND
      RETURN
      END
C
      SUBROUTINE FOPT
      RETURN
      END
C
      SUBROUTINE PVLS
      RETURN
      END
```

Makefile

```
PGM    = exp
FC     = f77
FFLAGS = -O
SRC    = $(PGM).f
EXE    = $(PGM).exe
LIBS   = $(AUTO_DIR)/lib/*.o
RM     = rm -f
#
all: clean $(EXE) run
#
$(EXE): $(SRC:.f=.o)
$(FC) $(FFLAGS) $(SRC:.f=.o) -o $@ $(LIBS)
#
run: first second
#
first: $(EXE)
@echo " "
@echo "Demo $(PGM) is started"
@echo " "
@echo "$(PGM) : first run : compute solution branch containing fold"
@cp r.$(PGM).1 fort.2
@touch fort.3
@$(EXE)
@cp fort.7 p.$(PGM)
@cp fort.8 q.$(PGM)
@cp fort.9 d.$(PGM)
@echo " Saved as *.$(PGM)"
@$(RM) fort.*
#
```

r.exp.1

```
2 4 0 1                        NDIM,IPS,IRS,ILP
1   1                          NICP,(ICP(I),I=1,NICP)
 5 4 3 1 1 3 2 0               NTST,NCOL,IAD,ISP,ISW,IPLT,NBC,NINT
50   0.0 4.0 0.0 50.0          NMX,RL0,RL1,A0,A1
50 10 2 8 5 3 0                NPR,MXBF,IID,ITMX,ITNW,NWTN,JAC
 1.e-6 1.e-6 1.e-4             EPSL,EPSU,EPSS
0.01 0.001 1.0 1               DS,DSMIN,DSMAX,IADS
0                              NTHL,(/,I,THL(I)),I=1,NTHL)
0                              NTHU,(/,I,THU(I)),I=1,NTHU)
1                              NUZR,(/,I,PAR(I)),I=1,NUZR)
  1  2.
```

q.exp

```
  1   19    5    3    1    1   21    3     59    5    4  100
    0.0000000000E+00  -2.5400193050E-28   4.0001091088E+00
    5.0413306178E-02   1.9688222065E-01   3.8041759126E+00
    1.0082661236E-01   3.8286645953E-01   3.5668745507E+00
    1.5123991853E-01   5.5573216144E-01   3.2829394737E+00
    2.0165322471E-01   7.1301770175E-01   2.9480725088E+00
    2.5018727968E-01   8.4725515962E-01   2.5751571496E+00
    2.9872133464E-01   9.6218085712E-01   2.1527534092E+00
    3.4725538961E-01   1.0554613113E+00   1.6840075999E+00
    3.9578944457E-01   1.1249993513E+00   1.1755113926E+00
    4.4423319077E-01   1.1690295031E+00   6.3828072652E-01
    4.9267693697E-01   1.1865703437E+00   8.4311078622E-02
    5.4112068316E-01   1.1771612119E+00  -4.7191551630E-01
    5.8956442936E-01   1.1410487251E+00  -1.0156810991E+00
    6.4075124035E-01   1.0749264131E+00  -1.5617190061E+00
    6.9193805133E-01   9.8188498909E-01  -2.0659892005E+00
    7.4312486232E-01   8.6429393184E-01  -2.5198546141E+00
    7.9431167330E-01   7.2485488175E-01  -2.9189739029E+00
    8.4573375498E-01   5.6563915198E-01  -3.2643082784E+00
    8.9715583665E-01   3.9005236537E-01  -3.5564406234E+00
    9.4857791833E-01   2.0071768661E-01  -3.7998531288E+00
    1.0000000000E+00   3.8411471564E-28  -4.0001091083E+00
```

p.exp

```
   0    0.0000E+00  4.0000E+00  0.0000E+00  5.0000E+01
   0    EPSL= 1.0000E-06  EPSU = 1.0000E-06  EPSS = 1.0000E-04
   0    DS  = 1.0000E-02  DSMIN= 1.0000E-03  DSMAX= 1.0000E+00
   0    NDIM=   2   IPS =   4   IRS =   0   ILP =   1
   0    NTST=   5   NCOL=   4   IAD =   3   ISP =   1
   0    ISW =   1   IPLT=   3   NBC =   2   NINT=   0
   0    NMX=   50   NPR =  50   MXBF=  10   IID =   2
   0    ITMX=   8   ITNW=   5   NWTN=   3   JAC=   0  NUZR=   1
   0    User-specified parameter:          1
   0    Active continuation parameter:     1
   0
   0     PT  TY LAB    PAR(1)      INTEGRAL U(1)   MAX U(1)      MAX U(2)
   1      1   9   1   0.00000E+00   0.00000E+00   0.00000E+00   0.00000E+00
   1      2   0   0   9.57018E-03   7.98279E-04   1.19747E-03   4.78891E-03
   1      3   0   0   2.39225E-02   1.99833E-03   2.99780E-03   1.19852E-02
   1      4   0   0   4.54445E-02   3.80437E-03   5.70590E-03   2.28089E-02
   1      5   0   0   7.77124E-02   6.52697E-03   9.79066E-03   3.91108E-02
   1      6   0   0   1.26080E-01   1.06418E-02   1.59662E-02   6.37150E-02
   1      7   0   0   1.98551E-01   1.68849E-02   2.53402E-02   1.00969E-01
   1      8   0   0   3.07069E-01   2.64144E-02   3.96607E-02   1.57655E-01
   1      9   0   0   4.69391E-01   4.10981E-02   6.17532E-02   2.44583E-01
   1     10   0   0   7.11732E-01   6.40690E-02   9.63921E-02   3.79569E-01
   1     11   0   0   1.07219E+00   1.00924E-01   1.52120E-01   5.93634E-01
   1     12   0   0   1.42795E+00   1.41155E-01   2.13188E-01   8.23876E-01
   1     13   0   0   1.77733E+00   1.85476E-01   2.80806E-01   1.07350E+00
   1     14  -4   2   2.00000E+00   2.16936E-01   3.28952E-01   1.24822E+00
   1     15   0   0   2.49304E+00   2.99469E-01   4.55976E-01   1.69721E+00
   1     16   0   0   2.94174E+00   4.01371E-01   6.13199E-01   2.23398E+00
   1     17   0   0   3.30042E+00   5.30089E-01   8.14985E-01   2.88665E+00
   1     18   0   0   3.49429E+00   6.87210E-01   1.06466E+00   3.64921E+00
   1     19   5   3   3.51383E+00   7.61924E-01   1.18657E+00   4.00011E+00
   1     20   0   0   3.36507E+00   9.95030E-01   1.56701E+00   5.05340E+00
   1     21   0   0   3.04297E+00   1.21473E+00   1.93274E+00   5.99852E+00
   1     22   0   0   2.66628E+00   1.42154E+00   2.28360E+00   6.85569E+00
   1     23   0   0   2.27928E+00   1.62442E+00   2.63283E+00   7.67335E+00
   1     24  -4   4   2.00000E+00   1.77497E+00   2.89525E+00   8.26877E+00
   1     25   0   0   1.59587E+00   2.01109E+00   3.30800E+00   9.18846E+00
   1     26   0   0   1.24155E+00   2.25066E+00   3.73614E+00   1.01095E+01
   1     27   0   0   9.44505E-01   2.49327E+00   4.17518E+00   1.10349E+01
   1     28   0   0   7.04832E-01   2.73800E+00   4.62595E+00   1.19649E+01
   1     29   0   0   5.17569E-01   2.98382E+00   5.08064E+00   1.28981E+01
   1     30   0   0   3.75005E-01   3.22992E+00   5.53973E+00   1.38332E+01
   1     31   0   0   2.68618E-01   3.47599E+00   6.00855E+00   1.47701E+01
   1     32   0   0   1.90639E-01   3.72139E+00   6.47350E+00   1.57073E+01
   1     33   0   0   1.34202E-01   3.96616E+00   6.93983E+00   1.66451E+01
   1     34   0   0   9.37538E-02   4.21081E+00   7.39289E+00   1.75856E+01
   1     35   0   0   6.51128E-02   4.45446E+00   7.85956E+00   1.85257E+01
   1     36   0   0   4.49720E-02   4.69748E+00   8.32664E+00   1.94668E+01
   1     37   0   0   3.09138E-02   4.93989E+00   8.79679E+00   2.04102E+01
   1     38   0   0   2.11478E-02   5.18190E+00   9.26479E+00   2.13541E+01
   1     39   0   0   1.44066E-02   5.42347E+00   9.73303E+00   2.22993E+01
   1     40   0   0   9.77843E-03   5.66370E+00   1.02097E+01   2.32321E+01
   1     41   0   0   6.60585E-03   5.90505E+00   1.06802E+01   2.41780E+01
   1     42   0   0   4.44930E-03   6.14592E+00   1.11505E+01   2.51238E+01
   1     43   0   0   2.97625E-03   6.38949E+00   1.15971E+01   2.60397E+01
   1     44   0   0   1.98818E-03   6.63144E+00   1.20692E+01   2.69823E+01
   1     45   0   0   1.32706E-03   6.87210E+00   1.25394E+01   2.79196E+01
   1     46   0   0   8.82361E-04   7.11263E+00   1.30506E+01   2.88833E+01
   1     47   0   0   5.85165E-04   7.35381E+00   1.35249E+01   2.98277E+01
   1     48   0   0   3.87693E-04   7.59414E+00   1.39979E+01   3.07690E+01
   1     49   0   0   2.56194E-04   7.83481E+00   1.44740E+01   3.17179E+01
   1     50   9   5   1.69018E-04   8.07507E+00   1.49476E+01   3.26599E+01
```

4. SlideCont 2.0: `Makefile`

```
#-----------------------------------------------------------------------
#
#     **************************************************
#     *                                                *
#     * Linear oscillator with normal and dry friction *
#     *                                                *
#     **************************************************
#
#-----------------------------------------------------------------------
#
PGM       = dryf
RM        = rm -f
#
all: superclean run
#
run: 1 2 3 4 5
#
#
1:
@echo " "
@echo "1: Forward continuation of a sliding cycle with one standard segment"
@cp sc.$(PGM).1 sc.$(PGM)
@cp $(PGM).f.1 $(PGM).f
@cp $(PGM).dat.1 $(PGM).dat
@"@scdat" $(PGM)
@cp q.$(PGM) q.dat.1
@"@sc" $(PGM)
@"@sv" $(PGM).1
@rm fort.* sc.$(PGM) q.$(PGM) $(PGM).f $(PGM).dat
#
```

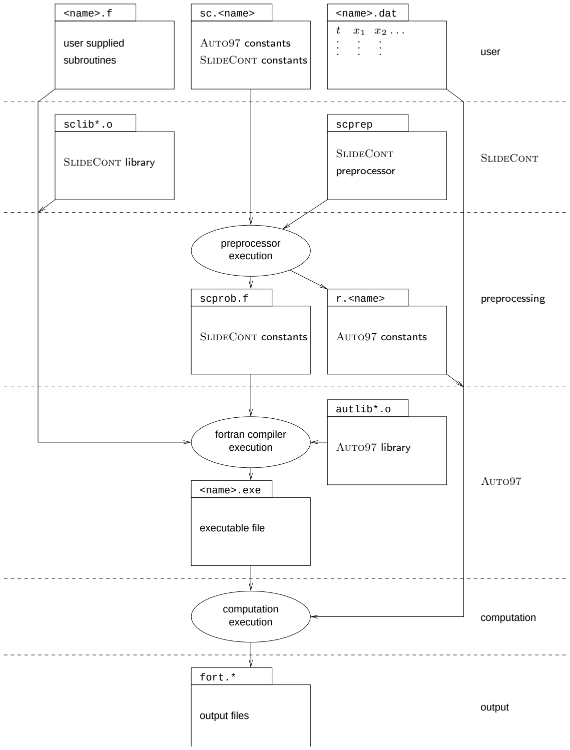
sc.dfyf.1

```
2 132 1 1                       NDIM,IPS,IRS,ILP
3 2 60 61                       NICP,(ICP(I),I=1,NICP)
35 4 3 1 1 0 0 0                NTST,NCOL,IAD,ISP,ISW,IPLT,NBC,NINT
50 0.0 3.0 0.0 100.0            NMX,RL0,RL1,A0,A1
10 0 2 8 7 5 0                  NPR,MXBF,IID,ITMX,ITNW,NWTN,JAC
1.e-8 1.e-8 1.e-7               EPSL,EPSU,EPSS
-1.e-3 1.e-10 1.0 1             DS,DSMIN,DSMAX,IADS
0                               NTHL,((ITHL(I),THL(I)),I=1,NTHL)
0                               NTHU,((ITHU(I),THU(I)),I=1,NTHU)
0                               NUZR,((IUZR(I),UZR(I)),I=1,NUZR)
0 2                             SCISTART,SCIDIFF
2 3 4                           SCNPSI,(SCIPSI(I),I=1,SCNPSI)
0                               SCNFIXED,(SCIFIXED(I),I=1,SCNFIXED)
```

```
C-----------------------------------------------------------------------
C-----------------------------------------------------------------------
C                                dryf.f
C-----------------------------------------------------------------------
C-----------------------------------------------------------------------
C
C     SlideCont
C
C     An AUTO97 driver for sliding bifurcation analysis
C
C     BY  Fabio Dercole & Yuri Kuznetsov
C
C
C     VERSION 2.0 (last revision 4/2004)
C
C-----------------------------------------------------------------------
C
C     *****************************************************************
C     *                                                               *
C     * Self-excited autoparametric system with dry friction (planar) *
C     *                                                               *
C     *****************************************************************
C
C-----------------------------------------------------------------------
C
c     ----------------------------------------------------
      SUBROUTINE SCFUNC(SCNDIM,X,PAR,SCIDIFF,FI,
     +  DFIDX,DFIDP,DFIDXDX,DFIDXDP,I)
c     ----------------------------------------------------
c
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
        INCLUDE 'dryf.h'
c
        INTEGER SCNDIM,SCIDIFF
c
        DIMENSION X(SCNDIM),PAR(*),FI(SCNDIM)
        DIMENSION DFIDX(SCNDIM,SCNDIM),DFIDP(SCNDIM,*)
        DIMENSION DFIDXDX(SCNDIM,SCNDIM,SCNDIM)
        DIMENSION DFIDXDP(SCNDIM,SCNDIM,*)
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'enter in SCFUNC',I
          END IF
        END IF
c
c       ----------------------
c       Vector field selection
c       ----------------------
        IF (I.EQ.1) THEN
c
c         --------------
c         Vector field 1
c         --------------
c
c         Right-hand side
c         ---------------
          FI(1)=X(2)
          FI(2)=-X(1)+PAR(1)*PAR(3)-PAR(2)*PAR(4)*(PAR(5)-X(2))
c
c         First derivatives
c         -----------------
          IF (SCIDIFF.GE.1) THEN
c
c           with respect to state
            DFIDX(1,2)=1.0D0
            DFIDX(2,1)=-1.0D0
            DFIDX(2,2)=PAR(2)*PAR(4)
c
c           with respect to parameters
            DFIDP(2,1)=PAR(3)
            DFIDP(2,2)=-PAR(4)*(PAR(5)-X(2))
            DFIDP(2,3)=PAR(1)
            DFIDP(2,4)=-PAR(2)*(PAR(5)-X(2))
            DFIDP(2,5)=-PAR(2)*PAR(4)
c
          END IF
c
c         Second derivatives
c         ------------------
          IF (SCIDIFF.GE.2) THEN
c
c           with respect to state twice
c
c           with respect to state and parameters
            DFIDXDP(2,2,2)=PAR(4)
            DFIDXDP(2,2,4)=PAR(2)
c
          END IF
c
        ELSE IF (I.EQ.2) THEN
c
c         --------------
c         Vector field 2
```

```
c         --------------
c
c         Right-hand side
c         ---------------
          FI(1)=X(2)
          FI(2)=-X(1)-PAR(1)*PAR(3)-PAR(2)*PAR(4)*(PAR(5)-X(2))
c         First derivatives
c         -----------------
          IF (SCIDIFF.GE.1) THEN
c
c           with respect to state
            DFIDX(1,2)=1.0D0
            DFIDX(2,1)=-1.0D0
            DFIDX(2,2)=PAR(2)*PAR(4)
c
c           with respect to parameters
            DFIDP(2,1)=-PAR(3)
            DFIDP(2,2)=-PAR(4)*(PAR(5)-X(2))
            DFIDP(2,3)=-PAR(1)
            DFIDP(2,4)=-PAR(2)*(PAR(5)-X(2))
            DFIDP(2,5)=-PAR(2)*PAR(4)
c
          END IF
c
c         Second derivatives
c         ------------------
          IF (SCIDIFF.GE.2) THEN
c
c           with respect to state twice
c
c           with respect to state and parameters
            DFIDXDP(2,2,2)=PAR(4)
            DFIDXDP(2,2,4)=PAR(2)
c
          END IF
c
        ELSE
          PRINT *,'SCFUNC error: I=',I
          STOP
        END IF
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'exit from SCFUNC',I
          END IF
        END IF
c
      RETURN
      END
c
c
c     ---------------------------------------------
      SUBROUTINE SCBOUND(SCNDIM,X,PAR,SCIDIFF,
     +  H,DHDX,DHDP,DHDXDX,DHDXDP)
c     ---------------------------------------------
c
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
        INCLUDE 'dryf.h'
c
        INTEGER SCNDIM,SCIDIFF
c
        DIMENSION X(SCNDIM),PAR(*),DHDX(SCNDIM),DHDP(*)
        DIMENSION DHDXDX(SCNDIM,SCNDIM),DHDXDP(SCNDIM,*)
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'enter in SCBOUND'
          END IF
        END IF
c
c       ----------------------
c       Discontinuity boundary
c       ----------------------
c
c       boundary function
c       -----------------
        H=X(2)-PAR(5)
c
c       First derivatives
c       -----------------
        IF (SCIDIFF.GE.1) THEN
c
c         with respect to state
          DHDX(2)=1.0D0
c
c         with respect to parameters
          DHDP(5)=-1.0D0
c
        END IF
c
c       Second derivatives
c       ------------------
        IF (SCIDIFF.GE.2) THEN
c
c         with respect to state twice
c
c         with respect to state and parameters
c
        END IF
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'exit from SCBOUND'
          END IF
        END IF
c
      RETURN
      END
c
c
c     ----------------------------------
      SUBROUTINE SCSTPNT(SCNDIM,X,PAR,T,I)
c     ----------------------------------
c
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
        INCLUDE 'dryf.h'
c
        INTEGER SCNDIM
c
        DIMENSION X(SCNDIM),PAR(*)
c
c       Debug
```

```
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'enter in SCSTPNT',I
          END IF
        END IF
c
c       Initialize parameters
c       ---------------------
c
PAR(1)=0.1D0
PAR(2)=0.03D0
PAR(3)=4.0D0
PAR(4)=4.0D0
PAR(5)=0.5D0
PAR(60)=1.02448D0
PAR(61)=5.46477D0
c
c       Initialize solution
c       -------------------
c
        IF (I.EQ.1) THEN
c
c         Init state of vector field 1
c         ----------------------------
c
        ELSE IF (I.EQ.2) THEN
c
c         Init state of vector field 2
c         ----------------------------
c
        ELSE IF (I.EQ.3) THEN
c
c         Init state
c         ----------
c
        END IF
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'exit from SCSTPNT',I
          END IF
        END IF
c
      RETURN
      END
c
c
c     ---------------------------------
      SUBROUTINE SCPVLS(SCNDIM,X,PAR,I)
c     ---------------------------------
c
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
c
        INCLUDE 'dryf.h'
c
        INTEGER SCNDIM
c
        DIMENSION X(SCNDIM),PAR(*)
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'enter in SCPVLS'
          END IF
        END IF
c
c       User functions
c       --------------
c       PAR(?)=?
c
c       Debug
        IF (DEBUG) THEN
          IF (DBGLEV.GE.DBGL2) THEN
            PRINT *,'exit from SCPVLS'
          END IF
        END IF
c
      RETURN
      END
c
c
c     -----------------
      SUBROUTINE SCBCND
c     -----------------
c
c       Dummy subroutine
c
      RETURN
      END
c
c
c     -----------------
      SUBROUTINE SCICND
c     -----------------
c
c       Dummy subroutine
c
      RETURN
      END
c
c     -----------------
      SUBROUTINE SCFOPT
c     -----------------
c
c       Dummy subroutine
c
      RETURN
      END
```



4. Examples

4.1. A simple planar example [Tondl, 1970]

$$\ddot{x}_1 + x_1 = \varepsilon\alpha_0 \text{sign}(v - \dot{x}_1) - \mu\alpha_1(v - \dot{x}_1).$$

It is equavalent to (1) with $x_2 = \dot{x}_1$,

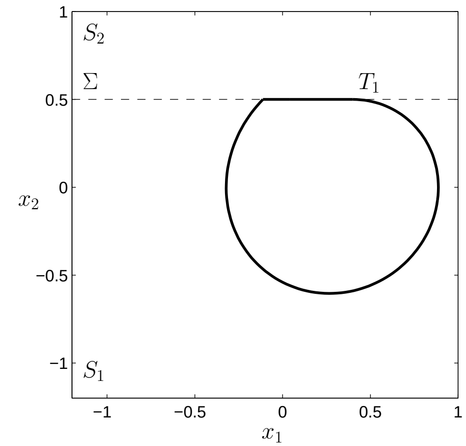
$$S_1 = \{x \in \mathbb{R}^2 : x_2 < v\},\ S_2 = \{x \in \mathbb{R}^2 : x_2 > v\}$$

and

$$f^{(1)}(x) = \begin{pmatrix} x_2 \\ -x_1 + \varepsilon\alpha_0 - \mu\alpha_1(v - x_2) \end{pmatrix},$$

$$f^{(2)}(x) = \begin{pmatrix} x_2 \\ -x_1 - \varepsilon\alpha_0 - \mu\alpha_1(v - x_2) \end{pmatrix}.$$

A sliding cycle:



$\varepsilon = 0.1, \mu = 0.03, \alpha_0 = \alpha_1 = 4, v = 0.5$

Continuation of the sliding cycle in PAR(2)$=\mu$

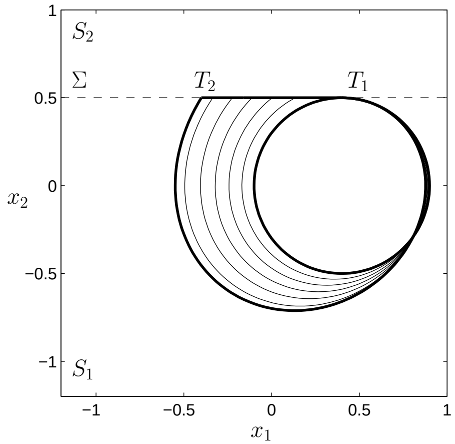
BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(43)	...
1	50		2	2.65719E-02	...	5.50782E+00	4.73382E-01	...
1	100		3	2.30473E-02	...	5.55611E+00	4.32653E-01	...
1	150		4	1.93686E-02	...	5.61187E+00	3.88868E-01	...
1	200		5	1.58932E-02	...	5.67093E+00	3.45696E-01	...
1	250		6	1.23734E-02	...	5.73921E+00	2.99217E-01	...
1	300		7	8.80398E-03	...	5.82108E+00	2.47461E-01	...
1	350		8	5.12502E-03	...	5.92802E+00	1.84932E-01	...
1	400		9	9.60983E-04	...	6.12808E+00	7.81608E-02	...
1	423	UZ	10	1.13542E-15	...	6.28319E+00	-1.55730E-08	...
1	425	EP	11	7.34401E-06	...	6.29677E+00	-6.79324E-03	...

Label 10: **grazing bifurcation** (O)

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	...	PAR(44)
1	50		12	3.00584E-02	...	5.46407E+00	...	2.87101E-01
1	100		13	3.38037E-02	...	5.42081E+00	...	2.45180E-01
1	150		14	3.76239E-02	...	5.38008E+00	...	2.02723E-01
1	200		15	4.14958E-02	...	5.34182E+00	...	1.59781E-01
1	250		16	4.53988E-02	...	5.30595E+00	...	1.16404E-01
1	300		17	4.93144E-02	...	5.27237E+00	...	7.26398E-02
1	350		18	5.32267E-02	...	5.24098E+00	...	2.85347E-02
1	383	UZ	19	5.57355E-02	...	5.22189E+00	...	8.24871E-09
1	400	EP	20	5.83763E-02	...	5.20261E+00	...	-3.02850E-02

Label 19: **switching (buckling) bifurcation** (SW)

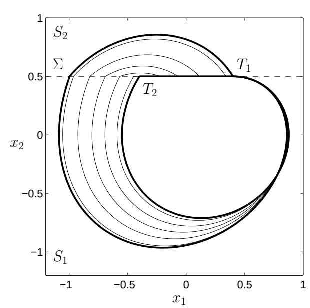
The family of sliding cycles without crossings:



Continuation of the sliding cycle with a crossing:

BR	PT	TY	LAB	PAR(2)	...	PAR(11)	PAR(12)	PAR(45)
1	60		2	5.90369E-02	...	5.19791E+00	1.52241E-01	7.61637E-01
1	120		3	6.24130E-02	...	5.17465E+00	3.10107E-01	7.20834E-01
1	180		4	6.60618E-02	...	5.15083E+00	4.81592E-01	6.74597E-01
1	240		5	7.00156E-02	...	5.12647E+00	6.66284E-01	6.21526E-01
1	300		6	7.44073E-02	...	5.10105E+00	8.67216E-01	5.58318E-01
1	360		7	7.93300E-02	...	5.07446E+00	1.08361E+00	4.81218E-01
1	420		8	8.48984E-02	...	5.04663E+00	1.31315E+00	3.84667E-01
1	480		9	9.18210E-02	...	5.01510E+00	1.57172E+00	2.48336E-01
1	540		10	1.01313E-01	...	4.97688E+00	1.87536E+00	2.60810E-02
1	547	UZ	11	1.02312E-01	...	4.97317E+00	1.90398E+00	2.02644E-12
1	600	EP	12	1.10002E-01	...	4.94653E+00	2.10506E+00	-2.20614E-01

Label 11: **crossing bifurcation** (CC)

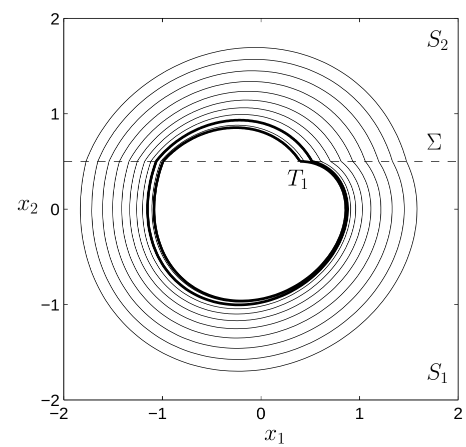


Continuation of the crossing cycle:

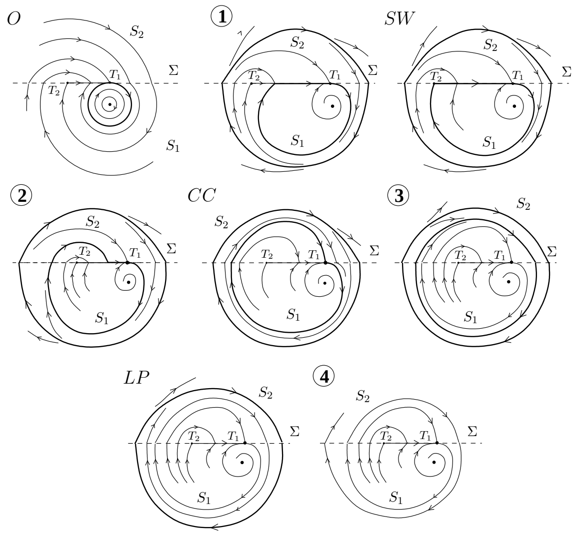
BR	PT	TY	LAB	PAR(2)	...	PAR(12)	PAR(41)
1	20		2	7.16260E-02	...	2.55611E+00	1.07609E+00
1	40		3	7.65905E-02	...	2.50756E+00	9.33090E-01
1	60		4	8.18743E-02	...	2.45264E+00	7.93320E-01
1	80		5	8.73065E-02	...	2.39121E+00	6.58564E-01
1	100		6	9.25997E-02	...	2.32381E+00	5.30933E-01
1	120		7	9.73634E-02	...	2.25165E+00	4.12404E-01
1	140		8	1.01168E-01	...	2.17637E+00	3.04205E-01
1	160		9	1.03627E-01	...	2.09946E+00	2.06403E-01
1	180	LP	10	1.04455E-01	...	2.02413E+00	1.20594E-01
1	200		11	1.03528E-01	...	1.94532E+00	3.94702E-02
1	211	UZ	12	1.02312E-01	...	1.90398E+00	-1.02862E-11
1	220	EP	13	1.00829E-01	...	1.86710E+00	-3.35819E-02

Label 10: **limit point of crossing cycles** (LP)
Label 12: **crossing bifurcation** (CC)

The family of crossing cycles:



Phase portraits



4.2. Forced dry-friction oscillations

$$\ddot{x}_1 + x_1 = \alpha_1 \, \mathrm{sgn}(1 - \dot{x}_1) - \alpha_2(1 - \dot{x}_1) + \alpha_3(1 - \dot{x}_1)^3 + \alpha_4 \cos(\alpha_5 t).$$

[Yoshitake & Sueoka, 2002]

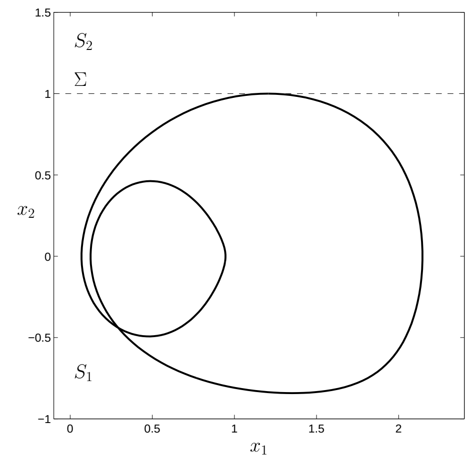
It is equavalent to (1) with $x \in \mathbb{R}^4, H(x,\alpha) = x_2 - 1$,

$$f^{(1,2)}(x,\alpha) = \begin{pmatrix} x_2 \\ -x_1 + g_{1,2}(x,\alpha) + \alpha_4 x_3 \\ x_3 - \alpha_5 x_4 - x_3(x_3^2 + x_4^2) \\ \alpha_5 x_3 + x_4 - x_4(x_3^2 + x_4^2) \end{pmatrix},$$

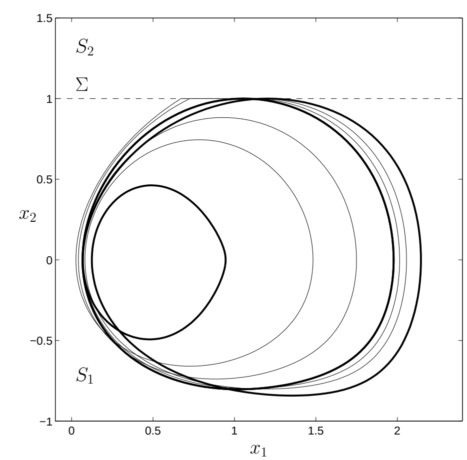
where $g_{1,2}(x,\alpha) = \pm\alpha_1 - \alpha_2(1 - x_2) + \alpha_3(1 - x_2)^3$.

There is a grazing $\frac{8\pi}{\alpha_5}$-cycle at

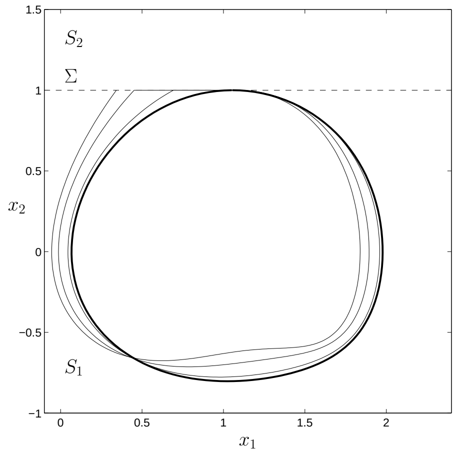
$$\alpha_1 = \alpha_2 = 1.5, \alpha_3 = 0.45, \alpha_4 = 0.1, \alpha_5 = 1.7078\ldots$$



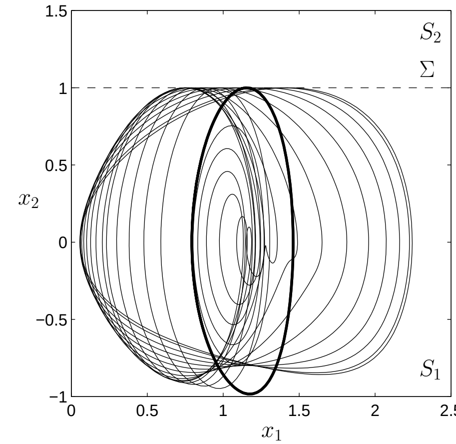
A family of sliding cycles originated from the $\frac{8\pi}{\alpha_5}$-periodic grazing cycle:



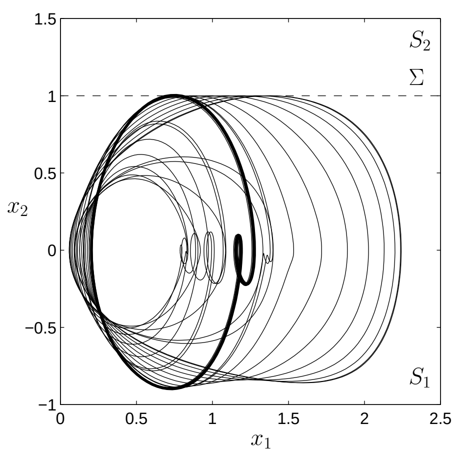
A family of sliding cycles originated from the $\frac{4\pi}{\alpha_5}$-periodic grazing cycle:



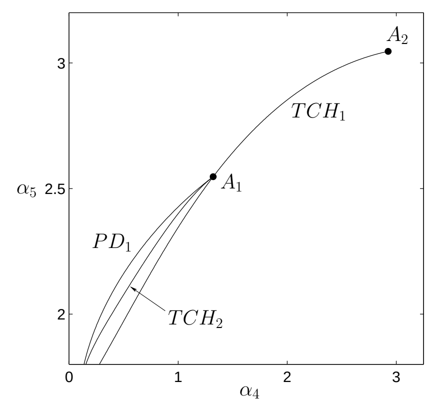
A family of $\frac{4\pi}{\alpha_5}$-periodic grazing cycles:



A family of $\frac{8\pi}{\alpha_5}$-periodic grazing cycles:



Sliding bifurcation curves:



TCH_1 - grazing bifurcation of the $\frac{4\pi}{\alpha_5}$-cycle
TCH_2 - grazing bifurcation of the $\frac{8\pi}{\alpha_5}$-cycle
$A_{1,2}$ - codimension-2 flip-grazing points