Lecture 5

Numerical continuation of connecting orbits of iterated maps and ODEs

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1. Point-to-point connections

- Consider a diffeomorphism

\[ x \mapsto f(x), \quad x \in \mathbb{R}^n, \]

having fixed points \( x^- \) and \( x^+ \), \( f(x^\pm) = x^\pm \).

**Def. 1** An orbit \( \Gamma = \{ x_k \}_{k \in \mathbb{Z}} \) where \( x_{k+1} = f(x_k) \), is called **heteroclinic** between \( x^- \) and \( x^+ \) if

\[ \lim_{k \to \pm \infty} x_k = x^\pm. \]

If \( x^\pm = x^0 \), it is called **homoclinic** to \( x^0 \).

- Introduce **unstable** and **stable invariant sets**

\[ W^u(x^-) = \{ x \in \mathbb{R}^n : \lim_{k \to \infty} f^{-k}(x) = x^- \}, \]

\[ W^s(x^+) = \{ x \in \mathbb{R}^n : \lim_{k \to +\infty} f^k(x) = x^+ \}. \]

Then \( \Gamma \subset W^u(x^-) \cap W^s(x^+) \).
• **Def. 2** A homoclinic orbit $\Gamma$ is called **regular** if $f_x(x^0)$ has no eigenvalues with $|\mu| = 1$ and the intersection of $W^u(x^0)$ and $W^s(x^0)$ along $\Gamma$ is transversal.

\[ x_0 \leftarrow f(x,\alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \]
regular homoclinic orbits exist in open parameter intervals.
Consider a family of ODEs

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \]

having equilibria \( x^- \) and \( x^+ \), \( f(x^\pm, \alpha) = 0 \).

**Def. 3** An orbit \( \Gamma = \{ x = x(t) : t \in \mathbb{R} \} \), where \( x(t) \) is a solution to the ODE system at some \( \alpha \), is called **heteroclinic** between \( x^- \) and \( x^+ \) if

\[
\lim_{t \to \pm \infty} x(t) = x^\pm.
\]

If \( x^\pm = x^0 \), it is called **homoclinic** to \( x^0 \).

- Introduce **unstable** and **stable invariant sets**

\[
W^u(x^-) = \{ x(0) \in \mathbb{R}^n : \lim_{t \to -\infty} x(t) = x^- \},
\]
\[
W^s(x^+) = \{ x(0) \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = x^+ \}.
\]

Then \( \Gamma \subset W^u(x^-) \cap W^s(x^+) \).
• The intersection of $W^u(x^0)$ and $W^s(x^0)$ cannot be transversal along a homoclinic orbit $\Gamma$, since $\dot{x}(t) \in T_{x(t)}W^u(x^0) \cap T_{x(t)}W^s(x^0)$.

• Homoclinic orbits exist in generic ODE families only at isolated parameter values.
Def. 4 A homoclinic orbit $\Gamma$ is called regular if

- $f_x(x^0)$ has no eigenvalues with $\Re(\lambda) = 0$;
- $\dim(T_{x(t)}W^u(x^0) \cap T_{x(t)}W^s(x^0)) = 1$;
- The intersection of the traces of $W^u(x^0)$ and $W^s(x^0)$ along $\Gamma$ is transversal in the $(x, \alpha)$-space.
2. Continuation of homoclinic orbits of maps

- **Homoclinic problem**

\[
\begin{align*}
\begin{cases}
    f(x^0, \alpha) - x^0 &= 0, \\
    x_{k+1} - f(x_k, \alpha) &= 0, \ k \in \mathbb{Z}, \\
    \lim_{k \to \pm\infty} x_k - x^0 &= 0.
\end{cases}
\end{align*}
\]

- **Truncate** with the projection boundary conditions:

\[
\begin{align*}
\begin{cases}
    f(x^0, \alpha) - x^0 &= 0, \\
    x_{k+1} - f(x_k, \alpha) &= 0, \ k \in [-K, K-1], \\
    L_s^T (x^0, \alpha)(x_{-K} - x^0) &= 0, \\
    L_u^T (x^0, \alpha)(x_{+K} - x^0) &= 0,
\end{cases}
\end{align*}
\]

where the columns of $L_s$ and $L_u$ span the orthogonal complements to $T^u = T_{x^0}W^u(x^0)$ and $T^s = T_{x^0}W^s(x^0)$, resp.
• Assume the eigenvalues of $A = f_x(x^0, \alpha)$ are arranged as follows:

$$|\mu_{n_s}| \leq \cdots \leq |\mu_1| < 1 < |\lambda_1| \leq \cdots \leq |\lambda_{n_u}|$$

If $V^* = \{v_1^*, \ldots, v_{n_s}^*\}$ and $W^* = \{w_1^*, \ldots, w_{n_u}^*\}$ span the stable and unstable eigenspaces of $A^\top$, then **Fredholm’s Alternative** implies: $L_s = [V^*]$ and $L_u = [W^*]$.

• Let $(\mu, \lambda)$ satisfy $|\mu_1| < \mu < 1 < \lambda < |\lambda_1|$ and $\nu = \max(\mu, \lambda^{-1})$.

**Th. 1 (Beyn–Kleinkauf)** *There is a locally unique solution to the truncated problem for a regular homoclinic orbit with an error that is $O(\nu^{2K})$.*

• The truncated system is an ALCP in $\mathbb{R}^{2nK + 2n + 1}$ to which the standard continuation methods are applicable.
3. Continuation of homoclinic orbits of ODEs

- **Homoclinic problem**

\[
\begin{aligned}
    f(x^0, \alpha) &= 0, \\
    \dot{x}(t) - f(x(t), \alpha) &= 0, \\
    \lim_{t \to \pm \infty} x(t) - x^0 &= 0, \quad t \in \mathbb{R}, \\
    \int_{-\infty}^{\infty} \langle \dot{y}(t), x(t) - y(t) \rangle dt &= 0,
\end{aligned}
\]

where \( y \) is a reference homoclinic solution.

- **Truncate** with the **projection boundary conditions**:

\[
\begin{aligned}
    f(x^0, \alpha) &= 0, \\
    \dot{x}(t) - f(x(t), \alpha) &= 0, \quad t \in [-T, T] \\
    L_s^T(x^0, \alpha)(x(-T) - x^0) &= 0, \\
    L_u^T(x^0, \alpha)(x(+T) - x^0) &= 0, \\
    \int_{-T}^{T} \langle \dot{y}(t), x(t) - y(t) \rangle dt &= 0,
\end{aligned}
\]

where the columns of \( L_s \) and \( L_u \) span the orthogonal complements to \( T^u = T_{x^0} W^u(x^0) \) and \( T^s = T_{x^0} W^s(x^0) \), resp.

- The truncated system is a BVCP to which the standard discretization and continuation methods are applicable when \( \alpha \in \mathbb{R}^2 \).
• Assume the eigenvalues of $A = f_x(x^0, \alpha)$ are arranged as follows:

$$\Re(\mu_{n_s}) \leq \cdots \leq \Re(\mu_1) < 0 < \Re(\lambda_1) \leq \cdots \leq \Re(\lambda_{n_u})$$

If $V^* = \{v_1^*, \ldots, v_{n_s}^*\}$ and $W^* = \{w_1^*, \ldots, w_{n_u}^*\}$ span the stable and unstable eigenspaces of $A^T$, then $L_s = [V^*]$ and $L_u = [W^*]$.

• Let $(\mu, \lambda)$ satisfy $\Re(\mu_1) < \mu < 0 < \lambda < \Re(\lambda_1)$ and $\omega = \min(|\mu|, \lambda)$.

**Th. 2 (Beyn)** There is a locally unique solution to the truncated problem for a regular homoclinic orbit with the $(x(\cdot), \alpha)$-error that is $O(e^{-2\omega T})$. 
Remarks:

1. If $W^u$ is one-dimensional, one can use the explicit boundary conditions

$$x(-T) - (x^0 + \varepsilon w_1) = 0,$$
$$\langle w^*_1, x(T) - x^0 \rangle = 0,$$

where $Aw_1 = \lambda_1 w_1$ and $A^T w^*_1 = \lambda_1 w^*_1$, without the integral phase condition.

2. Under implementation in MATCONT with possibilities to start

(i) from a large period cycle;

(ii) by homotopy.

(iii) from a codim 2 bifurcations of equilibria, i.e. BT and ZH;

3. Th. 3 (L.P. Shilnikov) There is always at least one limit cycle arbitrary close to $\Gamma$ near the bifurcation. There are infinitely many cycles nearby when $\mu_1$ and $\lambda_1$ are both complex, or when one of them is complex and has the smallest absolute value of the real part.
4. Continuation of invariant subspaces

Th. 4 (Smooth Schur Block Factorization)
Any parameter-dependent matrix \( A(s) \in \mathbb{R}^{n \times n} \) with nontrivial stable and unstable eigenspaces can be written as

\[
A(s) = Q(s) \begin{bmatrix}
R_{11}(s) & R_{12}(s) \\
0 & R_{22}(s)
\end{bmatrix} Q^T(s),
\]

where \( Q(s) = [Q_1(s) \quad Q_2(s)] \) such that

- \( Q(s) \) is orthogonal, i.e. \( Q^T(s)Q(s) = I_n \);
- the eigenvalues of \( R_{11}(s) \in \mathbb{R}^{m \times m} \) are the unstable eigenvalues of \( A(s) \), while the eigenvalues of \( R_{22}(s) \in \mathbb{R}^{(n-m) \times (n-m)} \) are the remaining \((n - m)\) eigenvalues of \( A(s) \);
- the columns of \( Q_1(s) \in \mathbb{R}^{n \times m} \) span the eigenspace \( \mathcal{E}(s) \) of \( A(s) \) corresponding to its \( m \) unstable eigenvalues;
- the columns of \( Q_2(s) \in \mathbb{R}^{n \times (n-m)} \) span the orthogonal complement \( \mathcal{E}^\perp(s) \);
- \( Q_i(s) \) and \( R_{ij}(s) \) have the same smoothness as \( A(s) \).

Then holds the invariant subspace relation:

\[
Q_2^T(s)A(s)Q_1(s) = 0.
\]
CIS-algorithm [Dieci & Friedman]

- Define

\[
\begin{bmatrix}
T_{11}(s) & T_{12}(s) \\
T_{21}(s) & T_{22}(s)
\end{bmatrix} = Q^T(0)A(s)Q(0)
\]

for small \(|s|\), where \(T_{11}(s) \in \mathbb{R}^{m\times m}\).

- Compute by Newton’s method \(Y \in \mathbb{R}^{(n-m)\times m}\) satisfying the Riccati matrix equation

\[
YT_{11}(s) - T_{22}(s)Y + YT_{12}(s)Y = T_{21}(s).
\]

- Then \(Q(s) = Q(0)U(s)\) where

\[
U(s) = [U_1(s) \ U_2(s)]
\]

with

\[
U_1(s) = \begin{pmatrix} I_m \\ Y \end{pmatrix} (I_{n-m} + Y^TY)^{-\frac{1}{2}},
\]

\[
U_2(s) = \begin{pmatrix} -Y^T \\ I_{n-m} \end{pmatrix} (I_{n-m} + YY^T)^{-\frac{1}{2}},
\]

so that columns of \(Q_1(s) = Q(0)U_1(s)\) and \(Q_2(s) = Q(0)U_2(s)\) form orthogonal bases in \(\mathcal{E}(s)\) and \(\mathcal{E}^\perp(s)\).
5. Detection of higher-order singularities

- For homoclinic orbits to fixed points, $x^0$ can exhibit one of codim 1 bifucations. LP's of the (truncated) ALCP correspond to homoclinic tangencies:

- For homoclinic orbits to equilibria, there are many codim 2 cases:

  1. fold or Hopf bifurcations of $x^0$;

  2. special eigenvalue configurations (e.g. $\sigma = R(\mu_1) + R(\lambda_1) = 0$ or $\mu_1 - \mu_2 = 0$);

  3. change of global topology of $W^s$ and $W^n$ (orbit and inclination flips);

  4. higher nontransversality.
6. Cycle-to-cycle connections in 3D ODEs

- Consider
  \[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}^p. \]

- Let \( O^- \) be a limit cycle with only one (trivial) multiplier satisfying \( |\mu| = 1 \) and having \( \dim W^-_u = m_u^- \).

- Let \( O^+ \) be a limit cycle with only one (trivial) multiplier satisfying \( |\mu| = 1 \) and having \( \dim W^+_s = m_s^+ \).

- Let \( x^\pm(t) \) be periodic solutions (with minimal periods \( T^\pm \)) corresponding to \( O^\pm \) and \( M^\pm \) the corresponding monodromy matrices, i.e. \( M(T^\pm) \) where
  \[ \dot{M} = f_x(x^\pm(t), \alpha)M, \quad M(0) = I_n. \]

- Then \( m_s^+ = n_s^+ + 1 \) and \( m_u^- = n_u^- + 1 \), where \( n_s^+ \) and \( n_u^- \) are the numbers of eigenvalues of \( M^\pm \) satisfying \( |\mu| < 1 \) and \( |\mu| > 1 \), resp.
Isolated families of connecting orbits

- **Beyn’s equality**: \( p = n - m_s^+ - m_u^- + 2 \).
- The cycle-to-cycle connections in \( \mathbb{R}^3 \):

  - heteroclinic orbit

  - homoclinic orbit \( \Rightarrow \) infinite number of cycles
Truncated BVCP

- The connecting solution $u(t)$ is truncated to an interval $[\tau_-, \tau_+]$.

- The points $u(\tau_+)$ and $u(\tau_-)$ are required to belong to the linear subspaces that are tangent to the stable and unstable invariant manifolds of $O^+$ and $O^-$, respectively:

$$
\begin{align*}
L^T_+(u(\tau_+) - x^+(0)) &= 0, \\
L^T_-(u(\tau_-) - x^-(0)) &= 0.
\end{align*}
$$

- Generically, the truncated BVP composed of the ODE, the above projection BC’s, and a phase condition on $u$, has a unique solution family $(\tilde{u}, \tilde{\alpha})$, provided that the ODE has a connecting solution family satisfying the phase condition and Beyn’s equality.
**Th. 5 (Pampel–Dieci–Rebaza)** If $u$ is a generic connecting solution to the ODE at parameter value $\alpha$, then the following estimate holds:

$$
\|(u|_{[\tau_-,\tau_+]}, \alpha) - (\hat{u}, \hat{\alpha})\| \leq Ce^{-2\min(\mu_-|\tau_-|, \mu_+|\tau_+|)}
$$

where

- $\| \cdot \|$ is an appropriate norm in the space $C^1([\tau_-, \tau_+], \mathbb{R}^n) \times \mathbb{R}^p$,
- $u|_{[\tau_-,\tau_+]}$ is the restriction of $u$ to the truncation interval,
- $\mu_{\pm}$ are determined by the eigenvalues of the monodromy matrices $M_{\pm}$.

**Adjoint variational equation:**

$$
\dot{w} = - f_x^T(x^{\pm}(t), \alpha)w, \quad w \in \mathbb{R}^n.
$$

Let $N(t)$ be the solution to

$$
\dot{N} = - f_x^T(x^{\pm}(t), \alpha)N, \quad N(0) = I_n.
$$

Then $N(T^{\pm}) = [M^{-1}(T^{\pm})]^T$. 
The defining BVCP in 3D

Cycle-related equations:

- Periodic solutions:
  \[
  \begin{align*}
  \dot{x}^\pm - f(x^\pm, \alpha) &= 0, \\
  x^\pm(0) - x^\pm(T^\pm) &= 0.
  \end{align*}
  \]

- Adjoint eigenfunctions: \( \mu^+ = \frac{1}{\mu_u^+}, \mu^- = \frac{1}{\mu_u^-} \).
  \[
  \begin{align*}
  \dot{w}^\pm + f_u^T(x^\pm, \alpha)w^\pm &= 0, \\
  w^\pm(T^\pm) - \mu^\pm w^\pm(0) &= 0, \\
  \langle w^\pm(0), w^\pm(0) \rangle - 1 &= 0,
  \end{align*}
  \]
  or equivalently
  \[
  \begin{align*}
  \dot{w}^\pm + f_u^T(x^\pm, \alpha)w^\pm + \lambda^\pm w^\pm &= 0, \\
  w^\pm(T^\pm) - s^\pm w^\pm(0) &= 0, \\
  \langle w^\pm(0), w^\pm(0) \rangle - 1 &= 0,
  \end{align*}
  \]
  where \( \lambda^\pm = \ln |\mu^\pm|, \ s^\pm = \text{sign}(\mu^\pm) \).

- Projection BC: \( \langle w^\pm(0), u(\tau^\pm) - x^\pm(0) \rangle = 0 \).
Connection-related equations:

- The equation for the connection:
  \[ \dot{u} - f(u, \alpha) = 0. \]

- We need the base points \( x^\pm(0) \) to move freely and independently upon each other along the corresponding cycles \( O^\pm \).

- We require the end-point of the connection to belong to a plane orthogonal to the vector \( f(x^+(0), \alpha) \), and the starting point of the connection to belong to a plane orthogonal to the vector \( f(x^-(0), \alpha) \):
  \[ \langle f(x^\pm(0), \alpha), u(\tau_\pm) - x^\pm(0) \rangle = 0. \]
The defining BVCP in 3D:

\[
\begin{align*}
\dot{x}^\pm - T^\pm f(x^\pm, \alpha) &= 0, \\
x^\pm(0) - x^\pm(1) &= 0, \\
\dot{w}^\pm + T^\pm f_u^T(x^\pm, \alpha)w^\pm + \lambda^\pm w^\pm &= 0, \\
w^\pm(1) - s^\pm w^\pm(0) &= 0, \\
\langle w^\pm(0), w^\pm(0) \rangle - 1 &= 0, \\
\dot{u} - T f(u, \alpha) &= 0, \\
\langle f(x^+(0), \alpha), u(1) - x^+(0) \rangle &= 0, \\
\langle f(x^-(0), \alpha), u(0) - x^-(0) \rangle &= 0, \\
\langle w^+(0), u(1) - x^+(0) \rangle &= 0, \\
\langle w^-(0), u(0) - x^-(0) \rangle &= 0, \\
\|u(0) - x^-(0)\|^2 - \varepsilon^2 &= 0.
\end{align*}
\]

There is an efficient **homotopy method** to find a starting solution.