TUTORIAL III:
One-parameter bifurcation analysis of limit cycles with
MATCONT

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This session is devoted to the numerical continuation of limit cycles in systems of autonomous ODEs depending on one parameter

\[ \dot{x} = f(x, \alpha), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}, \]

and detection of their bifurcations. We will also switch to the continuation of the limit cycle at the Hopf bifurcation and to the continuation of the doubled cycle at the period-doubling bifurcation.

1 Fold and Neimark-Sacker bifurcations of cycles in a chemical model

Consider the following chemical model by Steinmetz and Larter:

\[
\begin{align*}
\dot{A} &= -k_1 ABX - k_3 ABY + k_7 - k_{-7} A, \\
\dot{B} &= -k_1 ABX - k_3 ABY + k_8, \\
\dot{X} &= k_1 ABX - 2k_2 X^2 + 2k_3 ABY - k_4 X + k_6, \\
\dot{Y} &= -k_3 ABY + 2k_2 X^2 - k_5 Y.
\end{align*}
\]  

(1)

We will fix all parameters but \(k_7\) and study periodic solutions (limit cycles) of (1) when this parameter varies.

1.1 System specification

Specify a new ODE system – say StLar – in MATCONT:

\[
\begin{align*}
A' &= -k_1 A B X - k_3 A B Y + k_7 - k_{-7} A \\
B' &= -k_1 A B X - k_3 A B Y + k_8 \\
X' &= k_1 A B X - 2k_2 X^2 + 2k_3 A B Y - k_4 X + k_6 \\
Y' &= -k_3 A B Y + 2k_2 X^2 - k_5 Y
\end{align*}
\]

where \((A, B, X, Y)\) are the coordinates and \((k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_{-7}, k_8)\) are the parameters. Use (default) \(t\) for time and generate symbolically partial derivatives of order 1, 2, and 3.

1.2 Fold bifurcation of limit cycles

1.2.1 Continuation of an equilibrium

We will continue an equilibrium of (1) and detect its Hopf bifurcation, from which the continuation of a limit cycle can be started.

Input Type|Initial point|Equilibrium and Type|Curve|Equilibrium in the main MatCont window.

Input the following numerical data in the appeared Starter window:

\[
\begin{align*}
A &= 31.78997 \\
B &= 1.45468 \\
X &= 0.01524586 \\
Y &= 0.1776113 \\
k_1 &= 0.1631021 \\
k_2 &= 1250 \\
k_3 &= 0.046875 \\
k_4 &= 20 \\
k_5 &= 1.104 \\
k_6 &= 0.001 \\
k_7 &= 4.235322 \\
k_{-7} &= 0.1175 \\
k_8 &= 0.5
\end{align*}
\]
These values correspond to an unstable equilibrium \((A,B,X,Y)\) in the system. Activate the parameter \(k_7\).

Open **Window|Numeric** and change its appearance via the **Window|Layout** command. Namely, select EIGENVALUES to be shown in the window.

Use **Window|Graphic|2Dplot** to open the corresponding window and select the coordinates \((A,B)\) as abscissa and ordinate, respectively, with the visibility limits (as always, make sure that the lower limit is at all times lower than the upper limit): 

<table>
<thead>
<tr>
<th>Abscissa:</th>
<th>32</th>
<th>38</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordinate:</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

respectively.

Start **Compute|Forward**. The equilibrium curve will be continued and you get a Hopf bifurcation point labeled by \(\mathcal{H}\). The message in MATLAB Command Window

\[
\text{label} = \mathcal{H}, \quad x = (34.808899, 1.328517, 0.015246, 0.177611, 4.590046)
\]

First Lyapunov coefficient = 1.527549e-02

at \(k_7 = 4.500046\ldots\) indicates a subcritical Hopf bifurcation. Indeed, there are two eigenvalues of the equilibrium with \(\text{Re} \lambda_1,\lambda_2 \approx 0\) at this parameter value visible in the **Numeric** window. The critical frequency \(\text{Im} \lambda_1 \neq 0\), while the first Lyapunov coefficient is positive. Thus, there should exist an unstable limit cycle, bifurcating from the equilibrium. Resume computations and terminate them when the curve leaves the graphic window.

Rename the computed curve via **Select|Curve** and **Actions|Rename** into 

Equilibrium(+) 

1.2.2 Cycle continuation

Click **Select|Initial point** and select \(\mathcal{H}\): Hopf point in the Equilibrium(+) curve as initial. The **Starter** and **Continuer** windows for the continuation of the limit cycle from the Hopf point will appear.

Select yes in all **Monitor Singularities** fields of the **Starter** window. Increase the MaxStep-size to 1.0 in the **Continuer** window to allow larger steps along the curve and set MaxNumPoints to 50. The **Starter** and **Continuer** windows should look like in Figure 1.

Change the layout of the **Numeric** window via **Window|Layout** by selecting all MULTIPLIERS to be shown. The absolute values (modulae) and arguments in angular grads will be displayed.

**Compute|Forward** will produce a family of cycles with a cycle limit point labeled by LPC at \(k_7 = 4.74838\ldots\). The **Numeric** window corresponding to LPC is shown in Figure 2.

In the MATLAB Command Window, the following message appears:

Limit point cycle (period = 1.036108e+01, parameter = 4.748384e+00)

Normal form coefficient = -2.714838e-01

The critical cycle has (approximately) a double multiplier \(\mu = 1\) and the normal form coefficient is nonzero. Thus, the limit cycle manifold has a fold here. Resume the computations. The continuation algorithm will automatically follow the second (stable) cycle branch after the LPC point. Verify this by looking at the nontrivial multipliers in the **Numeric** window (\(|\mu| < 1\) for all such multipliers). The computations produce Figure 3 (after **Plot|Redraw diagram** in the **2Dplot** window.

To visualize LPC in another way, open another **2Dplot** window and select parameter \(k_7\) and Period of the cycle as abscissa and ordinate, respectively. Use the visibility limits (make sure that at all times the lower limit is lower than the upper limit)

<table>
<thead>
<tr>
<th>Abscissa:</th>
<th>4.5</th>
<th>4.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordinate:</td>
<td>7</td>
<td>12</td>
</tr>
</tbody>
</table>

You will get a curve with a limit point, clearly indicating the presence of two limit cycles with different periods for \(k_7 < 4.74838\) near LPC (see Figure 4). Close both **2Dplot** windows.
Figure 1: The **Starter** and **Continuer** windows for the cycle continuation.
Figure 2: The **Numeric** window at the LPC-point.
Figure 3: The family of limit cycles bifurcating from the Hopf point \( H \): LPC is a fold bifurcation of the cycle.
Figure 4: Period of the cycle versus $k_7$. 
1.3 Neimark-Sacker bifurcation

1.3.1 Backward continuation of the equilibrium

Select the first computed equilibrium curve \textit{Equilibrium(+)} in the dialog box appearing after the \textbf{Select|Curve} command. Select \textbf{Type|Curve|Equilibrium} for continuation.

Open a new \textit{2Dplot} window but select now \textit{B} and \textit{X} as abscissa and ordinate, respectively, with the visibility limits (make sure that at all times the lower limit is lower than the upper limit)

\begin{align*}
\text{Abscissa:} & \quad 25 \quad 27 \\
\text{Ordinate:} & \quad 0 \quad 0.03
\end{align*}

\textbf{Compute|Backward} and \textbf{Extend} once until you get another \textit{Hopf bifurcation} at \(k_7 = 0.712475\ldots\) with the message

\texttt{label = H , x = ( 1.808301 25.573303 0.015246 0.177611 0.712475 )}

First Lyapunov coefficient = \(-2.371880\times10^{-02}\)

in the MATLAB Command Window. The first Lyapunov coefficient is negative now. This means that a stable limit cycle bifurcates from the equilibrium, when it looses stability. Resume computations and terminate them when the equilibrium curve leaves the graphic window. Rename the computed curve into \textit{Equilibrium(-)}.

1.3.2 Cycle continuation

Select the Hopf point in the curve \textit{Equilibrium(-)} as initial. \textsc{matcont} will prepare to continue a limit cycle curve from the Hopf point (curve type \textit{H|LC}). Select \textit{yes} in all \textit{Monitor Singularities} fields of the \textbf{Starter} window. Activate \(k_7\) and the Period in the \textbf{Starter} window. Set

\begin{align*}
\text{MaxStepsize} & \quad 1 \\
\text{MaxNumPoints} & \quad 25
\end{align*}

in the \textbf{Continuer} window. Notice that in the \textbf{Numeric} window the eigenvalues were replaced by multipliers of the cycle.

Click \textbf{Compute|Forward} to start the continuation of the limit cycle. At \(k_7 = 0.716434\ldots\) the message \textit{Neimark-Sacker} indicates a \textit{torus} bifurcation. In the MATLAB Command Window, the following message appears:

\texttt{Neimark-Sacker (period = 1.091213e+01, parameter = 7.164336e-01)}

Normal form coefficient = \(-4.912065\times10^{-08}\).

Indeed, there are two complex multipliers with (approximately) \(|\mu| = 1\) and one trivial at \(\mu = 1\). This can be seen in the \textbf{Numeric} window that stays open, see Figure 5. The normal form coefficient is small but nonzero, indicating that a stable two-dimensional \textit{invariant torus} bifurcates from the limit cycle.

Resume computations further to see that after the \textit{NS}-point the cycle becomes unstable (with two multipliers satisfying \(|\mu| > 1\). You should get a family of limit cycles bifurcating from the Hopf point as in Figure 6.

Close the \textbf{Numeric} window.

1.3.3 Dynamics on a stable torus

Click \textbf{Select|Initial point}... and select the \textit{NS: Neimark-Sacker} point in the computed limit cycle curve. The critical parameter values will be read in from the archive. Immediately after that select \textbf{Type|Initial point|Point} and \textbf{Type|Curve|Orbit}. A point on the critical cycle will thus be selected as the initial for orbit integration.

Increase slightly the value of the parameter \(k_7\) and perturb the initial point by altering the value of \(B\), namely set:
Figure 5: The **Numeric** window at the **NS**-point.
Figure 6: Family of limit cycles bifurcating from the second Hopf point
k7 \hspace{0.5cm} 0.7165 \\
B \hspace{0.5cm} 26.2 \\

Change Method of integration to \texttt{ode23s} and alter

Interval \hspace{0.5cm} 1000 \\
Rel.Tol.\hspace{0.5cm} 1e-6 \\
Abs.Tol.\hspace{0.5cm} 1e-9 \\

in the Integrator window.

To speed up visualization, click Options|Output and set Plot after 1000 points, see Figure 7.

![Output window](image)

Figure 7: Output window.

Clear the graphic window and Compute|Forward followed by Compute|Extend. The integration will take some time. After a transient, the orbit will exhibit modulated oscillations with two frequencies near the limit cycle. This is a motion on a stable two-dimensional torus born via the Neimark-Sacker bifurcation. See Figure 8. Open another 2Dplot window with time \( t \) and coordinate \( B \) as abscissa and ordinate, with the visibility limits (make sure that at all times the lower limit is lower than the upper limit).

![Dynamics on a stable 2-torus](image)

Figure 8: Dynamics on a stable 2-torus
You will see the high-frequency oscillations with low-frequency modulation as in Figure 9.

Restore the original Plot after 1 point setting using Options\Output menu of the main MatCont window.

2 Period-doubling bifurcation in an adaptive control model

Consider the following adaptive control system of Lur’e type

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= z, \\
\dot{z} &= -\alpha z - \beta y - x + x^2.
\end{align*}
\]

We will fix $\alpha = 1$ and continue limit cycles of (2) using $\beta$ as a bifurcation parameter.

2.1 System specification

As usual, define a new system in MATCONT with the coordinates $x, y, z$, parameters $\alpha, \beta$, and select derivatives of order 1, 2, and 3 to be generated symbolically.

2.2 Equilibrium continuation

Select Type|Initial point|Equilibrium and Type|Curve|Equilibrium. In the Starter window, set $\alpha$ equal to 1 and activate $\beta$.

Open a 3Dplot window and select $\beta, x$ and $y$ as variables along the coordinate axes with the visibility limits (make sure that at all times the lower limit is lower than the upper limit)

Abscissa: 0.4 1.2
Ordinate: -0.5 1.2
Applicate: -0.6 0.7

respectively.

Compute|Forward results in an equilibrium curve with a Hopf bifurcation (labeled H) detected with the following message in the MATLAB Command Window:

\[\text{label = H , } x = ( 0.000000 \ 0.000000 \ 0.000000 \ 1.000000 )\]
\[\text{First Lyapunov coefficient = -3.000000e-01}\]

The Hopf bifurcation occurs at $\beta = 1$ and is supercritical, thus generating a stable limit cycle. Resume the continuation and eventually stop it, when the curve leaves the plotting region.
2.3 Cycle continuation starting from the H-point

Click Select|Initial point and choose H: Hopf point in the computed EP_EP(1) curve. The Starter and Continuer windows appear, corresponding to the limit cycle curve H_LC. Make sure that beta and Period are activated.

In the Starter window, increase the number of mesh points nst to 40 and select yes in all Minor Singularities fields. In the Continuer window, set MaxStepsize to 0.7 and MaxNumPoints to 80.

With Window|Numeric, open a Numeric window and make the cycle MULTIPLIERS visible using the Window|Layout command there.

Compute|Forward will produce a branch of limit cycles of increasing amplitude with two period-doubling bifurcations (labeled PD). The Numeric windows at both PD bifurcations are shown in Figure 10.

Figure 10: The Numeric windows at both period-doubling bifurcations. The bifurcation parameter values are $\beta_{PD,1} = 0.6273246$... and $\beta_{PD,2} = 0.5417461$.

Resume the computations at both PD points. The MATLAB Command Window will contain the messages

Period Doubling (period = 8.386068e+00, parameter = 6.273245e-01)  
Normal form coefficient = -3.405330e-03

Period Doubling (period = 9.864971e+00, parameter = 5.417461e-01)  
Normal form coefficient = -7.655596e-04

indicating that in both cases the normal form coefficients are negative, so that stable double-period cycles are involved.

You get a cycle manifold shown in Figure 11. The period of the cycle rapidly increases when $\beta$ approaches $\beta_{hom} = 0.515489$... To see this, open a 2Dplot window (see Figure 12) with the axes beta and Period and the plotting region

Abscissa: 0.51 0.53 
Ordinate: 0.0 35.0

Actually, the limit cycle approaches a homoclinic orbit to the second equilibrium $(x,,y,z) = (1,0,0)$ of (2), which is a global bifurcation. Below $\beta_{hom}$ no cycle exists. Near the homoclinic orbit

---

1 At $\beta = \beta_{hom}$, this equilibrium is a saddle-focus with the saddle quantity $\sigma < 0$. Thus, according to Shilnikov’s Theorem, from this homoclinic orbit bifurcates only one limit cycle - the one we computed.
Figure 11: The cycle branch in system (2) rooted at the Hopf point. Two critical cycles undergoing the period-doubling bifurcations are labeled by $PD$.

Figure 12: The cycle period near the homoclinic bifurcation.
bifurcation, the computation of the cycle and its multipliers becomes inaccurate. The loss of accuracy can be seen in the **Numeric** window, where no multiplier close to 1 is present at the last computed point.

Close the **2Dplot** window.

### 2.4 Continuation of the double cycle from the PD-point

Using **Select|Initial point**, take the first PD: **Period Doubling** point in the computed above curve as initial.

Adjust the **3Dplot** window by setting the visibility limits for **beta** as follows (make sure that at all times the lower limit is lower than the upper limit):

Abscissa: 0.5 0.63

In the **Starter** window, set the number of mesh points **ntst** to 80, increase the **amplitude** to 0.005, and select **yes** only to **Branching** in the **Monitor Singularities** section.

In the **Continuer** window, increase **MaxTestIters** to 30, and make sure that **MaxStepsize** remains equal to 0.7.

Click **Compute|Forward** and continue the doubled cycle bifurcating from the first PD-point at $\beta = \beta_{PD,1}$, see Figure 13. This cycle remains stable until at (approximately) $\beta = \beta_{PD,2}$ a **branching point cycle** (**BPC**) is detected. This is not surprising, since the PD-bifurcation for the original limit cycles corresponds to a branching point for the doubled cycles.

Stop computations at **BPC** and exit **MATCONT**.

![Figure 13: The branch of stable doubled cycles connects two PD points.](image-url)
3 Additional Problems

A. Study numerically limit cycles in the system

\[
\begin{align*}
\dot{x} &= \mu x - y - xz, \\
\dot{y} &= x + \mu y, \\
\dot{z} &= -z + y^2 + x^2 z,
\end{align*}
\]

that is another model of a feedback control system.

B. Consider the following predator-prey model

\[
\begin{align*}
\dot{x} &= x - \frac{xy}{1 + \alpha x}, \\
\dot{y} &= -y + \frac{xy}{1 + \alpha x} - \delta y^2.
\end{align*}
\]

1. Prove that the following polynomial system

\[
\begin{align*}
\dot{x} &= x(1 + \alpha x) - xy, \\
\dot{y} &= -(y + \delta y^2)(1 + \alpha x) + xy,
\end{align*}
\]

has the same orbits in the positive quadrant as the original model. Derive an explicit condition on \((\alpha, \delta)\) for system (3) to have a positive equilibrium with one zero eigenvalue. Derive a similar condition for (3) to have a positive equilibrium with a pair of purely imaginary eigenvalues.

2. Introduce new variables

\[
\begin{align*}
\xi &= \ln x, \\
\eta &= \ln y,
\end{align*}
\]

in which the original system becomes:

\[
\begin{align*}
\dot{\xi} &= 1 - \frac{\exp(\eta)}{1 + \alpha \exp(\xi)}, \\
\dot{\eta} &= -1 + \frac{\exp(\xi)}{1 + \alpha \exp(\xi)} - \delta \exp(\eta).
\end{align*}
\]

Fix \(\alpha = 0.3\) and study numerically limit cycles of (4), when the parameter \(\delta\) varies. \(\text{Hint:} \) Begin with finding an equilibrium at \(\delta = 0.35\) by integration.

3. Relate your analytical and numerical results.

C. Consider the following predator-double prey system:

\[
\begin{align*}
\dot{x} &= x(2.4 - x - 6y - 4z), \\
\dot{y} &= y(\beta - x - y - 10z), \\
\dot{z} &= -z(1 - 0.25x - 4y + z),
\end{align*}
\]

modelling dynamics of two prey populations affected by a predator.

1. Starting from \(\beta = 1.77\), find its positive equilibrium by orbit integration.

2. Continue the equilibrium with respect to \(\beta\) until it exhibits a Hopf bifurcation.

3. Continue a limit cycle from the Hopf point and monitor the dependence of its period upon \(\beta\). \(\text{Hint:} \) Use rather big ntst.

4. Plot the cycle at different period values and try to understand its asymptotic shape. \(\text{Hint:} \) Compute and analyze system’s equilibria located in the coordinate planes.
D. Consider the famous Lorenz system

\[
\begin{align*}
\dot{x} &= \sigma(-x + y), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]

with the standard parameter values \(\sigma = 10, b = \frac{8}{3}\), and \(r > 0\). Use MATCONT to analyse its simplest limit cycles numerically.

1. Compute a critical parameter value \(r_{BP}\), at which the trivial equilibrium \((0, 0, 0)\) undergoes a symmetric pitchfork bifurcation.

2. Compute a critical parameter value \(r_H\), corresponding to Hopf bifurcation of nontrivial equilibria.

3. Continue the limit cycle born at the Hopf bifurcation with respect to \(r\) and find out what happens to it. Plot the cycle period as function of \(r\).

4. Demonstrate by numerical integration that for \(r = 400\) the system has a stable symmetric limit cycle and determine its period approximately. Continue this cycle with respect to \(r\) and find the critical parameter value \(r_{BPC}\) corresponding to a branch point of cycles (symmetry breaking). Hint: Use Select cycle button in the P.O Starter window.

Derive explicit formulas for \(r_{BP}\) and \(r_H\) as functions of \((\sigma, b)\) and verify the numerical values obtained with MATCONT.