Lecture 1: Location and analysis of equilibria

1.1 Multivariate Taylor expansions

Let $F : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto F(x)$, be a smooth map. Its multivariate Taylor expansion at $x_0 \in \mathbb{R}^n$ for $h \in \mathbb{R}^n$ with small $\|h\|$ can be written as

$$F(x_0 + h) = F(x_0) + A(x_0)h + \frac{1}{2}B(x_0; h, h) + O(\|h\|^3),$$

where $\|h\| = \sqrt{\langle h, h \rangle}$ with

$$\langle x, y \rangle := x^T y, \quad x, y \in \mathbb{R}^n,$

and $A(x_0) = DF(x_0)$, while $B(x_0; r, s) = D^2F(x_0)(r, s)$. In components:

$$(A(x_0)h)_i = \sum_{j=1}^{n} \frac{\partial F_i(x_0)}{\partial x_j} h_j,$$

$$B_i(x_0; r, s) = \sum_{j,k=1}^{n} \frac{\partial^2 F_i(x_0)}{\partial x_j \partial x_k} r_j s_k,$$

for $r, s \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$. Given $x_0$, the mapping $h \mapsto A(x_0)h$ is linear, while $(r, s) \mapsto B(x_0; r, s)$ is a bilinear form.

Lemma 1 Let $x, x_0 \in \mathbb{R}^n$ and $h := x - x_0$. Then for all $x$ sufficiently close to $x_0$, i.e. $\|h\|$ small enough, hold

$$F(x) = F(x_0) + \int_0^1 A(x_0 + th)h \, dt,$$

$$F(x) = F(x_0) + A(x_0)h + \int_0^1 (1 - t)B(x_0 + th; h, h) \, dt.$$
Proof:
Introduce \( f(t) := F(x_0 + th) \). Then \( \dot{f}(t) = A(x_0 + th)h \) and \( \ddot{f}(t) = B(x_0 + th; h, h) \). This gives
\[
\int_0^1 A(x_0 + th) h \, dt = \int_0^1 \dot{f}(t) \, dt = f(1) - f(0) = F(x_0 + h) - F(x_0),
\]
from which (1.1) follows. Similarly,
\[
\int_0^1 (1-t) B(x_0 + th; h, h) \, dt = \int_0^1 (1-t) \dot{f}(t) dt
= \dot{f}(0) + f(1) - f(0)
= -A(x_0)h + F(x_0 + h) - F(x_0),
\]
implying (1.2).

Define
\[
\|A\|(x) := \sup_{h \neq 0} \frac{\|A(x)h\|}{\|h\|} \quad \text{and} \quad \|B\|(x) := \sup_{h \neq 0} \frac{\|B(x; h, h)\|}{\|h\|^2}.
\]
Then
\[
\|A(x)h\| \leq \|A\|(x)\|h\| \quad \text{and} \quad \|B(x; h, h)\| \leq \|B\|(x)\|h\|^2
\]
for all \( x, h \in \mathbb{R}^n \).

1.2 Newton’s method

Consider an “algebraic” equation
\[
F(x) = 0,
\]
where \( F: \mathbb{R}^n \to \mathbb{R}^n \) is a smooth map.

Theorem 1 Let \( x_* \) be a solution to (1.3), i.e. \( F(x_*) = 0 \), and suppose that \( A(x_*) \) is invertible. Then there exist \( \varepsilon, C > 0 \) such that for any \( x_0 \) with \( \|x_0 - x_*\| \leq \varepsilon \) the Newton iterations
\[
x_{k+1} = x_k - A^{-1}(x_k)F(x_k), \quad k = 0, 1, 2, \ldots,
\]
are well defined, satisfy
\[
\|x_k - x_*\| \leq \varepsilon, \quad \|x_{k+1} - x_*\| \leq C\|x_k - x_*\|^2,
\]
for all \( k \geq 0 \), implying that the sequence \( \{x_k\} \) converges to \( x_* \) quadratically.

Proof:
Provided \( A^{-1}(x_k) \) exists, it follows from (1.4) that
\[
x_{k+1} - x_* = A^{-1}(x_k)(A(x_k)(x_k - x_*) - F(x_k) + F(x_*)).
\]
Using (1.2) and making a substitution in the integral, we see that

$$A(x_k)(x_k - x^*_s) - F(x_k) + F(x^*_s) = \int_0^1 tB(x^*_s + th; h, h) \, dt,$$

where $h = x_k - x^*_s$. Thus

$$x_{k+1} - x^*_s = A^{-1}(x_k)\int_0^1 tB(x^*_s + th; h, h) \, dt. \quad (1.7)$$

Since $F$ is smooth and $A(x^*_s)$ is invertible, there exist $\delta, M > 0$ such that for all $x$ with $\|x - x^*_s\| \leq \delta$ the matrix $A(x)$ is also invertible and

$$\|A^{-1}(x)\| \leq \frac{1}{M}.$$ 

Let

$$L = \sup_{\|x - x^*_s\| \leq \delta} \|B\|(x).$$

From (1.7) it follows that

$$\|x_{k+1} - x^*_s\| \leq \frac{L}{2M} \|h\|^2 = \frac{L}{2M} \|x_k - x^*_s\|^2 \quad (1.8)$$

for $\|x_k - x^*_s\| \leq \delta$. Therefore, if $\{x_k\}$ is well defined and (1.5) holds for some $0 < \varepsilon < \delta$, then (1.6) also holds with

$$C = \frac{L}{2M}.$$ 

Now take $\varepsilon$ satisfying

$$0 < \varepsilon < \min\left(\delta, \frac{2M}{L}\right).$$

Then, if $x_k$ satisfies $\|x_k - x^*_s\| \leq \varepsilon$, it is also true that $\|x_k - x^*_s\| \leq \delta$, so that $A^{-1}(x_k)$ exists and $x_{k+1}$ is well defined. Moreover, (1.8) gives

$$\|x_{k+1} - x^*_s\| \leq \frac{L}{2M} \|x_k - x^*_s\| \|x_k - x^*_s\| \leq \frac{L}{2M} \frac{2M}{L} \varepsilon = \varepsilon.$$ 

We have $\varepsilon > 0$ such that for any $x_0$ with $\|x_0 - x^*_s\| \leq \varepsilon$ holds $\|x_k - x^*_s\| \leq \varepsilon$ for all $k \geq 1$ (by induction). Thus, both estimates (1.5) and (1.6) are established.

Introducing

$$\xi_k = \frac{L}{2M} \|x_k - x^*_s\|,$$

we get from (1.8) that $\xi_{k+1} \leq \xi_k^2$ for $k \geq 0$, which implies

$$\xi_k \leq \xi_0^{2^k}.$$ 

Here $\xi_0 < 1$, since

$$\frac{L}{2M} \|x_0 - x^*_s\| \leq \frac{L}{2M} \varepsilon < \frac{L}{2M} \frac{2M}{L} = 1.$$ 

Thus $\xi_k \to 0$ as $k \to \infty$, meaning that $\{x_k\}$ converges to $x^*_s$ quadratically.
1.3 Approximation of 1D invariant manifolds

Let \( x_* = 0 \) be a hyperbolic equilibrium of a smooth ODE system

\[
\dot{x} = Ax + \frac{1}{2} B(x, x) + O(\|x\|^3), \quad x \in \mathbb{R}^n.
\]  

Suppose that \( A \) has one simple eigenvalue \( \lambda > 0 \) and \((n-1)\) eigenvalues with \( \Re(\lambda) < 0 \). Then there exists a one-dimensional invariant manifold \( W^u(0) \) that is tangent at the equilibrium 0 to the one-dimensional unstable eigenspace \( E^u \) of \( A \) spanned by the eigenvector \( q \in \mathbb{R}^n \):

\[ Aq = \lambda q, \quad \langle q, q \rangle = 1. \]

The manifold \( W^u(0) \) can be parametrized near the origin by

\[ x = \xi q + \frac{1}{2} \xi^2 s + O(\|\xi\|^3), \]

with \( \xi \in \mathbb{R} \) and some fixed vector \( s \in \mathbb{R}^n \) satisfying

\[ \langle p, s \rangle = 0, \]

where \( p \in \mathbb{R}^n \) is the adjoint eigenvector:

\[ A^T p = \lambda p, \quad \langle p, q \rangle = 1. \]

The condition (1.11) means that \( s \) belongs the \((n-1)\)-dimensional stable eigenspace \( E^s \) of \( A \), so that the quadratic term in (1.10) does not contain any component in the \( q \)-direction.

The restriction of (1.9) to its \( W^u(0) \) can be written as

\[ \xi = \lambda \xi + a \xi^2 + O(\|\xi\|^3), \]

where \( a \in \mathbb{R} \). Notice that both \( a \) and \( s \) are unknown at this stage.
Using the invariancy of $W^u(0)$, we obtain

$$
\dot{x} = \dot{\xi}q + \xi \dot{s} + \ldots
= (\lambda \xi + a \xi^2 + \ldots)q + \xi(a \xi + a \xi^2 + \ldots)s + \ldots
= \lambda \xi q + \xi^2(aq + \lambda s) + \ldots,
$$
as well as

$$
\dot{x} = A(\xi q + \frac{1}{2} \xi^2 s + \ldots) + \frac{1}{2} B(\xi q + \ldots, \xi q + \ldots) + \ldots
= \lambda \xi q + \frac{1}{2} \xi^2(Aq + B(q, q)) + \ldots,
$$

where $\ldots$ denote the $O(|\xi|^3)$-terms. Collecting the $\xi^2$ terms, we obtain the following non-singular linear system

$$(A - 2\lambda I_n)s = 2aq - B(q, q). \tag{1.13}$$

Since

$$
\langle p, (A - 2\lambda I_n)s \rangle = \langle (A - 2\lambda I_n)^T p, s \rangle = \langle A^T p, s \rangle - 2\lambda \langle p, s \rangle
= \lambda \langle p, s \rangle - 2\lambda \langle p, s \rangle
= -\lambda \langle p, s \rangle
= 0,
$$
due to (1.11), we must have $\langle p, 2aq - B(q, q) \rangle = 0$. This implies

$$
a = \frac{1}{2} \langle p, B(q, q) \rangle
$$
and (1.13) finally gives

$$
s = (A - 2\lambda I_n)^{-1}(\langle p, B(q, q) \rangle q - B(q, q)).$$