Lecture 1: Location and analysis of equilibria

1.1 Multivariate Taylor expansions

Let $F : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto F(x)$, be a smooth map. Its **multivariate Taylor expansion** at $x_0 \in \mathbb{R}^n$ for $h \in \mathbb{R}^n$ with small $\|h\|$ can be written as

$$F(x_0 + h) = F(x_0) + A(x_0)h + \frac{1}{2}B(x_0; h, h) + O(\|h\|^3),$$

where $\|h\| = \sqrt{(h, h)}$ with

$$(x, y) := x^T y, \quad x, y \in \mathbb{R}^n,$$

and $A(x_0) = DF(x_0)$, while $B(x_0; r, s) = D^2F(x_0)(r, s)$. In components:

$$(A(x_0)h)_i = \sum_{j=1}^{n} \frac{\partial F_i(x_0)}{\partial x_j} h_j,$$

$$B_i(x_0; r, s) = \sum_{j,k=1}^{n} \frac{\partial^2 F_i(x_0)}{\partial x_j \partial x_k} r_j s_k,$$

for $r, s \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$. Given $x_0$, the mapping $h \mapsto A(x_0)h$ is linear, while $(r, s) \mapsto B(x_0; r, s)$ is a bilinear form.
Lemma 1 Let \( x, x_0 \in \mathbb{R}^n \) and \( h := x - x_0 \). Then for all \( x \) sufficiently close to \( x_0 \), i.e. \( \|h\| \) small enough, hold

\[
F(x) = F(x_0) + \int_0^1 A(x_0 + th) h \, dt, \quad (1.1)
\]

\[
F(x) = F(x_0) + A(x_0) h + \int_0^1 (1 - t) B(x_0 + th; h, h) \, dt. \quad (1.2)
\]

Proof:

Introduce \( f(t) = F(x_0 + th) \). Then \( \dot{f}(t) = A(x_0 + th) h \), in particular \( \dot{f}(0) = A(x_0) h \), and \( \ddot{f}(t) = B(x_0 + th; h, h) \). This gives

\[
\int_0^1 A(x_0 + th) h \, dt = \int_0^1 \dot{f}(t) \, dt = f(1) - f(0) = F(x_0 + h) - F(x_0),
\]

from which (1.1) follows. Similarly,

\[
\int_0^1 (1 - t) B(x_0 + th; h, h) \, dt = \int_0^1 (1 - t) \ddot{f}(t) \, dt \]
\[
= \left[ \dot{f}(t)(1 - t) \right]_0^1 + \int_0^1 \ddot{f}(t) \, dt \]
\[
= -\dot{f}(0) + f(1) - f(0) \]
\[
= -A(x_0) h + F(x_0 + h) - F(x_0),
\]

implying (1.2).

Define

\[
\|A\|(x) := \sup_{h \neq 0} \frac{\|A(x) h\|}{\|h\|} \quad \text{and} \quad \|B\|(x) := \sup_{h \neq 0} \frac{\|B(x; h, h)\|}{\|h\|^2}.
\]

Then

\[
\|A(x) h\| \leq \|A\|(x) \|h\| \quad \text{and} \quad \|B(x; h, h)\| \leq \|B\|(x) \|h\|^2
\]

for all \( x, h \in \mathbb{R}^n \).

1.2 Newton’s method

Consider an “algebraic” equation

\[
F(x) = 0, \quad (1.3)
\]

where \( F : \mathbb{R}^n \to \mathbb{R}^n \) is a smooth map.
1.2. NEWTON’S METHOD

**Theorem 1** Let \( x_\ast \) be a solution to (1.3), i.e. \( F(x_\ast) = 0 \), and suppose that \( A(x_\ast) \) is invertible. Then there exists \( \varepsilon > 0 \) such that for any \( x_0 \) with \( \| x_0 - x_\ast \| \leq \varepsilon \) the Newton iterations

\[
    x_{k+1} = x_k - A^{-1}(x_k)F(x_k), \quad k = 0, 1, 2, \ldots, \tag{1.4}
\]

are well defined, satisfy \( \| x_k - x_\ast \| \leq \varepsilon \) for all \( k \geq 0 \), and converge quadratically to \( x_\ast \), i.e.

\[
    \| x_{k+1} - x_\ast \| \leq C\| x_k - x_\ast \|^2
\]

for all \( k \geq 0 \) with some \( C > 0 \).

**Proof:**
Provided \( A^{-1}(x_k) \) exists, it follows from (1.4) that

\[
    x_{k+1} - x_\ast = A^{-1}(x_k)(A(x_k)(x_k - x_\ast) - F(x_k) + F(x_\ast)).
\]

Using (1.2) and making a substitution in the integral, we see that

\[
    A(x_k)(x_k - x_\ast) - F(x_k) + F(x_\ast) = \int_0^1 tB(x_\ast + th; h, h) \, dt,
\]

where \( h = x_k - x_\ast \). Thus

\[
    x_{k+1} - x_\ast = A^{-1}(x_k)\int_0^1 tB(x_\ast + th; h, h) \, dt. \tag{1.5}
\]

Since \( F \) is smooth and \( A(x_\ast) \) is invertible, there exist \( \delta, M > 0 \) such that for all \( x \) with \( \| x - x_\ast \| \leq \delta \) the matrix \( A(x) \) is also invertible and

\[
    \| A^{-1}(x) \| \leq \frac{1}{M}.
\]

Let

\[
    L = \sup_{\| x - x_\ast \| \leq \delta} \| B \|(x).
\]

From (1.5) it follows that

\[
    \| x_{k+1} - x_\ast \| \leq \frac{L}{2M} \| h \|^2 = \frac{L}{2M} \| x_k - x_\ast \|^2
\]

for \( \| x_k - x_\ast \| \leq \delta \). Therefore, if \( \{ x_k \} \) is well defined and converges to \( x_\ast \), it converges quadratically with \( C = \frac{L}{2M} \).
Take \( \varepsilon \) satisfying

\[ 0 < \varepsilon < \min \left( \delta, \frac{2M}{L} \right). \]

Then, if \( x_k \) satisfies \( \|x_k - x_*\| \leq \varepsilon \), it is also true that \( \|x_k - x_*\| \leq \delta \), so that \( A^{-1}(x_k) \) exists and \( x_{k+1} \) is well defined. Moreover,

\[
\|x_{k+1} - x_*\| \leq \frac{L}{2M} \|x_k - x_*\| \|x_k - x_*\| < \frac{L}{2M} \frac{2M}{L} \varepsilon = \varepsilon.
\]

We have \( \varepsilon > 0 \) such that for any \( x_0 \) with \( \|x_0 - x_*\| \leq \varepsilon \) holds \( \|x_k - x_*\| \leq \varepsilon \) for all \( k \geq 1 \) (by induction).

Introducing

\[ \xi_k = \frac{L}{2M} \|x_k - x_*\|, \]

we get \( \xi_{k+1} \leq \xi_k^2 \) for \( k \geq 0 \), which implies

\[ \xi_k \leq \xi_0^k. \]

Here \( \xi_0 < 1 \), since

\[
\frac{L}{2M} \|x_0 - x_*\| \leq \frac{L}{2M} \varepsilon < \frac{L}{2M} \frac{2M}{L} = 1.
\]

Thus \( \xi_k \to 0 \) as \( k \to \infty \), meaning that \( \{x_k\} \) converges to \( x_* \).

\[ \square \]

### 1.3 Approximation of 1D invariant manifolds

Let \( x_* = 0 \) be a hyperbolic equilibrium of a smooth ODE system

\[
\dot{x} = Ax + \frac{1}{2} B(x, x) + O(\|x\|^3), \quad x \in \mathbb{R}^n. \tag{1.6}
\]

Suppose that \( A \) has one simple eigenvalue \( \lambda > 0 \) and \((n - 1)\) eigenvalues with \( \Re(\lambda) < 0 \). Then there exists a one-dimensional invariant manifold \( W^u(0) \) that is tangent at the equilibrium 0 to the eigenvector \( q \in \mathbb{R}^n \):

\[ Aq = \lambda q, \quad \langle q, q \rangle = 1. \]

The manifold \( W^u(0) \) can be parametrized near the origin by

\[ x = \xi q + \frac{1}{2} \xi^2 s + O(|\xi|^3), \tag{1.7} \]
1.3. APPROXIMATION OF 1D INVARIANT MANIFOLDS

with \( \xi \in \mathbb{R} \) and some fixed vector \( s \in \mathbb{R}^n \) satisfying

\[
\langle p, s \rangle = 0,
\]

(1.8)

where \( p \in \mathbb{R}^n \) is the adjoint eigenvector:

\[
A^T p = \lambda p, \quad \langle p, q \rangle = 1.
\]

The condition (1.8) means that \( s \) belongs the \((n - 1)\)-dimensional stable eigenspace of \( A \), so that the quadratic term in (1.7) does not contain any component in the \( q \)-direction.

The restriction of (1.6) to its \( W^u(0) \) can be written as

\[
\dot{\xi} = \lambda \xi + a \xi^2 + O(|\xi|^3),
\]

(1.9)

where \( a \in \mathbb{R} \). Notice that both \( a \) and \( s \) are unknown at this stage.

Using the invariancy of \( W^u(0) \), we obtain

\[
\dot{x} = \dot{\xi} q + \xi \dot{s} + \ldots
\]

as well as

\[
\dot{x} = A(\xi q + \frac{1}{2} \xi^2 s + \ldots) + \frac{1}{2} B(\xi q + \ldots, \xi q + \ldots) + \ldots
\]

\[
= \xi A q + \frac{1}{2} \xi^2 A s + \frac{1}{2} \xi^2 B(q, q) + \ldots
\]

\[
= \lambda \xi q + \frac{1}{2} \xi^2 (A q + B(q, q)) + \ldots,
\]

where “…” denote the \( O(|\xi|^3) \)-terms. Collecting the \( \xi^2 \) terms, we obtain the following non-singular linear system

\[
(A - 2\lambda I_n)s = 2aq - B(q, q).
\]

(1.10)

Since

\[
\langle p, (A - 2\lambda I_n)s \rangle = \langle (A - 2\lambda I_n)^T p, s \rangle = \langle A^T p, s \rangle - 2\lambda \langle p, s \rangle
\]

\[
= \lambda \langle p, s \rangle - 2\lambda \langle p, s \rangle = -\lambda \langle p, s \rangle
\]

\[
= 0,
\]
due to (1.8), we must have \( \langle p, 2aq - B(q,q) \rangle = 0 \). This implies
\[
a = \frac{1}{2} \langle p, B(q,q) \rangle
\]
and (1.10) finally gives
\[
s = (A - 2\lambda I_n)^{-1}(\langle p, B(q,q) \rangle q - B(q,q)).
\]