Lecture 4: Bordering technique. Detection of limit and branching points.

4.1 Bordering technique I

Consider a smooth one-parameter family of $N \times N$ matrices $A(s)$, such that $A(0)$ is singular with rank $A(0) = N - 1$.

Lemma 9 The matrix

$$M(0) = \begin{pmatrix} A(0) & p \\ q^T & 0 \end{pmatrix},$$

where $A(0)q = A^T(0)p = 0$ with $\|q\| = \|p\| = 1$, is nonsingular.

Proof: Suppose that

$$M(0) \begin{pmatrix} X \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $X \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ such that

$$\begin{pmatrix} X \\ \beta \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
This is equivalent to the system

\[
\begin{aligned}
    A(0)X + \beta p &= 0, \\
    \langle q, X \rangle &= 0.
\end{aligned}
\]  

(4.21)

Computing the scalar product of the first equation in (4.21) with \( p \), we obtain

\[
0 = \langle p, A(0)X + \beta p \rangle = \langle A^T(0) p, X \rangle + \beta \langle p, p \rangle = \beta \|p\|^2 = \beta,
\]

where \( A^T(0)p = 0 \) is taken into account. We conclude that \( \beta = 0 \) and so the first equation in (4.21) actually has the form

\[ A(0)X = 0. \]

This implies that \( X = \gamma q \) with some \( \gamma \in \mathbb{R} \). Substituting \( X = \gamma q \) in the second equation of (4.21), we see that

\[
\langle q, \gamma q \rangle = \gamma \|q\|^2 = \gamma = 0,
\]

yielding \( X = 0 \). Thus

\[
\begin{pmatrix}
    X \\
    \beta
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0
\end{pmatrix},
\]

a contradiction. Therefore, \( M(0) \) is nonsingular. \( \Box \)

Lemma 9 ensures by continuity that the matrix

\[
M(s) = \begin{pmatrix}
    A(s) & p \\
    q^T & 0
\end{pmatrix}
\]  

(4.22)

is nonsingular for all \( s \) with \( |s| \) sufficiently small. For such values of \( s \), introduce the nonsingular \textbf{bordered system}:

\[
M(s) \begin{pmatrix}
    w \\
    g
\end{pmatrix} = \begin{pmatrix}
    0 \\
    1
\end{pmatrix}.
\]  

(4.23)

At \( s = 0 \), the explicit solution to this system is obvious:

\[
\begin{pmatrix}
    w(0) \\
    g(0)
\end{pmatrix} = \begin{pmatrix}
    q \\
    0
\end{pmatrix}.
\]

Thus, \( g(0) = 0 \). If

\[
\begin{pmatrix}
    w \\
    g
\end{pmatrix} = \begin{pmatrix}
    w(s) \\
    g(s)
\end{pmatrix}
\]
is the solution of (4.23), then Cramer’s rule gives

\[
g(s) = \frac{\det A(s)}{\det M(s)},
\]

(4.24)

implying that \( g(s) \) is as smooth as \( A(s) \). The following lemma shows how the derivative \( \dot{g}(0) \) can be computed explicitly.

**Lemma 10** It holds that

\[
\dot{g}(0) = -\langle p, \dot{A}(0)q \rangle.
\]

**Proof:**

Differentiating (4.23) w.r.t. \( s \) yields

\[
M(s) \begin{pmatrix} w(s) \\ g(s) \end{pmatrix} + M(s) \begin{pmatrix} \dot{w}(s) \\ \dot{g}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

implying

\[
M(0) \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} = -\dot{M}(0) \begin{pmatrix} w(0) \\ g(0) \end{pmatrix}.
\]

Thus,

\[
M(0) \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} = -\dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix}.
\]

(4.25)

Further notice that the transposed matrix

\[
M^T(0) = \begin{pmatrix} A^T(0) & q \\ p^T & 0 \end{pmatrix}
\]

is also nonsingular, so that the linear system

\[
M^T(0) \begin{pmatrix} \varphi \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

(4.26)

has the unique solution, namely

\[
\begin{pmatrix} \varphi \\ h \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}.
\]
Computing now the scalar product of this solution with both sides of (4.25), we obtain
\[
\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, M(0) \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} \right\rangle = -\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, \dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix} \right\rangle
\]
or
\[
\left\langle M^T(0) \begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} \dot{w}(0) \\ \dot{g}(0) \end{pmatrix} \right\rangle = -\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, \dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix} \right\rangle.
\]
Taking into account (4.26), we see that
\[
\dot{g}(0) = -\left\langle \begin{pmatrix} p \\ 0 \end{pmatrix}, \dot{M}(0) \begin{pmatrix} q \\ 0 \end{pmatrix} \right\rangle.
\]
Since
\[
\dot{M}(0) = \begin{pmatrix} \dot{A}(0) & 0 \\ 0 & 0 \end{pmatrix},
\]
we get
\[
\dot{g}(0) = -\langle p, \dot{A}(0)q \rangle.
\]
This completes the proof. \qed

4.2 Detection of local bifurcations

Consider a system of ODEs depending on one parameter
\[
\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \quad \alpha \in \mathbb{R},
\]
where \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is smooth. The continuation of a branch in its equilibrium manifold leads to ALCP (3.7) with
\[
x = \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1}
\]
and \( F(x) = f(u, \alpha) \). Assume that this branch is parametrized by \( u = u(s) \) and \( \alpha = \alpha(s) \) and that \( s = 0 \) corresponds to either a quadratic limit point w.r.t. \( \alpha \) or a simple branching point of (3.7). We will construct a regular test-function \( \Psi(s) \) to detect each bifurcation, i.e. a smooth scalar function satisfying
\[
\Psi(0) = 0, \quad \dot{\Psi}(0) \neq 0.
\]
4.2. DETECTION OF LOCAL BIFURCATIONS

4.2.1 Limit point detection

Assume that \( s = 0 \) corresponds to a limit point w.r.t. \( \alpha \). We can also select such a parametrization of the equilibrium branch near the limit point by \( s \) that the tangent vector at \( s = 0 \) will have the form

\[
\begin{pmatrix}
\dot{u}(0) \\
\dot{\alpha}(0)
\end{pmatrix} = \begin{pmatrix}
q \\
0
\end{pmatrix},
\]

with \( q \in \mathbb{R}^n \) satisfying

\[ A(0)q = 0, \quad \|q\| = 1, \]

where \( A(0) = f_u(u(0), \alpha(0)) \). Introduce

\[ \Psi_{LP}(s) = g(s), \]

where \( g(s) \) is defined by solving the bordered system (4.23). In that system, matrix \( M(s) \) is given by (4.22) with \( A(s) = f_u(u(s), \alpha(s)) \), vector \( q \) is defined above, and \( p \in \mathbb{R}^n \) satisfies \( A^T(0)p = 0, \quad \|p\| = 1. \)

**Theorem 5** At a quadratic limit point holds

\[ \Psi_{LP}(0) = 0 \quad \text{and} \quad \dot{\Psi}_{LP}(0) \neq 0. \]

**Proof:**
Clearly, \( \Psi_{LP}(0) = g(0) = 0 \). Using Lemma 10, we obtain

\[ \dot{g}(0) = -\langle p, \dot{A}(0)q \rangle = -\langle p, f_{uu}(u(0), \alpha(0))[q, q] \rangle = -\langle p, B(q, q) \rangle. \]

Since \( \langle p, B(q, q) \rangle \neq 0 \) at a quadratic limit point, \( \dot{\Psi}_{LP}(0) = \dot{g}(0) \neq 0. \)

4.2.2 Branching point detection

Suppose that \( s = 0 \) corresponds to a simple branching point of ALCP (3.7) in the solution branch \( \Gamma_1 \) parametrized by \( x^{(1)}(s) \) such that

\[ \|\dot{x}^{(1)}(0)\| = 1. \]

As in Theorem 3, define the \((N + 1) \times (N + 1)\) matrix

\[ D(s) = \begin{pmatrix}
F_{x}(x^{(1)}(s)) \\
[\dot{\alpha}^{(1)}(s)]^T
\end{pmatrix} \]
and introduce
\[ \Psi_{BP}(s) = g(s), \]
where \( g(s) \) is still defined by solving the bordered system (4.23) but now
\[
M(s) = \begin{pmatrix}
D(s) & P \\
Q^T & 0
\end{pmatrix}
\]
with vectors \( Q, P \in \mathbb{R}^{N+1} \) satisfying \( D(0)Q = D^T(0)P = 0 \) and \( \|Q\| = \|P\| = 1 \), so that \( M(s) \) is a \((N + 2) \times (N + 2)\) nonsingular matrix for small \(|s|\).

**Theorem 6** At a simple branching point holds
\[
\Psi_{BP}(0) = 0 \quad \text{and} \quad \dot{\Psi}_{BP}(0) \neq 0.
\]

**Proof:**
We have already seen in the proof of Theorem 3 that matrix \( D(0) \) is singular. Its null-space \( N(D(0)) \) is one-dimensional and is spanned by \( Q = q^{(2)} \). Thus, \( g(0) = 0 \).

The null-space \( N(D^T(0)) \) is also one-dimensional and spanned by \( P \in \mathbb{R}^{N+1} \), where \( J^T \varphi = 0 \) and \( \|\varphi\| = 1 \) implying \( \|P\| = 1 \).

Now, Lemma 10 allows us to write
\[
\dot{g}(0) = -\langle P, \dot{D}(0)Q \rangle.
\]
Since \( Q = q^{(2)} \) and \( \dot{x}^{(1)}(0) = q^{(1)} \), we have
\[
\dot{D}(0)Q = \dot{D}(0)q^{(2)} = \begin{pmatrix} F_{xx} [\dot{x}^{(1)}(0), q^{(2)}] \\ \dot{x}^{(1)}(0)^T q^{(2)} \end{pmatrix} = \begin{pmatrix} B(q^{(1)}, q^{(2)}) \\ [\dot{x}^{(1)}(0)]^T q^{(2)} \end{pmatrix},
\]
so that
\[
\langle P, \dot{D}(0)q^{(2)} \rangle = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \cdot \begin{pmatrix} B(q^{(1)}, q^{(2)}) \\ [\dot{x}^{(1)}(0)]^T q^{(2)} \end{pmatrix} = \langle \varphi, B(q^{(1)}, q^{(2)}) \rangle.
\]
This gives
\[
\dot{g}(0) = -\langle \varphi, B(q^{(1)}, q^{(2)}) \rangle = -b_{12} \neq 0,
\]
because the branching point is simple. \( \square \)