4. Computation of normal forms for LP and Hopf bifurcations

4.1. Normal forms on center manifolds

4.2. Fredholm’s Alternative

4.3. Critical LP-coefficient

4.4. Critical H-coefficient

4.5. Approximation of multilinear forms by finite differences
4.1. Normal forms on center manifolds

- **LP:** \[ \dot{\xi} = \beta + b\xi^2, \quad b \neq 0 \]

\[
\begin{array}{c}
\text{Equilibria: } \beta + b\xi^2 = 0 \Rightarrow \xi_{1,2} = \pm \sqrt{-\frac{\beta}{b}} \\
\text{H: } \dot{\xi} = (\beta + i\omega)\xi + c|\xi|^2, \quad l_1 = \frac{1}{\omega}\Re(c) \neq 0
\end{array}
\]

\[
\begin{array}{c}
\text{Limit cycle: } \\
\left\{ \begin{array}{l}
\dot{\rho} = \rho(\beta + \Re(c)\rho^2), \quad \Rightarrow \rho_0 = \sqrt{-\frac{\beta}{\Re(c)}} \\
\dot{\phi} = \omega + \Im(c)\rho^2
\end{array} \right.
\end{array}
\]
4.2. Fredholm’s Alternative

- **Lemma 1** The linear system $Ax = b$ with $b \in \mathbb{R}^n$ and a singular $n \times n$ real matrix $A$ is solvable if and only if $\langle p, b \rangle = 0$ for all $p$ satisfying $A^T p = 0$.

Indeed, $\mathbb{R}^n = L \oplus R$ with $L \perp R$, where

$$L = \mathcal{N}(A^T) = \{ p \in \mathbb{R}^n : A^T p = 0 \}$$

and

$$R = \{ x \in \mathbb{R}^n : x = Ay \text{ for some } y \in \mathbb{R}^n \}.$$  

The proof is completed by showing that the orthogonal complement $L^\perp$ to $L$ coincides with $R$.

- In the complex case:

\[
\begin{align*}
\mathbb{R}^n & \Rightarrow \mathbb{C}^n \\
\langle p, b \rangle & = \bar{p}^T b \\
A^T & \Rightarrow A^* = \bar{A}^T
\end{align*}
\]
4.3. Critical LP-coefficient $b$

- Let $Aq = A^Tp = 0$ with $\langle q, q \rangle = \langle p, q \rangle = 1$.

- Write the RHS at the bifurcation as
  \[ F(u) = Au + \frac{1}{2}B(u, u) + O(\|u\|^3), \]
  and locally represent the center manifold $W^c_0$ as the graph of a function $H : \mathbb{R} \to \mathbb{R}^n$,
  \[ u = H(\xi) = \xi q + \frac{1}{2}h_2\xi^2 + O(\xi^3), \quad \xi \in \mathbb{R}, \ h_2 \in \mathbb{R}^n. \]
  The restriction of $\dot{u} = F(u)$ to $W^c_0$ is
  \[ \dot{\xi} = G(\xi) = b\xi^2 + O(\xi^3). \]

- The invariance of the center manifold $H_\xi(\xi)\dot{\xi} = F(H(\xi))$ implies
  \[ H_\xi(\xi)G(\xi) = F(H(\xi)) \]
  Substitute all expansions into this homological equation and collect the coefficients of the $\xi^j$-terms.
We have

\[ A(\xi q + \frac{1}{2} h_2 \xi^2) + \frac{1}{2} B(\xi q, \xi q) + O(|\xi|^3) \]
\[ = b\xi^2 q + b\xi^3 h_2 + O(|\xi|^4) \]

• The \( \xi \)-terms give the identity: \( Aq = 0 \).

• The \( \xi^2 \)-terms give the equation for \( h_2 \):

\[ Ah_2 = -B(q, q) + 2bq. \]

It is singular and its Fredholm solvability

\[ \langle p, -B(q, q) + 2bq \rangle = 0 \]

implies

\[ b = \frac{1}{2} \langle p, B(q, q) \rangle \]
4.4. Critical H-coefficient $c$

- $Aq = i\omega_0 q, A^T p = -i\omega_0 p, \langle q, q \rangle = \langle p, q \rangle = 1$.

- Write

$$F(u) = Au + \frac{1}{2} B(u, u) + \frac{1}{3!} C(u, u, u) + O(\|u\|^4)$$

and locally represent the center manifold $W^c_0$ as the graph of a function $H : \mathbb{C} \to \mathbb{R}^n$,

$$u = H(\xi, \bar{\xi}) = \xi q + \bar{\xi} \bar{q} + \sum_{2 \leq j + k \leq 3} \frac{1}{j!k!} h_{j,k} \xi^j \bar{\xi}^k + O(|\xi|^4).$$

The restriction of $\dot{u} = F(u)$ to $W^c_0$ is

$$\dot{\xi} = G(\xi, \bar{\xi}) = i\omega_0 \xi + c\xi|\xi|^2 + O(|\xi|^4).$$

- The invariance of $W^c_0$

$$H_\xi(\xi, \bar{\xi}) \dot{\xi} + H_{\bar{\xi}}(\xi, \bar{\xi}) \dot{\bar{\xi}} = F(H(\xi, \bar{\xi}))$$

implies

$$H_\xi(\xi, \bar{\xi}) G(\xi, \bar{\xi}) + H_{\bar{\xi}}(\xi, \bar{\xi}) G(\xi, \bar{\xi}) = F(H(\xi, \bar{\xi})).$$
• Quadratic $\xi^2$- and $|\xi|^2$-terms give
\[
  h_{20} = (2i\omega_0 I_n - A)^{-1}B(q, q), \\
  h_{11} = -A^{-1}B(q, \bar{q}).
\]

• Cubic $w^2\overline{w}$-terms give the singular system
\[
(i\omega_0 I_n - A)h_{21} = C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) - 2cq.
\]

The solvability of this system implies
\[
c = \frac{1}{2}\langle p, C'(q, q, \bar{q}) + B(\bar{q}, (2i\omega_0 I_n - A)^{-1}B(q, q)) - 2B(q, A^{-1}B(q, \bar{q})) \rangle
\]

• The first Lyapunov coefficient
\[
l_1 = \frac{1}{\omega_0} \Re(c).
\]
4.5. Approximation of multilinear forms by finite differences

- Finite-difference approximation of directional derivatives:

\[
B(q, q) = \frac{1}{h^2} \left[ f(u_0 + hq, \alpha_0) + f(u_0 - hq, \alpha_0) \right] + O(h^2)
\]

\[
C(r, r, r) = \frac{1}{8h^3} \left[ f(u_0 + 3hr, \alpha_0) - 3f(u_0 + hr, \alpha_0) + 3f(u_0 - hr, \alpha_0) - f(u_0 - 3hr, \alpha_0) \right] + O(h^2).
\]

- Polarization identities:

\[
B(q, r) = \frac{1}{4} \left[ B(q + r, q + r) - B(q - r, q - r) \right],
\]

\[
C(q, q, r) = \frac{1}{6} \left[ C(q + r, q + r, q + r) - C(q - r, q - r, q - r) \right] - \frac{1}{3} C(r, r, r).
\]
5. Detection of codim 2 bifurcations

- codim 2 cases along the LP-curve:
  - **Bogdanov-Takens (BT)**: $\lambda_{1,2} = 0 \quad (\psi_{BT} = \langle p, q \rangle$ with $\langle q, q \rangle = \langle p, p \rangle = 1$)
  - **fold-Hopf (ZH)**: $\lambda_1 = 0, \lambda_{2,3} = \pm i\omega_0 \quad (\psi_{ZH} = \det(2A \odot I_n))$
  - **cusp (CP)**: $\lambda_1 = 0, b = 0 \quad (\psi_{CP} = b)$

- Critical cases along the H-curve:
  - **Bogdanov-Takens (BT)**: $\lambda_{1,2} = 0 \quad (\psi_{BT} = \kappa)$
  - **fold-Hopf (ZH)**: $\lambda_{1,2} = \pm i\omega_0, \lambda_3 = 0 \quad (\psi_{ZH} = \det A)$
  - **double Hopf (HH)**: $\lambda_{1,2} = \pm i\omega_0, \lambda_{3,4} = \pm i\omega_1 \quad (\psi_{HH} = \det(2A^\perp \odot I_{n-2})$
  - **Bautin (GH)**: $\lambda_{1,2} = \pm i\omega_0, l_1 = 0 \quad (\psi_{GH} = l_1)$