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UvA Lecture 1:

One-Parameter Bifurcations of Fixed Points of Maps
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1. Critical cases

Consider a one-parameter system

\[ x \mapsto \tilde{x} = f(x, \alpha), \quad x, \tilde{x} \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^1. \]

The fixed-point manifold

\[ f(x, \alpha) - x = 0 \]

is a smooth curve in \( \mathbb{R}^{n+1} \), provided rank \( J = n \), where \( J = [f_x - I_n | f_\alpha] \).

Let \( \{\mu_1, \mu_2, \ldots, \mu_n\} \) be eigenvalues of \( f_x \). Critical cases: (a) LP; (b) PD; (c) NS
2. Fold (limit point) bifurcation

Example 1 (Fold normal form)
Consider

\[ x \mapsto \alpha + x + x^2 \equiv f(x, \alpha) \equiv f_\alpha(x), \quad x, \alpha \in \mathbb{R}^1. \]

At \( \alpha = 0 \) this system has a nonhyperbolic fixed point \( x_0 = 0 \) with \( \mu = f_x(0,0) = 1. \)

For \( \alpha < 0 \) there are two hyperbolic fixed points

\[ x_{1,2}(\alpha) = \pm \sqrt{-\alpha}. \]

For \( \alpha > 0 \) there are no fixed points in the system.
**Lemma 1** The system

\[ x \mapsto \alpha + x + x^2 + O(x^3) \]

is locally topologically equivalent near the origin to the system

\[ x \mapsto \alpha + x + x^2. \square \]

**Th. 1** Suppose that a one-dimensional system

\[ x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^1, \alpha \in \mathbb{R}^1, \]

with smooth \( f \), has at \( \alpha = 0 \) the fixed point \( x = 0 \), and let \( \mu = f_x(0,0) = 1 \). Assume that the following conditions are satisfied:

\[(A.1) \quad f_{xx}(0,0) \neq 0 \quad \text{(nondegeneracy)}\]
\[(A.2) \quad f_{\alpha}(0,0) \neq 0 \quad \text{(transversality)}\]

Then there are invertible smooth coordinate and parameter changes transforming the system into

\[ \eta \mapsto \beta + \eta \pm \eta^2 + O(\eta^3). \]
**Proof:** Expand

\[ f(x, \alpha) = f_0(\alpha) + f_1(\alpha)x + f_2(\alpha)x^2 + O(x^3). \]

Here \( f_0(0) = f(0,0) = 0, \ f_1(0) = f_x(0,0) = 1. \)

Thus \( f_1(\alpha) = 1 + g(\alpha), \) where \( g(0) = 0. \) Let

\[ \xi = x + \delta, \]

where \( \delta = \delta(\alpha) \) is to be defined later.

\[ \tilde{\xi} = \tilde{x} + \delta = f(x, \alpha) + \delta = f(\xi - \delta, \alpha) + \delta. \]

Therefore,

\[
\begin{align*}
\tilde{\xi} &= \left[ f_0(\alpha) - g(\alpha)\delta + f_2(\alpha)\delta^2 + O(\delta^3) \right] \\
&\quad + \xi + [g(\alpha) - 2f_2(\alpha)\delta + O(\delta^2)]\xi \\
&\quad + [f_2(\alpha) + O(\delta)]\xi^2 + O(\xi^3).
\end{align*}
\]

Write the coefficient in front of \( \xi \) as

\[ F(\alpha, \delta) = g(\alpha) - 2f_2(\alpha)\delta + \delta^2\varphi(\alpha, \delta), \]

where \( \varphi \) is some smooth function. We have

\[ F(0,0) = 0, \quad \frac{\partial F}{\partial \delta}(0,0) = -2f_2(0) = f_{xx}(0,0) \neq 0 \]

by \((A.1)\).
The Implicit Function Theorem gives (local) existence and uniqueness of a smooth function \( \delta = \delta(\alpha) \) such that \( \delta(0) = 0 \) and \( F(\alpha, \delta(\alpha)) \equiv 0 \). Now

\[
\tilde{\xi} = \left[ f'_0(0)\alpha + O(\alpha^2) \right] + \xi \\
+ \left[ f_2(0) + O(\alpha) \right] \xi^2 + O(\xi^3).
\]

Consider the \( \xi \)-independent term as a new parameter

\[
\mu(\alpha) = f'_0(0)\alpha + \alpha^2 \varphi(\alpha).
\]

We have \( \mu(0) = 0 \), \( \mu'(0) = f'_0(0) = f_\alpha(0,0) \neq 0 \), by (A.2). Therefore, the Inverse Function Theorem implies local existence and uniqueness of a smooth function \( \alpha = \alpha(\mu) \) with \( \alpha(0) = 0 \). Now the map reads

\[
\tilde{\xi} = \mu + \xi + a(\mu)\xi^2 + O(\xi^3),
\]

where \( a(\mu) \) is a smooth function, \( a(0) = f_2(0) \neq 0 \). Let \( \eta = |a(\mu)|\xi \) and \( \beta = |a(\mu)|\mu \). Then we get

\[
\tilde{\eta} = \beta + \eta + s\eta^2 + O(\eta^3),
\]

where \( s = \text{sign } a(0) = \pm 1 \). \( \square \)
3. Flip (period doubling) bifurcation

Example 2 (Flip normal form) Consider

\[ x \mapsto -(1 + \alpha)x + x^3 \equiv f(x, \alpha) \equiv f_\alpha(x), \quad x, \alpha \in \mathbb{R}^1. \]

At \( \alpha = 0 \) this system has a nonhyperbolic fixed point \( x_0 = 0 \) with \( \mu = f_\alpha(x(0,0)) = -1 \). The second iterate

\[ f_\alpha^2(x) = (1+\alpha)^2 x - [(1+\alpha)(2+2\alpha+\alpha^2)] x^3 + O(x^3). \]

Period-2 cycle \( \{x_1, x_2\}, x_{1,2} = \pm \sqrt{\alpha} \), appears
Lemma 2  The system
\[ x \mapsto -(1 + \alpha)x + x^3 + O(x^4) \]
is locally topologically equivalent near the origin to the system
\[ x \mapsto -(1 + \alpha)x + x^3. \quad \square \]

Th. 2  Suppose that a one-dimensional system
\[ x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^1, \alpha \in \mathbb{R}^1, \]
with smooth \( f \), has the fixed point \( x_0 = 0 \) for all sufficiently small \( |\alpha| \), and let \( \mu = f_x(0,0) = -1 \). Assume that the following conditions are satisfied:

\[(B.1) \quad \frac{1}{2}(f_{xx}(0,0))^2 + \frac{1}{3}f_{xxx}(0,0) \neq 0; \]
\[(B.2) \quad f_{x\alpha}(0,0) \neq 0. \]

Then there are smooth invertible coordinate and parameter changes transforming the system into
\[ \eta \mapsto -(1 + \beta)\eta \pm \eta^3 + O(\eta^4). \]
Remark: If the map $x \mapsto f(\alpha)$ has a fixed point $x_0$ that depends on $\alpha$, the condition $(B.2)$ has to be modified.

Proof:
Write the Taylor expansion of the function $f(x, \alpha)$ near $x = 0$ as

$$\tilde{x} = f(x, \alpha) = f_1(\alpha)x + f_2(\alpha)x^2 + f_3(\alpha)x^3 + O(x^4),$$

where $f_1(\alpha) = -[1 + g(\alpha)]$ for some smooth function $g$. Since $g(0) = 0$ and

$$g'(0) = f_{x\alpha}(0, 0) \neq 0,$$

according to $(B.2)$, the function $g$ is locally invertible and can be used to introduce a new parameter: $\beta = g(\alpha)$. Thus

$$\tilde{x} = \mu(\beta)x + a(\beta)x^2 + b(\beta)x^3 + O(x^4),$$

where $\mu(\beta) = -(1 + \beta)$, and the functions $a(\beta)$ and $b(\beta)$ are smooth,

$$a(0) = \frac{1}{2}f_{xx}(0, 0), \quad b(0) = \frac{1}{6}f_{xxx}(0, 0).$$
Perform a smooth change of coordinate:

\[ x = y + \delta y^2, \]

where \( \delta = \delta(\beta) \) is a smooth function to be defined. The transformation is invertible in some neighborhood of the origin, and its inverse

\[ y = x - \delta x^2 + 2\delta^2 x^3 + O(x^4). \]

Therefore

\[ \tilde{y} = \mu y + (a + \delta \mu - \delta \mu^2)y^2 + (b + 2\delta a - 2\delta \mu(\delta \mu + a) + 2\delta^2 \mu^3)y^3 + O(y^4). \]

Thus, the \( y^2 \)-term can be "killed" for all sufficiently small \( |\beta| \) by setting

\[ \delta(\beta) = \frac{a(\beta)}{\mu^2(\beta) - \mu(\beta)}. \]

This is valid since \( \mu^2(0) - \mu(0) = 2 \neq 0 \) and, thus, the denominator is nonzero for all \( |\beta| \) small.
The map takes the form
\[
\tilde{y} = \mu y + \left( b + \frac{2a^2}{\mu^2 - \mu} \right) y^3 + O(y^4)
\]
\[
= -(1 + \beta)y + c(\beta)y^3 + O(y^4)
\]
for some smooth function \( c(\beta) \),
\[
c(0) = a^2(0) + b(0) = \frac{1}{4}(f_{xx}(0,0))^2 + \frac{1}{6}f_{xxx}(0,0).
\]
Notice that \( c(0) \neq 0 \) by (B.1). The rescaling
\[
y = \frac{\eta}{\sqrt{|c(\beta)|}}
\]
brings the system into the desired form:
\[
\tilde{\eta} = -(1 + \beta)\eta + s\eta^3 + O(\eta^4),
\]
where \( s = \text{sign } c(0) = \pm 1. \) □

Notice that
\[
c(0) = -\frac{1}{12} \frac{\partial^3}{\partial x^3} f_\alpha^2(x) \bigg|_{(x,\alpha)=(0,0)},
\]
where \( f_\alpha(x) = f(x, \alpha) \)
Example 3 (Ricker’s equation) The map

\[ x \mapsto f(x, \alpha) = \alpha xe^{-x} \]

has at \( \alpha_1 = e^2 = 7.38907 \ldots \) the fixed point \( x_1 = 2 \) with the multiplier \( \mu_1 = -1 \) and

\[ c(\alpha_1) = \frac{1}{6} > 0. \]

Thus, a stable period-2 cycle bifurcates from \( x_1 \). It exists for \( \alpha > \alpha_1 \) but is stable only for \( |\alpha - \alpha_1| \) sufficiently small.