Chapter 1

Introduction

1.1 Fundamental definitions

1.1.1 Time, state space and evolution

We use real numbers (denoted by symbols like $t, t_0, s, \tau, \ldots$) to label the moments in time. In other words, *time* is represented by the set $\mathbb{R}$, equipped with the operation of addition, the natural order relation "$\geq$", and the absolute value $|\cdot|$ to measure the distance between the time moments. Relative to time t, the *future* is the set (t, ∞) and the *past* is the set $(-\infty, t)$. We use the notation $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

So far we think of time as *continuous*, as flowing. The stroboscopic view of time arises when one monitors (*e.g.* by opening ones eyes) rhythmically, say every time the clock strikes the hour or the calendar shows a specific day of the year. We represent this so-called *discrete* time by the set of integers $\mathbb{Z}$, with, relative to time zero, the future corresponding to positive and the past to negative integers. We denote the set of nonnegative integers by $\mathbb{Z}_+$ (or $\mathbb{N}$) and the set of nonpositive integers by $\mathbb{Z}_-$.

We shall use the symbol $\mathbb{T}$ to denote a set of real numbers that is closed under addition, i.e. $t, s \in \mathbb{T}$ implies $t + s \in \mathbb{T}$. In this book, $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z}$, or $\mathbb{Z}_+$.

The set of all conceivable states of a system is called the *state* or *phase space* and we shall usually denote it by the symbol X. Its elements will be denoted by symbols like $x, y, \xi, \eta, \ldots$. The key idea about the notion of "state" is that it should summarize, in a condensed way, the information about the past that is relevant for predicting the future. In most of this text we consider systems whose states can be described by specifying n real numbers and then X is (a nice subset of) $\mathbb{R}^n$ for some positive $n \in \mathbb{N}$. In this case, we represent $x \in X$ as a *vector* (one-column matrix):

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

with $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$. Sometimes it is more convenient to describe a state by *complex numbers* (in particular, replace $\mathbb{R}^n$ by $\mathbb{C}^n$), even if our ultimate interest is in real quantities only. Complex state spaces occur naturally in quantum physics.

It could also happen that a finite number of real numbers is insufficient to characterize the state of a system, i.e. X must have *infinite dimension.* Occasionally we shall use state spaces that contain infinite sequences of numbers or more general functions.

Naturally, the theory becomes richer if X is equipped with some additional structure. All the spaces we will work with are *metric spaces*, *i.e.* a *distance* (or *metric*) $\rho(x, y)$ between points $x, y \in X$ is defined such that it has all the required properties[1]. This allows to define the distance between a point $x \in X$ and a subset $S \subset X$ as $\rho(x, S) = \inf_{y \in S} \rho(x, y)$, as well as the ε-neighbourhood of S as the set of all points $x \in X$ with $\rho(x, S) < \varepsilon$. It is also convenient if X is *complete*[2] with respect to ρ.

If X is a (subset of) a complex linear space with a *norm* $\|\cdot\|$ satifying the standard requirements[3], the distance can be defined as $\rho(x, y) = \|x - y\|$. If the linear space X is complete in this norm, it is called a *Banach space.* Finally, in a linear space X endowed with a *scalar product*[4] $\langle\cdot, \cdot\rangle$ a norm can be defined via $\|x\| = \sqrt{\langle x, x\rangle}$. As an example, take $X = \mathbb{C}^n$. This is a linear space with the standard scalar product

$$\langle x, y\rangle = \sum_{k=1}^{n} \bar{x}_k y_k = \bar{x}^T y,$$

where $\bar{x}^T = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ is the one-row matrix with the elements which are complex conjugate to those of x and where the standard matrix multiplication is used. Then,

$$\rho(x, y) = \sqrt{\langle x - y, x - y\rangle}$$

is the standard distance in $\mathbb{C}^n$. For the real subspace $\mathbb{R}^n$ of $\mathbb{C}^n$ this leads to the familiar Euclidean distance.

Now we have to describe the *evolution* of the system, i.e. the change of its state in time. A natural way to do this is to consider a map $x : \mathbb{T} \to X$,

$$t \mapsto x(t),$$

such that $x(t)$ can be interpreted as the state of the system at time t. Here our understanding is that the system can only be in one state at any moment of time, the map x is single-valued, i.e. a function.

But not all functions will do, there are constraints. We will consider only *deterministic* systems: By assumption the specification of a state $x_0 = x(t_0)$ at time t_0

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[1] For any $x, y, z \in X$ should hold: (i) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$; (ii) $\rho(x, y) = \rho(y, x)$; (iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

[2] X is called complete if every Cauchy sequence in X has a limit that belongs to X. Recall that $\{x_k\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists integer N such that $\rho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

[3] For any $x, y \in X$ and $\lambda \in \mathbb{C}$ should hold: (i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$; (ii) $\|\lambda x\| = |\lambda| \, \|x\|$; (iii) $\|x + y\| \leq \|x\| + \|y\|$.

[4] In general, a scalar product has the following properties. For any $x, y, z \in X$ and $\lambda \in \mathbb{C}$ holds: (i) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$; (ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$; (iii) $\langle x, x\rangle > 0$ for $x \neq 0$.

uniquely determines the state $x(t)$ at any time t. This means that a single-valued map $\psi : \mathbb{T} \times \mathbb{T} \times X \to X$ should exist, such that

$$x(t) = \psi(t, t_0, x_0).$$

If $t > t_0$, this formula means that a future state is fixed by the combination of an *initial state* and the *input* from the environment, the external world (the word "input" should not be taken too literally, as it may refer to such things as cooling or harvesting).

To make things even simpler, we concentrate on *autonomous* systems, meaning that the input should not vary in time (in other words, we describe the input by constant *parameters*). The consequence is that it does not matter at what time t_0 we specify the initial state x_0, only time differences matter and the system is time-translation invariant:

$$x(t) = \varphi(t - t_0, x_0).$$

In that case, we can take without loss of generality $t_0 = 0$ as the moment to specify the initial state $x_0 = x(0)$ of the system.

Often, it is convenient to single out the role of time in the notation and to write

$$x(t) = \varphi^t(x_0),$$

i.e., for given sets $\mathbb{T}$ and X, we consider a collection $\Phi = \{\varphi^t\}_{t\in\mathbb{T}}$ of maps from X to X, parametrized by the elements of $\mathbb{T}$:

$$\varphi^t : X \to X.$$

Depending on the smoothness of φ^t for fixed t, one distinguishes *continuous* and *smooth* dynamical systems. (When time is continuous we usually have joint continuity in t and x.)

Let us immediately point out that $\varphi^t(x)$ need not be defined for all $(x, t) \in X \times \mathbb{T}$. For example, if $\mathbb{T} = \mathbb{R}$, then $\varphi^t(x)$ might exist only for $t \in (-a(x), b(x)) \subset \mathbb{R}$, for some $a(x), b(x) > 0$. Such systems are called *locally defined*. As we shall see later, this is not unusual for dynamical systems associated to nonlinear differential equations. Systems for which the map φ^t is defined for $t = 0$, (at least small) $t > 0$ as well as $t < 0$ are called *invertible systems*. For such systems, the initial state x_0 fixes not only future but also past states. If one can predict future states given an initial state, but past states are not reconstructible, the system will be called *noninvertible*. For such systems, φ^t has meaning (in the sense of being defined and single-valued) only when $t \geq 0$, so that $\mathbb{T} = \mathbb{Z}_+$ or $\mathbb{R}_+$.

Of course, $\varphi^t(x)$ may very well be defined for all $x \in X$ and all $t \in \mathbb{T}$. An important class of such globally defined systems are those associated to linear ordinary differential equations or invertible linear maps in $\mathbb{R}^n$.

Let us return to the properties of the maps $\Phi = \{\varphi^t\}_{t\in\mathbb{T}}$ defining the evolution of a deterministic system. There are two general properties that these maps possess, namely:

$$\varphi^0(x) = x; \tag{1.1}$$

$$\varphi^{t+s}(x) = \varphi^t(\varphi^s(x)), \tag{1.2}$$

for all $x \in X$ and $t, s \in \mathbb{T}$ such that all quantities at both sides of the above relations are defined[5]. Property (1.1) expresses that x is the prescribed state at time zero, while the so-called *group property* (1.2) expresses that the state after time $t + s$ when starting at x, is identical to the state at time t when starting at $\varphi^s(x)$ (see Figure 1.1). In case $\mathbb{T} = \mathbb{Z}_+, \mathbb{R}_+$ equation (1.2) is usually called the *semigroup property*. Obviously, (1.2) expresses the combined effect of uniqueness and translation invariance.

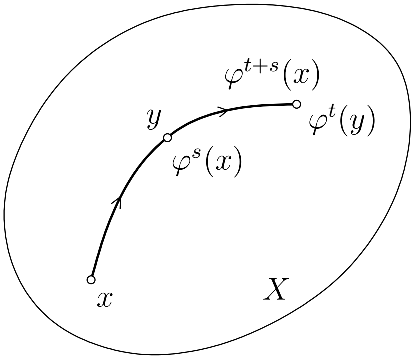

Figure 1.1: Group property: If $y = \varphi^s(x)$, then $\varphi^t(y) = \varphi^{t+s}(x)$.

Definition 1.1 *A* ***dynamical system*** *is a triple* $\{\mathbb{T}, X, \varphi^t\}$*, where* $\mathbb{T}$ *is a time set,* X *is a state space, and* $\{\varphi^t\}_{t\in\mathbb{T}}$ *is a family of evolution operators satisfying* (1.1) *and* (1.2)*.*

As often it is clear from the context what $\mathbb{T}$ and X are, we shall also call $\{\varphi^t\}$ a dynamical system. The map φ^t is called the *evolution operator* or the *t-shift map*. When time is continuous, the collection of evolution operators is called a *flow* (or *semiflow* if $\mathbb{T} = \mathbb{R}_+$). Thus, in the continuous time case the words "dynamical system" and "flow" are synonymous.

Example 1.2 (Translation dynamics)

There are many examples of dynamical systems with infinite-dimensional state spaces, among which we only mention *translation on BC*, i.e. $\{\mathbb{R}, BC(\mathbb{R}, \mathbb{R}), \theta^t\}$, where

$$[\theta^t(f)](x) = f(x + t),$$

for $x, t \in \mathbb{R}$. Here $BC(\mathbb{R}, \mathbb{R})$ is the Banach space of bounded continuous scalar functions on $\mathbb{R}$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{-\infty<x<+\infty} |f(x)|.$$

This is a continuous-time, invertible dynamical system. ◇

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[5] Using the symbol "$\circ$" for map composition and denoting the identity map by I, i.e. $I(x) = x$ for all $x \in X$, one can rewrite these properties as $\varphi^0 = I$ and $\varphi^{t+s} = \varphi^t \circ \varphi^s$ where defined, respectively.

1.1.2 The concept of a generator

Consider a dynamical system with $\mathbb{T} = \mathbb{Z}_+$. Define $f : X \to X$ by $f(x) = \varphi^1(x)$. We call f the *generator* of the dynamical system $\{\mathbb{Z}_+, X, \varphi^k\}$ to express that the collection $\Phi = \{\varphi^k\}_{k \in \mathbb{Z}_+}$ can be constructed from f by the operation of map composition. Indeed, (1.2) implies that

$$\varphi^2 = \varphi^1 \circ \varphi^1 = f \circ f$$

and, by induction for $k \in \mathbb{Z}_+$,

$$\varphi^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}. \tag{1.3}$$

The key point about this observation is that we can, vice versa, for any given map $f : X \to X$ define a dynamical system with $\mathbb{T} = \mathbb{Z}_+$ by means of these formulas. So, a discrete-time dynamical system corresponds to the iteration of a map and the super-index in φ^t can be rightfully interpreted as a power with respect to composition !

Example 1.3 (Linear maps in $\mathbb{R}^n$)

Let A be a $n \times n$ matrix with real elements $a_{ij},\ i, j = 1, 2, \ldots, n$. For $x \in \mathbb{R}^n$ the vector $Ax \in \mathbb{R}^n$ is defined by

$$(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, n.$$

Thus $x \mapsto Ax$ is a linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$. This map generates the discrete-time linear dynamical system $\{\mathbb{Z}, \mathbb{R}^n, A^k\}$. Here A^k denotes the map obtained by applying A k-times in a row. Of course, the corresponding matrix is the k-th power of the matrix A. ◇

Consider now an invertible dynamical system with $\mathbb{T} = \mathbb{Z}$ and define $f = \varphi^1$ as before. Then the map f has to be invertible. Indeed, by the fundamental properties (1.1) and (1.2) we have

$$\varphi^{-1} \circ f = \varphi^{-1} \circ \varphi^1 = \varphi^0 = I$$

and

$$f \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = \varphi^0 = I,$$

which shows that f^{-1} exists and is given by φ^{-1}. Now it is clear that any invertible map $f : X \to X$ defines an invertible dynamical system $\{\mathbb{Z}, X, \varphi^k\}$, where φ^k is specified for integer $k < 0$ by the formula

$$\varphi^k = \underbrace{f^{-1} \circ f^{-1} \circ \cdot \circ f^{-1}}_{|k| \text{ times}}. \tag{1.4}$$

Example 1.4 (Symbolic dynamics)

Important examples of discrete-time dynamical systems are provided by *symbolic dynamics.*

Take the set Ω_m of doubly-infinite sequences of symbols chosen from a collection of m symbols, say the natural numbers $\{1, 2, \dots, m\}$. An element $\omega \in \Omega_m$ is a sequence with a distinguished position zero

$$\omega = \dots \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \dots,$$

where ω_i is a natural number, $1 \leq \omega_i \leq m$. The space Ω_m is a compact and complete metric space with respect to a distance between $\omega, \theta \in \Omega_m$ defined by the formula:

$$\rho(\omega, \theta) = \sum_{j=-\infty}^{+\infty} \Delta_{\omega_j, \theta_j} 2^{-|j|},$$

where

$$\Delta_{\omega_j, \theta_j} = \begin{cases} 0 & \text{if} \quad \omega_j = \theta_j, \\ 1 & \text{if} \quad \omega_j \neq \theta_j. \end{cases}$$

According to this formula, two sequences are considered to be close if they have a long block of coinciding elements centered at position zero. The *shift map* σ

$$\omega \mapsto \theta = \sigma(\omega), \theta_j = \omega_{j+1}, \ j \in \mathbb{Z},$$

defines a discrete-time, invertible dynamical system $\{\mathbb{Z}, \Omega_m, \sigma^k\}$ called the *shift dynamic.*

We can also take any closed subset $\Omega \subset \Omega_m$ that is invariant under the shift σ, that is $\sigma(\Omega) = \Omega$. Then, clearly $\{\mathbb{Z}, \Omega, \sigma^k\}$ is again a dynamical system, called a *subshift dynamic.* An interesting class of subshifts arises when Ω is defined in the following way by a *transition matrix,* i.e. a matrix $A = (a_{ij})_{i,j=1,\dots,m}$ with elements $a_{ij} \in \{0, 1\}$,

$$\Omega = \{\omega \in \Omega_m : a_{\omega_j, \omega_{j+1}} = 1 \text{ for all } j \in \mathbb{Z}\}. \tag{1.5}$$

Subshift dynamics of this type are called *topological Markov chains.* As an example consider

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}. \tag{1.6}$$

Then any sequence $\omega \in \Omega$ can contain arbitrary long blocks of 1's. Such a block either extends indefinitely or terminates with a sequence $\dots 1, 2, 3, \dots, m, 1, \dots$, after which another block of 1's can follow, etc.

One may view the transition matrix A as the *adjacency matrix* of a directed graph and then note that any sequence $\omega \in \Omega$ corresponds to an infinite travel

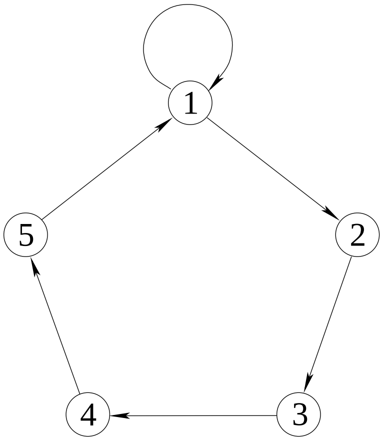

Figure 1.2: The directed graph with the adjacency matrix (1.6).

through this graph. Figure 1.2 shows a graph with the adjacency matrix A above, for $m = 5$. $\diamond$

In continuous time, the notion of generator is far more subtle. Assume that $T = \mathbb{R}_+$ or $\mathbb{R}$ and that X is a (subset of) a linear normed space. Also, let us supplement (1.1) and (1.2) by the continuity condition

$$\lim_{t \downarrow 0} \varphi^t(x) = x, \tag{1.7}$$

which excludes jumps. The question still is: What is the minimal amount of information needed to specify a dynamical system? And to get some feeling for the answer we again begin by working in the other direction, i.e. from a given/known dynamical system towards a condensed representation of the essential information.

In the discrete time case we considered the smallest possible time step (equal to 1), but now there is no such step, as time is continuous. The identity (1.7) states that we obtain only trivial (i.e., system unspecific) information when letting the time interval shrink to zero. The key idea is now to compute the *rate* of change of the state and to define (if possible) a function $f : X \to X$ by

$$f(x) = \lim_{t \downarrow 0} \frac{1}{t}(\varphi^t(x) - x). \tag{1.8}$$

Next we ask the question: Can we recover $\Phi = \{\varphi^t\}_{t \in T}$ from f? If the answer is positive, we call f the (*infinitesimal*) *generator* of the system. And, most importantly, we apply the "reconstruction procedure" to functions f, for which it is not yet known that they generate a dynamical system, in order to construct continuous-time dynamical systems from minimal and, from a modelling point of view, natural information. It is here that differential equations enter the scene.

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a *vector field*. The following theorem will allow us to conclude that smooth vector fields define continuous-time, invertible, smooth dynamical systems on $\mathbb{R}^n$.

Theorem 1.5 *Consider a system of autonomous ordinary differential equations*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ *is smooth in an open region* $U \subset \mathbb{R}^n$*. Then there is an open domain*

$$\Omega = \{(t, x_0) : x_0 \in U,\ t \in J(x_0) = (-\delta_1(x_0), \delta_2(x_0)) \text{ with } \delta_{1,2}(x_0) > 0\}$$

and a unique function $\psi : \Omega \mapsto U, x = \psi(t, x_0)$ *that is smooth with respect to* (t, x_0)*, and satisfies, for each* $x_0 \in U$*, the following conditions:*

(i) $\psi(0, x_0) = x_0$*;*
(ii) $\frac{\partial}{\partial t}\psi(t, x_0) = f(\psi(t, x_0))$ *for* $t \in J(x_0)$*.* □

If we now define

$$\varphi^t(x_0) = \psi(t, x_0),$$

then the triple $\{\mathbb{R}, \mathbb{R}^n, \varphi^t\}$ is a continuous-time, invertible, smooth dynamical system with the infinitesimal generator f. In general, this system is only locally defined in time. However, the theorem implies that for each $x_0 \in U$ there exists an open domain $U_0 \subset U$ and $\delta_0 > 0$ such that $\varphi^t(x_0)$ is defined for all $x_0 \in U_0$ and $|t| \leq \delta_0$.

As far as continuous time dynamical systems are concerned, we shall in most of this book restrict our attention to those defined on (a subset of) $\mathbb{R}^n$ and generated by a system of ordinary differential equations (*ODEs*). There is also a modelling related motivation to base the description of a dynamical system on differential equations. Many physical, chemical, biological, etc. laws give expressions for the *rate* of change in time of quantities that describe the state. Moreover, contributions of different mechanisms to the rate of change can be simply added, whereas such contributions may interact in subtle ways when one considers the changes over a finite time interval.

1.2 Orbits and phase portraits

Definition 1.6 *The* ***time-series*** *starting at* $x_0 \in X$ *is*

$$\Theta(x_0) = \{(t, x) : x = \varphi^t(x_0) \text{ for } t \in \mathbb{T} \text{ such that } \varphi^t(x_0) \text{ is defined}\}.$$

The orbit *starting at* $x_0 \in X$ *is*

$$\Gamma(x_0) = \{x : x = \varphi^t(x_0) \text{ for } t \in \mathbb{T} \text{ such that } \varphi^t(x_0) \text{ is defined}\}.$$

Note that $\Theta(x_0) \subset \mathbb{T} \times X$, while $\Gamma(x_0) \subset X$. Also note that the orbit starting at $y_0 \in \Gamma(x_0)$ coincides with $\Gamma(x_0)$ (see Exercise 1.5.5). Accordingly we could speak about orbits without specifying a particular starting point. The orbits are oriented by the increase (advance) of time. Finally note that the orbits of continuous time dynamical systems are *curves*, provided $\varphi^t(x)$ depends continuously on t.

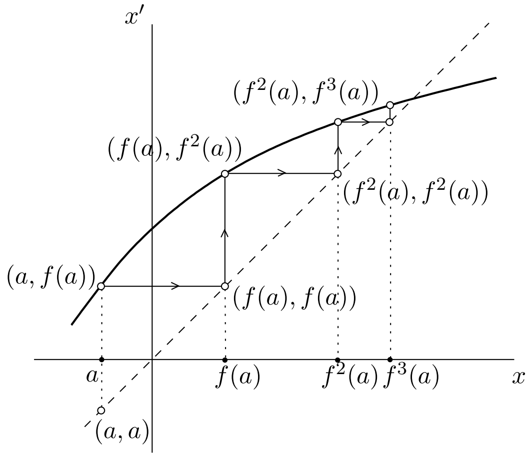

Figure 1.3: Staircase diagram.

Example 1.7 (Scalar maps)

Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x' = f(x)$ be a smooth map. There is a useful technique to visualize orbits of the corresponding discrete-time dynamical system: Staircase/cobweb diagrams, sometimes called *Lemeray's diagrams* (Figure 1.3). Consider the graph of f in the (x, x')-plane together with the line $x' = x$. Take a point (a, a) on this line and then produce the sequence of points

$$(a, a),\ (a, f(a)),\ (f(a), f(a)),\ (f(a), f^2(a)),\ (f^2(a), f^2(a)), \ldots$$

by "jumping" vertically and horizontally between the graph of f and the line $x' = x$. Now notice that the sequence obtained by taking the first coordinate of all odd points composes the forward part

$$a, f(a), f^2(a), \ldots,$$

of the orbit starting at $a \in \mathbb{R}$. If f is invertible, the backward part can be constructed in a similar manner by jumping between the line $x' = x$ and the graph of f, that is, in the opposite order. Notice that constant orbits

$$\ldots, a, a, a, a, \ldots$$

correspond to those points, where the graph of f intersects the line $x' = x$: $f(a) = a$. ◇

Example 1.8 (Scalar autonomous ordinary differential equations)

Another example, where one could plot and analyse orbits easily, is a continuous-time dynamical system generated by a scalar differential equation $\dot{x} = f(x)$ with a smooth $f : \mathbb{R} \to \mathbb{R}$. Looking at the graph of the function $y = f(x)$ in the (x, y)-plane (Figure 1.4), one concludes immediately that any orbit is either a root of the equation $f(x) = 0$ or an open segment of the x-axis, bounded by such roots or extending to infinity. The orientation of a nontrivial orbit is determined by the sign of $f(x)$ in the corresponding interval. ◇

The previous example serves to motivate the following definitions.

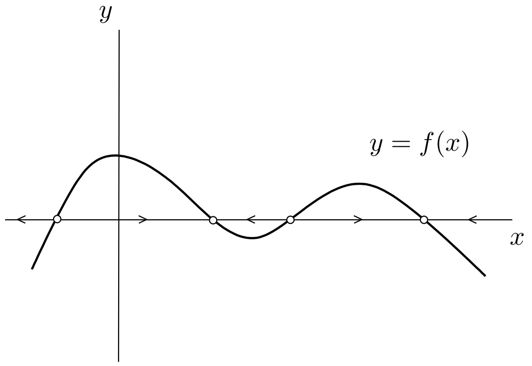

Figure 1.4: A scalar differential equation.

Definition 1.9 *The* ***phase portrait*** *of a dynamical system is the collection of its oriented orbits.*

Whenever the system is invertible, the phase portrait induces a partitioning of the state space (see Exercise 1.5.5). When we draw the phase portrait of a continuous-time planar dynamical system, we only sketch a few orbits qualitatively. However, we will do this in such a way that one can infer from the orbits drawn how the others should look. In particular, we will try to depict all orbits with exceptional properties, which bound "cells" composed of orbits having similar properties. Moreover, we draw at least one representative orbit in each cell.

Definition 1.10 *A point* $x^0 \in X$ *is called a* ***fixed point (equilibrium)*** *if* $\varphi^t(x^0) = x^0$ *for all* $t \in \mathbb{T}$.

We use the term "fixed point" primarily in the discrete-time case, while the term "equilibrium" is usually reserved for the continuous-time case. Both fixed points and equilibria are also called *steady states*. Fixed points of a discrete-time dynamical system generated by a map $x \mapsto f(x)$ satisfy the equation $x = f(x)$, while equilibria of a continuous-time system generated by an ODE $\dot{x} = f(x)$ satisfy $f(x) = 0$. In both cases one may think of $x, f(x) \in \mathbb{R}^n$. The shift dynamics with m symbols (see Example 1.4) has only m fixed points (find them!), while the equilibria of the translation dynamics from Example 1.2 are all constant functions $f(x) = f_0$.

Definition 1.11 *A* ***periodic orbit (cycle)*** *of period* $p > 0$ *is a nonequilibrium orbit* L_0, *such that for any point* $x_0 \in L_0$ *and for all* $t \in \mathbb{T}$ *holds*

$$\varphi^{t+p}(x_0) = \varphi^t(x_0).$$

Suppose p_0 is the minimal value of p for which the identity in the definition holds. In a continuous-time dynamical system generated by a smooth system of ODEs, each periodic orbit is a smooth closed curve that is traversed exactly once in a time interval of length p_0 (Figure 1.5(a)). In a discrete-time system generated by

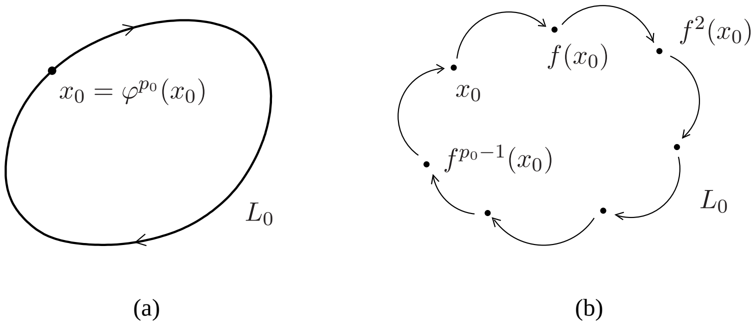

Figure 1.5: Periodic orbits.

a map f, periodic orbits are finite sets of points of the form

$$\{x_0, f(x_0), f^2(x_0), \ldots, f^{p_0-1}(x_0)\}$$

with $f^{p_0}(x_0) = x_0$ for an integer period p_0 (Figure 1.5(b)). The shift dynamic has an infinite number of periodic orbits. Indeed, any periodic sequence is a starting point of a periodic orbit, since it is mapped exactly onto itself by shifting it over the period.

There are other interesting types of special orbits. Let ρ be a metric in the state space X.

Definition 1.12 *An orbit* Γ_0 *is called* ***homoclinic to a fixed point (equilibrium)*** x^0 *if for any point* $x_0 \in \Gamma_0$*:* $\rho(\varphi^t(x_0), x^0) \to 0$*, for* $t \to \pm\infty$*.*

An orbit Γ_0 *is called* ***heteroclinic to fixed points (equilibria)*** x^1 *and* x^2 *if for any point* $x_0 \in \Gamma_0$*:* $\rho(\varphi^t(x_0), x^1) \to 0$*, for* $t \to -\infty$*, and* $\rho(\varphi^t(x_0), x^2) \to 0$*, for* $t \to +\infty$*.*

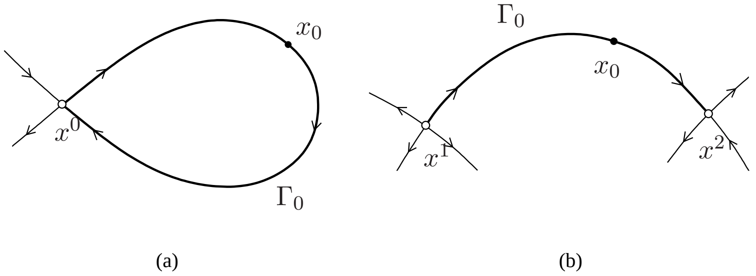

Figure 1.6: Homoclinic (a) and heteroclinic (b) orbits to equilibria.

Figure 1.6 illustrates these notions for continuous time systems in the plane. Similarly one can define orbits homoclinic to a periodic orbit L_0, as well as heteroclinic orbits that tend to different periodic orbits $L^{1,2}$ as $t \to \pm\infty$ (or to an equilibrium as $t \to -\infty$ and to a periodic orbit as $t \to +\infty$, etc). Together homoclinic and heteroclinic orbits are called *connecting orbits.*

Definition 1.13 *An* ***invariant set*** *of a dynamical system is a set* $S \subset X$ *entirely composed of orbits which are defined for all time, i.e.* $x \in S$ *if and only if* $\varphi^t(x) \in S$ *for all* $t \in T$*.*

Definition 1.14 (Lyapunov stability) *An invariant set S is called* ***(Lyapunov) stable*** *if for any neighbourhood $U \supset S$ there exists a neighbourhood $V \supset S$ such that $\varphi^t(x) \in U$ for all $x \in V$ and all $t > 0$.*

Definition 1.15 *An invariant set S is called* ***asymptotically stable*** *if it is stable and there exists a neighbourhood $U \supset S$ such that $\rho(\varphi^t(x), S) \to 0$ for all $x \in U$, as $t \to +\infty$. If we can take $U = X$, we call S* ***globally asymptotically stable****.*

The fixed point of a contraction $f : X \to X$ in a complete metric space X is globally asymptotically stable (see Theorem 3.20 in Chapter 3).

A compact[6] asymptotically stable invariant set is often called an *attractor*. If a bounded invariant set becomes asymptotically stable after time reversal $t \mapsto -t$, it is called a *repellor*. An attractor (in general, invariant set) is called *strange* if it has a Cantor (fractal) structure.

1.3 Equivalence of dynamical systems

Definition 1.16 *A dynamical system $\{T, X, \varphi^t\}$ is* ***equivalent*** *to a dynamical system $\{T, Y, \psi^t\}$ if there is an invertible surjective map $h : X \to Y$ which maps orbits of the first system onto orbits of the second system, preserving the direction of time.*

By "preserving the direction of time" we mean that, whenever $\varphi^{t_1}(x)$ and $\varphi^{t_2}(x)$ with $t_1 \leq t_2$ are mapped to $\psi^{s_1}(h(x))$ and $\psi^{s_2}(h(x))$, respectively, then we can choose s_1 and s_2 such that $s_1 \leq s_2$. Recall that a map $h : X \to Y$ is *surjective* if it maps X onto Y. If h and h^{-1} are continuous maps, *i.e.* h is a *homeomorphism*, the equivalence is called *topological equivalence*. If h and h^{-1} are continuously-differentiable maps, *i.e.* h is a *diffeomorphism*, the equivalence is called *smooth equivalence*.

Example 1.17 (Equivalence of planar systems)

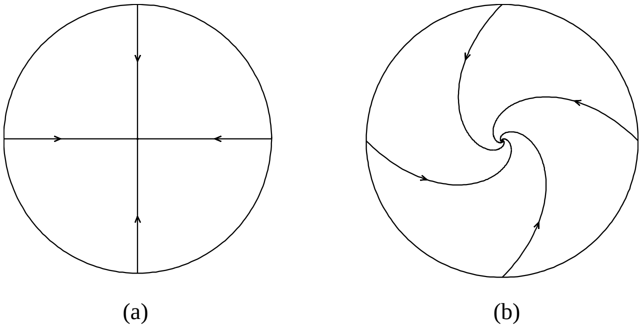
(a) (b)

Figure 1.7: Node (a) and focus (b) are topologically equivalent.

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[6] In finite-dimensional spaces, compact sets are those which are bounded and closed.

Two continuous-time dynamical systems in $\mathbb{R}^2$ with linear evolution operators, given by the matrices

$$\varphi^t = e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\psi^t = e^{-t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

are topologically equivalent (each has a globally asymptotically stable equilibrium at the origin, see Figure 1.7). A proof is indicated in Exercises 1.5.13 and 1.5.14. ◇

If h preserves not only the direction of time, but also the time parametrization along orbits, the systems are called (topologically, smooth) *conjugate*. In this case:

$$\psi^t = h \circ \varphi^t \circ h^{-1}, \tag{1.9}$$

where defined, see Figure 1.8. The two systems from Example 1.17 are not only topologically equivalent but also topologically conjugate.

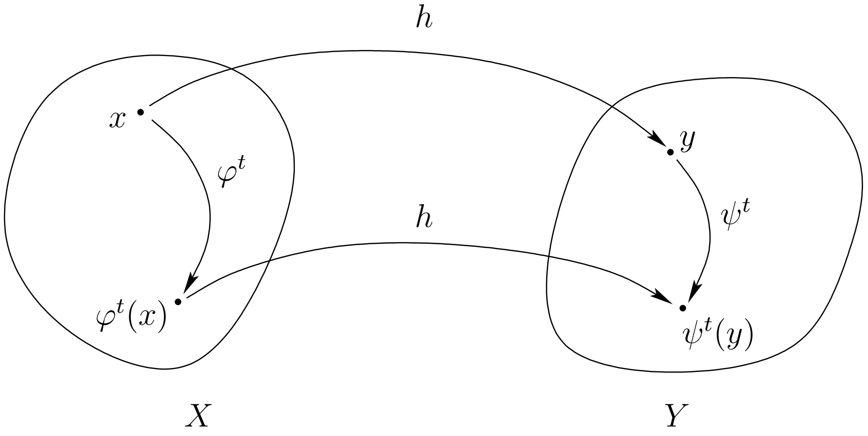

Figure 1.8: Conjugate systems: $h(\varphi^t(x)) = \psi^t(h(x))$.

Topologically (smoothly) equivalent discrete-time systems are always conjugate. Moreover, if $f = \varphi^1$ and $g = \psi^1$, then the topological conjugacy of the corresponding systems follows from

$$g = h \circ f \circ h^{-1}$$

by induction. When both f and g are invertible, their inverses are conjugate via the same map h:

$$g^{-1} = h \circ f^{-1} \circ h^{-1} .$$

Smooth conjugacy of the flows generated by two smooth autonomous systems of ODEs can be expressed in terms of the vector fields. Introduce the generators

$$\left.\frac{d}{dt}\varphi^t(x)\right|_{t=0} = f(x), \quad \left.\frac{d}{dt}\psi^t(y)\right|_{t=0} = g(y),$$

then differentiate $\psi^t \circ h = h \circ \varphi^t$ with respect to t at $t = 0$ to obtain

$$g(h(x)) = h_x(x)f(x).$$

Since h is a diffeomorphism, the Jacobian matrix h_x with elements

$$\frac{\partial h_i(x)}{\partial x_j}.$$

is invertible and we find the relation

$$f(x) = h_x^{-1}(x)g(h(x)). \tag{1.10}$$

Two smooth autonomous systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ for which the relation (1.10) holds for some diffeomorphism h are called *diffeomorphic*. Changing variables $y = h(x)$ in $\dot{y} = g(y)$ and using (1.10) we immediately obtain $\dot{x} = f(x)$. Above we have shown that smooth conjugacies lead to diffeomorphic ODE's. Conversely, it is obvious that the flows generated by two smoothly diffeomorphic ODE's are smoothly conjugate.

Two smooth autonomous systems $\dot{x} = f(x)$ and $\dot{x} = g(x)$ for which

$$f(x) = \mu(x)g(x),$$

where $\mu = \mu(x) > 0$ is a smooth positive function, are called *orbitally equivalent*. In this case, h in Definition 1.16 can be defined by $h(x) = x$ and the systems have identical orbits with different time-parametrization. If the system $\dot{x} = f(x)$ is diffeomorphic to a system that is orbitally equivalent to the system $\dot{y} = g(y)$, then these two systems are called *smoothly orbitally equivalent*.

If h is defined in some neighbourhood $U \subset X$ of an invariant set S of the first system, while h^{-1} is defined in the neighbourhood $V = h(U) \subset Y$ of the invariant set $h(S)$ of the second system, the equivalence is called *local equivalence* near the invariant sets. Notice that stability is a topologically invariant notion, i.e. the corresponding invariant sets (S and $h(S)$) of two topologically equivalent systems are both stable or both unstable (i.e., not stable).

The (overly ambitious) aim of the qualitative theory of dynamical systems is to provide a catalogue of equivalence classes, as well as rules to determine the equivalence class from the generator. The pragmatic and more realistic version concentrates on the neighbourhood of special orbits (like those introduced in the previous section). In Chapter 3 we shall see that "linearization" is a powerful method when trying to determine the phase portrait near an equilibrium or a fixed point. Clearly then, we should carry out the classification program for *linear* systems, and this is exactly what the next chapter is all about.

1.4 References

There are many texts on dynamical systems, where all fundamental notions introduced in this chapter are discussed, e.g. [Irwin 1980, Amann 1990, Hale & Koçak 1991, Alligood, Sauer & Yorke 1997, Perko 2001, Hasselblatt & Katok 2003], see also [Anosov, Bronshtein, Aranson & Grines 1988, Chapter 1].

1.5 Exercises

E 1.5.1 (Nonautonomous systems)

In the non-autonomous case we have a map

$$\psi : \mathbb{T} \times \mathbb{T} \times X \to X$$

such that $\psi(t, t_0, x_0)$ is the state of the system at time t, given that its state at time t_0 is x_0.

(i) Formulate the analogue of the properties (1.1) and (1.2). (*Hint*: First formulate them in words and only then re-express the statements in mathematical symbols.)

(ii) Show that ψ leads to an autonomous dynamical system on the extended phase space $Y = \mathbb{T} \times X$ via

$$\Phi(s, y) = (t + s, \psi(t + s, y)), \quad \text{where} \quad y = (t, x).$$

(iii) In the smooth continuous time case how would you define the infinitesimal generator of ψ^t and what will be its relation to the generator of the extended flow Φ^t ?

E 1.5.2 (Continuous-time system with discontinuous orbits)

For $x \in \mathbb{R}, t \geq 0$ define

$$\varphi^t(x) = \begin{cases} x + t + 1 & \text{if } \ x < 0 \leq x + t, \\ x + t & \text{otherwise.} \end{cases}$$

Show that this defines a dynamical system $\{\mathbb{R}_+, \mathbb{R}, \varphi^t\}$ for which orbits are right continuous (i.e. $\lim_{t \downarrow 0} \varphi^t(x) = x$ for all $x \in \mathbb{R}$) but not continuous. Can this system be extended to an invertible one on $X = \mathbb{R}$?

E 1.5.3 (Orbits of maps)

Consider the *Ricker map*[7]

$$x \mapsto x' = \alpha x e^{-x}.$$

(i) Set $\alpha = 6$. Pick an initial point, say $x_0 = 0.2$, and construct graphically the orbit of the map starting at this point (see Figure 1.9(a)). Do you see why one also speaks of cobweb (rather than staircase) diagrams? Calculate the fixed point.

(ii) Construct an orbit starting at the same initial point x_0, but for $\alpha = 9$ (see Figure 1.9(b)). Find (approximately) the coordinates of the two points on the period-2 cycle.

(iii) Explain first in graphical terms the difficulty of going backwards in time and then give an analytic reformulation.

E 1.5.4 (Orbits of ODEs)

Using one of the standard ODE solvers, construct numerically an orbit of the *Rössler system*[8]:

$$\begin{cases} \dot{x}_1 & = & -x_2 - x_3, \\ \dot{x}_2 & = & x_1 + A x_2, \\ \dot{x}_3 & = & B x_1 - C x_3 + x_1 x_3, \end{cases}$$

with $A = 0.36, B = 0.4, C = 4.5$, starting at $x_0 = (0.74, -0.41, 0.06)$ on the time interval $0 \leq t \leq 200$ (see Figure 1.10). Think about the accuracy of the approximation of the solution! What kind of invariant set do you expect in this system?

E 1.5.5 (Orbits and time-series)

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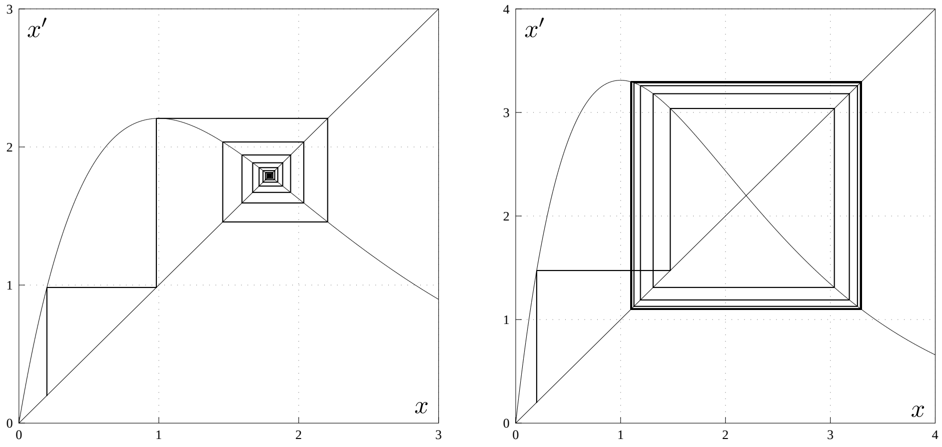
[7] Ricker, W. 'Stock and recruitment', *J. Fish. Res. Board Canada* **211** (1954), 559-663.

[8] Rössler, O.E. 'Continuous chaos–four prototype equations', In: Bifurcation Theory and Applications in Scientific Disciplines, *Ann. New York Acad. Sci.* **316**, New York, 1979, pp. 376–392.



(a) (b)

Figure 1.9: Orbits of Ricker map: (a) $\alpha = 6$; (b) $\alpha = 9$.

Consider an invertible dynamical system. Write $y \sim x$ when the point y belongs to the orbit starting at the point x.

(i) Show that "$\sim$" is an equivalence relation, *i.e.* prove that (1) $x \sim x$ (reflexivity); (2) $y \sim x$ implies $x \sim y$ (symmetry); (3) $x \sim y$ and $y \sim z$ implies $x \sim z$ (transitivity).

Accordingly, we can without any ambiguity speak about "orbits" without specifying a particular starting point. Every point belongs to exactly one orbit. A mathematical way of expressing this fact is to say that orbits *partition* the state space X.

(ii) Assume that x_0 and x_1 belong to the same orbit. Show that the time-series starting at, respectively, x_0 and x_1 are translates of each other.

E 1.5.6 (Higher order recursions)

Fibonacci's law of replication reads

$$x(t+1) = x(t) + x(t-1)$$

with two initial values $x(0), x(1)$. Write this as a dynamical system in $\mathbb{R}^2$. What would you do if $x(t+1)$ depends on $x(t)$ and k states $x(t-1), \ldots, x(t-k)$ from the past ?

E 1.5.7 (Generator of translation dynamics)

Compute the infinitesimal generator of the translation dynamics from Example 1.2. This generator is only defined on a subspace of $BC(\mathbb{R}, \mathbb{R})$ (which one ?).

E 1.5.8 (Periodic and connecting orbits of shift dynamics)

(i) Compute the number $N(k)$ of period-k cycles in the symbolic dynamics $\{\mathbb{Z}, \Omega_2, \sigma^k\}$.

(ii) For the symbolic dynamics $\{\mathbb{Z}, \Omega_2, \sigma^k\}$, describe all orbits homoclinic to the zero sequence and all heteroclinic orbits connecting the zero sequence to the sequence of ones.

E 1.5.9 (Different metrics on spaces of symbols)

Show that the metric in Example 1.4 is equivalent to any of the metrics

$$\rho_q(\omega, \theta) = \sum_{k=-\infty}^{+\infty} |\omega_k - \theta_k| \, q^{-|k|},$$

where q is a constant > 1. (*Hint:* Two metrics ρ_1, ρ_2 on a space X are called equivalent if for any $\omega \in X$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho_1(\omega, \theta) \leq \delta$ implies $\rho_2(\omega, \theta) \leq \varepsilon$ and vice versa).

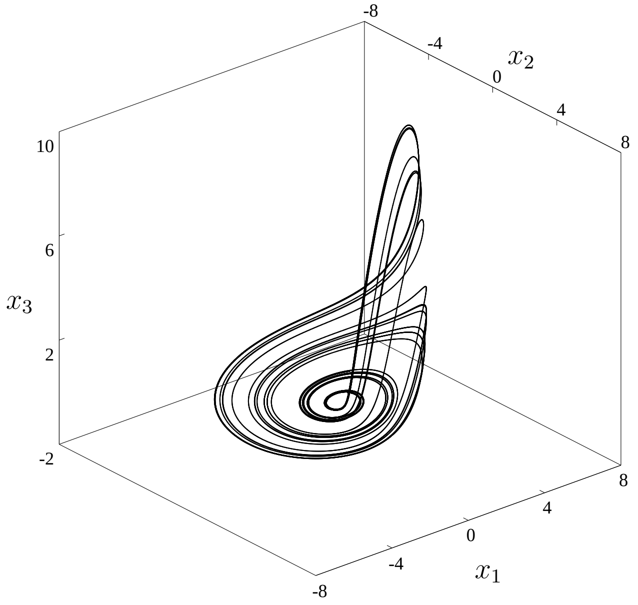


Figure 1.10: An orbit of the Rössler system.

E 1.5.10 (Topological equivalence of scalar ODEs)

Consider two *scalar* autonomous ODEs defined by functions f_1 and f_2, respectively. Give a graphical criterion for the topological equivalence of the dynamical systems generated by f_1 and f_2.

E 1.5.11 (Topological equivalence of monotone scalar maps)

Consider two *monotone scalar* functions f_1 and f_2. Give a graphical criterion for the topological equivalence of the discrete-time dynamical systems generated by iterating f_1 and f_2. *Hint*: Use the method of fundamental domains, see Section 2.3.2 in Chapter 2.

E 1.5.12 (Smooth equivalence of linear ODEs)

If two linear systems $\dot{x} = Ax$ and $\dot{y} = By$ are diffeomorphic through a linear change $y = Sx$ of coordinates, what is the relation between the eigenvalues (or more generally the Jordan normal forms) of A and B ?

E 1.5.13 (Topological equivalence of planar linear ODEs)

Elaborate Example 1.17 from Section 1.3.

(i) Show that two continuous-time dynamical systems in $\mathbb{R}^2$ with linear evolution operators, given by the matrices

$$\varphi^t = e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \psi^t = e^{-t}R(t), \quad R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

are topologically conjugate in $\mathbb{R}^2$. Compute a conjugating map h and its inverse h^{-1} explicitly. *Hint:* Assume that $h(x)$ is the identity $h(x) = x$ on the circle $\|x\| = 1$ and determine it elsewhere from the relation

$$h(e^{-t}x) = e^{-t}R(t)h(x).$$

(ii) Discuss the differentiability of thus constructed map h at the origin.

E 1.5.14 (Topological equivalence of planar linear ODEs revisited)

Reconsider Exercise 1.5.13 by working with generators and show topological conjugacy in the unit disc $U = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$.

(i) By computing the infinitesimal generators, show that the above evolution operators are associated with the following linear systems

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = -x_2, \end{cases} \tag{1.11}$$

and

$$\begin{cases} \dot{x}_1 = -x_1 - x_2, \\ \dot{x}_2 = x_1 - x_2. \end{cases} \tag{1.12}$$

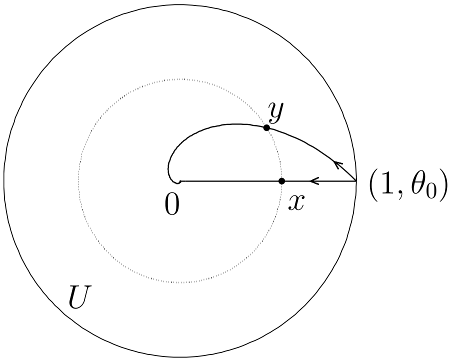


Figure 1.11: Construction of the homeomorphism in Exercise 1.5.14: $x \mapsto y = h(x)$.

(ii) Write equations (1.11) and (1.12) in polar coordinates (ρ, θ),

$$\begin{cases} x_1 = \rho \cos \theta, \\ x_2 = \rho \sin \theta. \end{cases}$$

Verify that the solution of (1.11) in these coordinates is given by

$$\begin{aligned} \rho(t) &= \rho_0 e^{-t}, \\ \theta(t) &= \theta_0, \end{aligned}$$

while that of (1.12) is

$$\begin{aligned} \rho(t) &= \rho_0 e^{-t}, \\ \theta(t) &= \theta_0 + t. \end{aligned}$$

Here (ρ_0, θ_0) corresponds to an initial point.

(iii) For any point $x \in U$ with polar coordinates (ρ_0, θ_0), $\rho_0 \neq 0$, construct a point $y \in U$ with polar coordinates (ρ_1, θ_1) as follows (see Figure 1.11). Consider the time τ required to move, along an orbit of system (1.11), from the point on the boundary with polar coordinates $(1, \theta_0)$ to the point x. Then consider an orbit of system (1.12) starting at the boundary point with polar coordinates $(1, \theta_0)$, and let $y = (\rho_1, \theta_1)$ be the point at which this orbit arrives after $\tau(\rho_0)$ units of time. Compute the thus defined map $x \mapsto y = h(x)$ explicitly in polar coordinates.

(iv) Define $h(0,0) = (0,0)$ and show that the resulting map $h : U \to U$ is a homeomorphism that maps orbits of (1.11) in U onto orbits of (1.12) in U, meaning that these two systems are topologically conjugate.

(v) Discuss the differentiability of h at the origin. *Hint*: Use the original coordinates (x_1, x_2).

(vi) Show how a similar procedure can be used to construct a conjugating homeomorphism h outside U in $\mathbb{R}^2$.

1.6 Appendix: Bifurcation of homoclinic orbits to saddle-saddle

The aim of this appendix is to demonstrate that notions introduced in this chapter are sufficient to describe rather complex dynamical phenomena. Of course, an understanding of how the following results are actually obtained will come only after studying all chapters of this (and perhaps some other) book. Thus, the reader should not be disappointed if some statements below are not immediately clear.

Consider a smooth ODE system in $\mathbb{R}^3$ having in the unit cube $\{\xi \in \mathbb{R}^3 : |\xi_k| \leq 1, k = 1, 2, 3\}$ the form

$$\begin{cases} \dot{\xi}_1 &= \beta + b\xi_1^2, \\ \dot{\xi}_2 &= \lambda_2\xi_2, \\ \dot{\xi}_3 &= \lambda_3\xi_3, \end{cases} \tag{1.13}$$

where β is a nonnegative parameter and $\lambda_2 > 0, \lambda_3 < 0, b > 0$. Its phase portrait at $\beta = 0$ in the unit cube is presented in Figure 1.12. This phase portrait can be constructed by taking into account that all the coordinate planes $\xi_k = 0, k = 1, 2, 3$, are invariant with respect to (1.13). The point $x^0 = (0, 0, 0)$ is obviously an equilibrium, called a *saddle-saddle*. The *stable invariant set* of x^0 is composed of all orbits approaching x^0 as $t \to +\infty$ and has the local representation

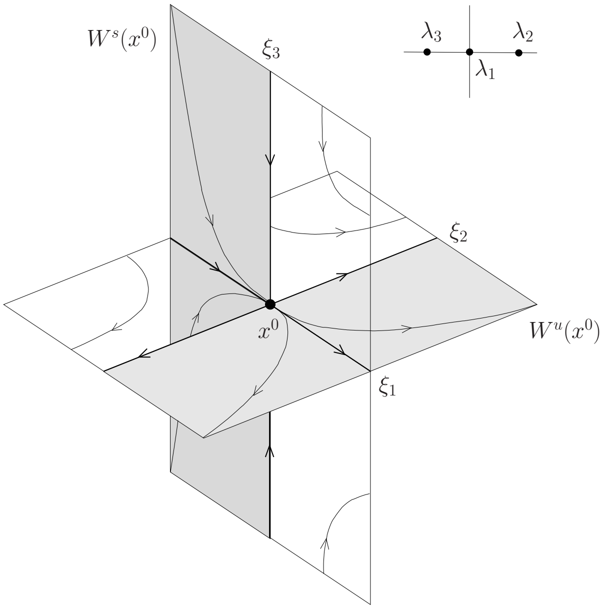


Figure 1.12: Saddle-saddle equilibrium at $\beta = 0$.

$$W^s(x^0) = \{\xi : \xi_1 \leq 0, \xi_2 = 0\},$$

while the *unstable invariant set* of x^0 is composed of all orbits approaching x^0 as $t \to -\infty$ and is locally represented by

$$W^s(x^0) = \{\xi : \xi_1 \leq 0, \xi_2 = 0\},\ W^u = \{\xi : \xi_1 \geq 0, \xi_3 = 0\}.$$

The sets can be extended outside the unit cube, where they become two-dimensional *invariant manifolds* (surfaces). Moreover, these surfaces can intersect transversally along several *homoclinic orbits* to x^0. Assume that exactly two such orbits, Γ_1 and Γ_2, exist at $\beta = 0$ (see Figure 1.13). Consider two faces of the cube:

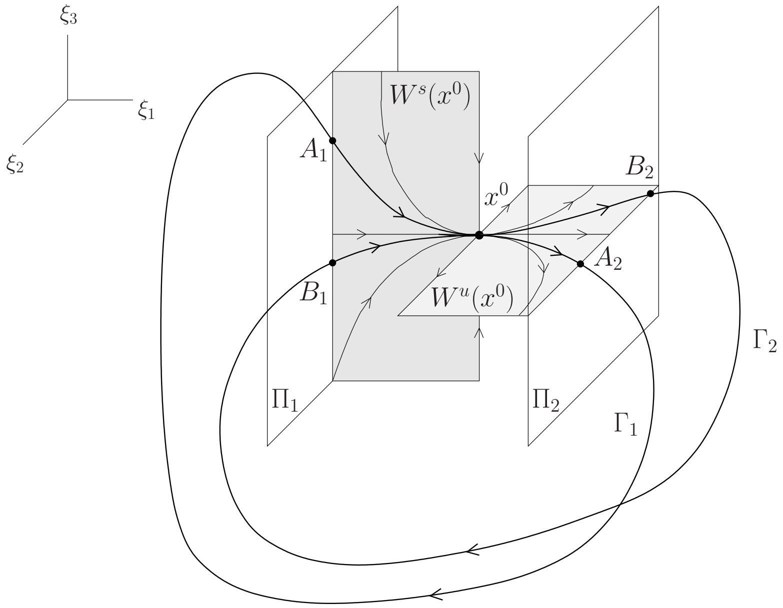


Figure 1.13: A saddle-saddle with two homoclinic orbits.

$$\Pi_1 = \{\xi : \xi_1 = -1, |\xi_{2,3}| \leq 1\},\ \Pi_2 = \{\xi : \xi_1 = 1, |\xi_{2,3}| \leq 1\}.$$

Let A_1, B_1 be the points where the orbits Γ_1, Γ_2 intersect with the plane Π_1 as they enter the unit cube while returning to the saddle-saddle. Similarly, denote by A_2, B_2 the intersection points of Γ_1, Γ_2 with the plane Π_2 as these orbits leave the cube.

For small $\beta > 0$, the system (1.13) has no equilibria in the cube: A *fold bifurcation* occurs that will be studied in detail in Chapters 5 and 6. In this case, the *Poincaré return map*

$$P_\beta : \Pi_1 \to \Pi_1,$$

can be defined by following orbits of the system that start in (a subset of) Π_1. This map can be represented as the composition of a "local" map $\Delta_\beta : \Pi_1 \to \Pi_2$ along orbits passing through the cube, and a "global" map $Q_\beta : \Pi_2 \to \Pi_1$:

$$P_\beta = Q_\beta \circ \Delta_\beta.$$

By analyzing the system with small $\beta > 0$ inside the cube, one can show that the square Π_1 is contracted by the map Δ_β in the ξ_3-direction and expanded in the ξ_2-direction. Thus, the intersection of its image $\Delta_\beta \Pi_1$ with the square Π_2 would include a horizontal strip $\Sigma = \Delta_\beta \Pi_1 \cap \Pi_2$ (see Figure 1.14(b)), which gets thinner and thinner as $\beta \to 0$ (explain why).

The strip Σ contains the points A_2 and B_2. Since the map Q_0 sends the point A_2 into the point A_1 and the point B_2 into the point B_1,

$$Q_0(A_2) = A_1,\ Q_0(B_2) = B_1,$$

there will be some neighbourhoods of A_2, B_2 in Π_2, that Q_β maps into neighborhoods of A_1 and B_1 in Π_1, respectively, for small $\beta > 0$. Therefore, the image $Q_\beta(\Sigma)$ will intersect the square Π_1 in two strips, Σ_1 and Σ_2, $\Sigma_1 \cup \Sigma_2 = Q_\beta \Sigma_0 \cap \Pi_1$ (see Figure 1.14(a)), containing A_1 and B_1, respectively. Due to the transversality assumption, Σ_1 and Σ_2 intersect the vertical axis at nonzero angles near the points A_1 and B_1, respectively.

Thus, the intersection of the image of Π_1 under the Poincaré map $P_\beta = Q_\beta \circ \Delta_\beta$ with Π_1 has the standard features of the so called *Smale horseshoe* (see Chapter 7). For example, applying the construction once more, we first obtain two strips inside Σ, and then two narrow strips inside each

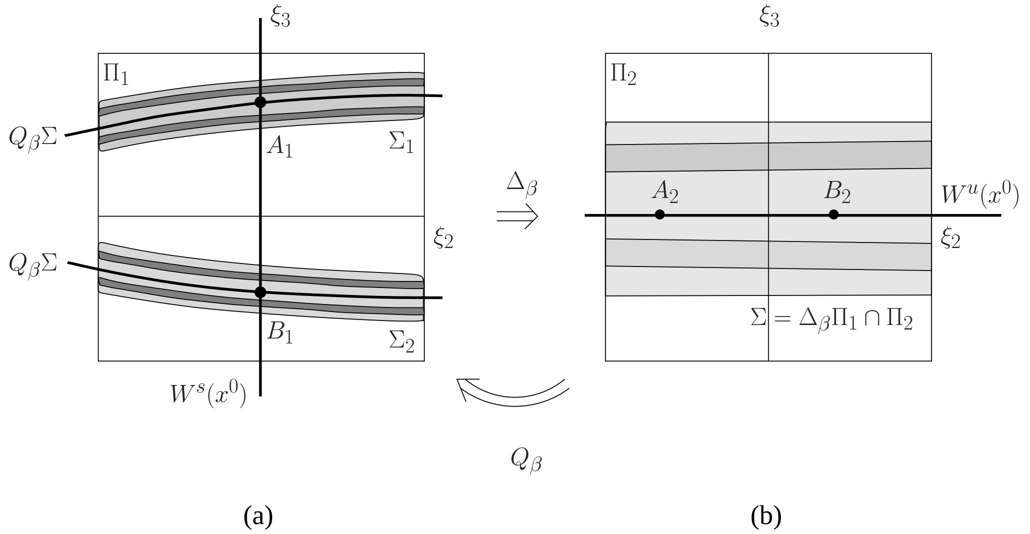


Figure 1.14: Cross-sections near a saddle-saddle.

$\Sigma_{1,2}$, and so forth (see Figure 1.14). Inverting the procedure, we get vertical strips with similar structure.

As we shall see in Chapter 7, the presence of the Smale horseshoe implies the existence of an invariant – for the Poincaré map P_β – set $\Lambda_\beta \subset \Pi_1$ that lies in the intersection of all strips. Restricted to this invariant Cantor set, P_β is *topologically equivalent* to the *shift map* σ on the set Ω_2 of all bi-infinite sequences of two symbols $\{1, 2\}$. In particular, this implies that the considered ODE system has an *infinite number* of periodic orbits (*cycles*) for small $\beta > 0$. These orbits are coded by periodic sequences.

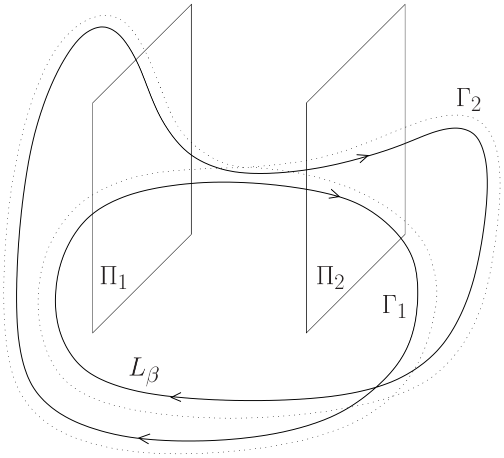


Figure 1.15: A cycle corresponding to the sequence $\{\ldots, 1, 2, 1, 2, \ldots\}$.

In the present context, the coding has a simple geometrical interpretation. Indeed, let γ be an orbit located in a neighbourhood U_0 of the union of the homoclinic orbits $\Gamma_{1,2}$ for all $t \in (-\infty, +\infty)$. Then it passes outside the cube near either Γ_1 or Γ_2. The elements of the corresponding sequence $\omega = \{\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots\}$ specify whether the orbit γ makes its ith passage near Γ_1 or Γ_2; in the former case, $\omega_i = 1$, while in the latter, $\omega_i = 2$. For example, the sequence

$$\{\ldots, 1, 1, 1, 1, 1, \ldots\}$$

corresponds to a unique cycle located near Γ_1. The sequence

$$\{\ldots, 2, 2, 2, 2, 2, \ldots\}$$

describes a periodic orbit located near Γ_2, while the periodic sequence

$$\{\ldots, 1, 2, 1, 2, 1, 2, \ldots\}$$

corresponds to a cycle making its first trip near Γ_1, its second near Γ_2, and so on (see Figure 1.15).

The described phenomenon was discovered by L.P. Shilnikov in the 1960s. He proved that it occurs in generic n-dimensional smooth ODEs when a *nonhyperbolic* equilibrium possessing two homoclinic orbits disappears. This Appendix is based on [Kuznetsov 2004, Ch.7], further details can be found in [Arnol'd, Afraimovich, Il'yashenko & Shil'nikov 1994, Ilyashenko & Li 1999].