are the eigenvectors of the Jacobian matrix and its transpose at $b = b_0$,

$$Jq = i\omega_0 q, \ \ J^T p = -i\omega_0 p.$$

To achieve the normalization $\langle p, q\rangle = 1$ we take

$$q = \left(-\frac{ia + a^2}{1 + a^2}, \ 1\right)^T, \quad p = \left(-\frac{i(1 + a^2)}{2a}, \ \frac{1 - ia}{2}\right)^T.$$

Now compose $x = x_0 + zq + \bar{z}\bar{q}$ and compute the function

$$H(z, \bar{z}) = \langle p, f(x_0 + zq + \bar{z}\bar{q}, a, 1 + a^2)\rangle.$$

The first few coefficients of its Taylor expansion at $(z, \bar{z}) = (0, 0)$,

$$H(z, \bar{z}) = i\omega_0 z + \sum_{2\leq j+k\leq 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4),$$

are

$$g_{20} = a - i, \quad g_{11} = \frac{(a - i)(a^2 - 1)}{1 + a^2}, \quad g_{21} = -\frac{a(3a - i)}{1 + a^2}.$$

Therefore, the first Lyapunov coefficient is given by

$$l_1 = \frac{1}{2\omega_0^2}\mathrm{Re}(ig_{20}g_{11} + \omega_0 g_{2,1}) = -\frac{2 + a^2}{2a(1 + a^2)} < 0.$$

Thus, a unique and stable limit cycle bifurcates in (5.47) from the equilibrium for $b > b_0$: Andronov-Hopf bifurcation is supercritical (see Figure 5.15). ◇

5.3 One-parameter bifurcations of fixed points

Consider a generic smooth one-parameter family of maps

$$x \mapsto f(x, \alpha) = f_{(\alpha)}(x), \quad x \in \mathbb{R}^n, \alpha \in \mathbb{R}.$$

As for equilibria in ODEs, we expect that local bifurcations happen to *nonhyperbolic* fixed points. Thus, we should consider three *critical cases* (or *singularities*, see Figure 5.16):

(a) the fixed point x_0 has eigenvalue $\mu_1 = 1$;

(b) the fixed point x_0 has eigenvalue $\mu_1 = -1$;

(c) the fixed point x_0 has a pair of complex-conjugate eigenvalues $\mu_{1,2} = \mathrm{e}^{\pm i\theta_0}$ with $0 < \theta_0 < \pi$.

Moreover, we can assume that in each case no other eigenvalue satisfies $|\mu| = 1$ and the critical eigenvalues of x_0 are algebraically simple. The codim 1 bifurcation associated to the critical case (a) is called a *fold* (or *limit point*) bifurcation of maps. The codim 1 bifurcation occuring at the critical case (b) is called a *flip* (or *period-doubling*) bifurcation, while the codim 1 bifurcation related to case (c) is referred to as a *Neimark-Sacker* bifurcation. Since local bifurcations of period-k cycles can be treated as bifurcations of fixed points of $f^k_{(\alpha)}$, we only consider local bifurcations of fixed points here.

First we study these bifurcations in the lowest possible phase-space dimension, i.e., $n = 1$ for the fold and flip and $n = 2$ for the Neimark-Sacker bifurcation.

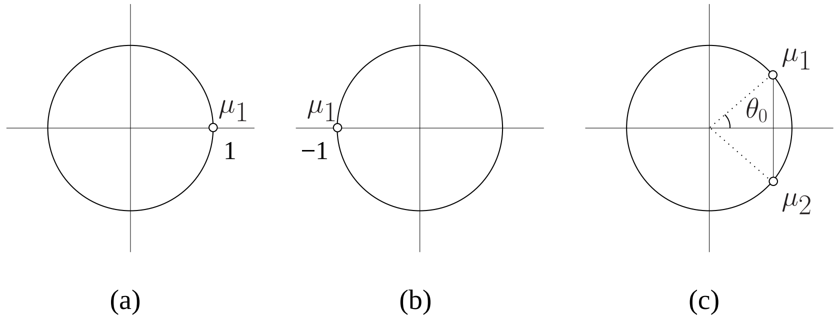

Figure 5.16: Critical cases for fixed points.

5.3.1 Fold bifurcation of scalar maps

Example 5.21 (Normal form of the fold bifurcation of maps)

Consider the one-dimensional dynamical system generated by the following map depending on one parameter:

$$x \mapsto f_{(\alpha)}(x) = \alpha + x + x^2. \tag{5.48}$$

Let $f(x, \alpha) = \alpha + x + x^2$. The map $f_{(\alpha)}$ is invertible in a neighbourhood of the origin for $|\alpha|$ small. The system has at $\alpha = 0$ a nonhyperbolic fixed point $x_0 = 0$ with $\mu = \frac{\partial f}{\partial x}(0, 0) = 1$. The behaviour of the system near $x = 0$ for small $|\alpha|$ is shown in Figure 5.17. For $\alpha < 0$ there are two fixed points in the system: $x_{1,2}(\alpha) = \pm\sqrt{-\alpha}$,

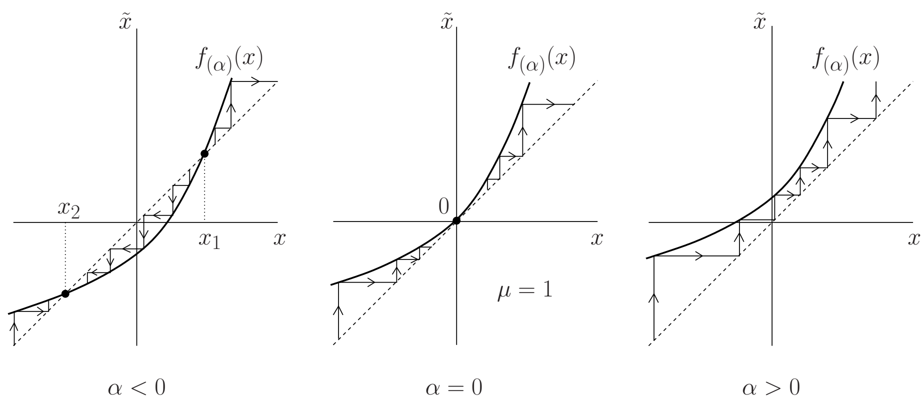

Figure 5.17: Fold bifurcation.

the left of which is stable, while the right one is unstable. For $\alpha > 0$ there are no fixed points in the system. While α crosses zero from negative to positive values, the two fixed points (stable and unstable) "collide", forming at $\alpha = 0$ a fixed point with $\mu = 1$, and disappear. This is a fold bifurcation in the discrete-time dynamical system. ◇

Theorem 5.22 *The smooth map*

$$y \mapsto \alpha + y + y^2 + O(y^3)$$

is locally topologically equivalent near the origin to the map

$$x \mapsto \alpha + x + x^2.$$

Proof: Write the first map as

$$y \mapsto g_{(\alpha)}(y) = \alpha + y + y^2 + y^3\varphi(y, \alpha). \tag{5.49}$$

where φ is a smooth function of (y, α). The number and stability of the fixed points of (5.49) in a neighbourhood of $y = 0$ for small $|\alpha|$ is the same as in the normal form (5.48). This can be proved exactly as in the continuous-time fold case (Theorem 5.6).

The construction of a conjugating homeomorphism is more involved, since it has to map not only fixed points of (5.49) onto fixed points of (5.48) but map all orbits onto orbits. Note that both $f_{(\alpha)}$ and $g_{(\alpha)}$ are invertible near the origin. We consider separately the cases $\alpha > 0,\ \alpha = 0$, and $\alpha < 0$.

Case $\alpha > 0$. Define a fundamental domain for $f_{(\alpha)}$:

$$D_f = [0, f_{(\alpha)}(0)] = [0, \alpha],$$

and the corresponding domain for $g_{(\alpha)}$:

$$D_g = [0, g_{(\alpha)}(0)] = [0, \alpha].$$

(See Figure 5.18.) Let the homeomorphism $h : \mathbb{R} \to \mathbb{R}$ under construction be $y = x$

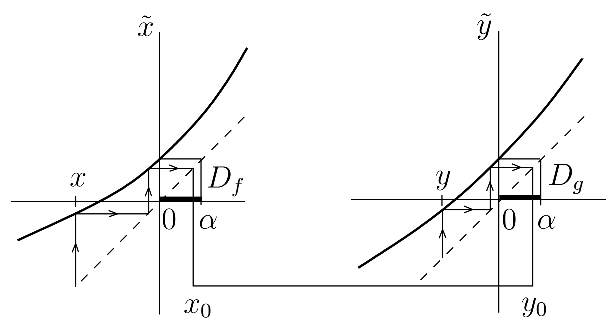

Figure 5.18: Conjugating homeomorphism for $\alpha > 0$.

for $x \in D_f$. This identifies D_g and D_f.

Any point $x \notin D_f$ from a small neighbourhood of $x = 0$ falls after a finite number of iterations (forward or backward) with $f_{(\alpha)}$ into D_f, provided $\alpha > 0$ is small. Let $K = K(x) \neq 0$ be the integer with the minimal $|K|$ such that

$$x_0 = f_{(\alpha)}^{K(x)}(x) \in D_f.$$

Take $y_0 = x_0$ and then set

$$y = g_{(\alpha)}^{-K(x)}(y_0).$$

The (α-dependent) map $h : x \mapsto y$ is a local homeomorphism defined for small $\alpha > 0$. Moreover, it maps orbits of $f_{(\alpha)}$ onto orbits of $g_{(\alpha)}$ in a non-shrinking neighbourhood of the origin for all small $\alpha > 0$.

Case $\alpha = 0$. In this case we fix a small $\varepsilon > 0$ and introduce two fundamental domains for $f_{(0)}$:

$$D_f^{\pm} = [\pm\varepsilon, f_{(0)}(\pm\varepsilon)]$$

and two similar domains for $g_{(0)}$:

$$D_g^{\pm} = [\pm\varepsilon, g_{(0)}(\pm\varepsilon)].$$

(See Figure 5.19.) Next we identify these domains pairwise by linear (orientation

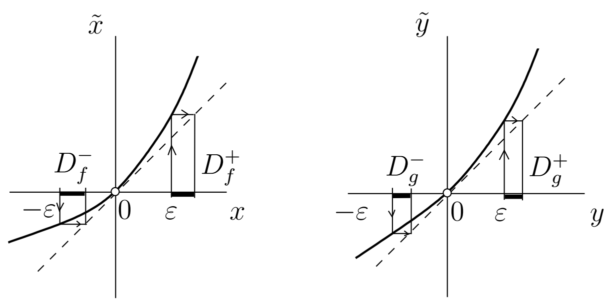

Figure 5.19: Fundamental domains for $\alpha = 0$.

preserving) maps defined for small $x, y > 0$ and $x, y < 0$, respectively. This gives a homeomorphism $h : D_f^{\pm} \to D_g^{\pm}$.

For $x \notin D_f^{\pm}$ with small $|x| \neq 0$, let $K^{\pm} = K^{\pm}(x)$ be an integer with the minimal $|K|$ such that

$$x_0 = f_{(\alpha)}^{K^{\pm}(x)}(x) \in D_f^{\pm}.$$

Set $y_0 = h(x_0)$, where h is the affine map mentioned above. Then take

$$y = g_{(\alpha)}^{-K^{\pm}(x)}(y_0).$$

Define $h(x) = y$ for $x \neq 0$. Set $h(0) = 0$ by continuity. The map h is defined in a small neighbourhood of $x = 0$ and maps orbits of f_0 onto orbits of g_0 in a small neighbourhood of $y = 0$.

Case $\alpha < 0$. Denote by $x_{1,2}(\alpha) = \mp\sqrt{-\alpha}$ the fixed points of $f_{(\alpha)}$ and by $y_{1,2}(\alpha)$ the corresponding fixed points of $g_{(\alpha)}$. Similarly to the case $\alpha > 0$, define and identify the fundamental domains

$$D_f = [f_{(\alpha)}(0), 0] = [\alpha, 0]$$

and

$$D_g = [g_{(\alpha)}(0), 0] = [\alpha, 0].$$

(See Figure 5.20). For any $x \notin D_f$ and such that $x \in (x_1(\alpha), x_2(\alpha))$, define an $y \in (y_1(\alpha), y_2(\alpha))$ as in the case $\alpha > 0$. Extend the map thus obtained by continuity to a map $h : [x_1(\alpha), x_2(\alpha)] \to [y_1(\alpha), y_2(\alpha)]$.

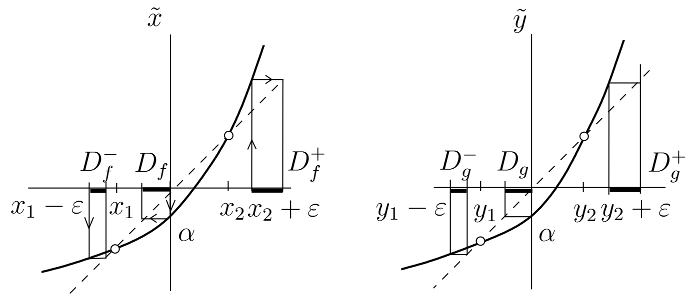

Figure 5.20: Fundamental domains for $\alpha < 0$.

Finally, let

$$D_f^{\mp} = [x_{1,2}(\alpha) \mp \varepsilon, f_{(\alpha)}(x_{1,2}(\alpha) + \varepsilon)],$$

and

$$D_g^{\mp} = [y_{1,2}(\alpha) \mp \varepsilon, g_{(\alpha)}(y_{1,2}(\alpha) + \varepsilon)],$$

where $\varepsilon > 0$ is sufficiently small (see Figure 5.20). Take for the map $h : D_f^{\mp} \to D_g^{\mp}$ a linear (orientation preserving) transformation of $D_f^{\mp}$ onto $D_g^{\mp}$. Then, repeat the standard procedure to complete the construction of a local homeomorphism. $\square$

Theorem 5.23 *Suppose that a one-dimensional map*

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}, \ \alpha \in \mathbb{R}, \tag{5.50}$$

where f is smooth, has at $\alpha = 0$ the fixed point $x_0 = 0$, and let $\mu = f_x(0,0) = 1$. Assume that the following conditions are satisfied:

(A.1) $f_{xx}(0,0) \neq 0$;

(A.2) $f_\alpha(0,0) \neq 0$.

Then there are smooth invertible coordinate and parameter changes transforming the map into

$$y \mapsto \beta + y \pm y^2 + O(y^3).$$

Proof: Expand $f(x, \alpha)$ in a Taylor series with respect to x at $x = 0$:

$$f(x, \alpha) = f_0(\alpha) + f_1(\alpha)x + f_2(\alpha)x^2 + O(x^3).$$

Two conditions are satisfied: $f_0(0) = f(0,0) = 0$ (*fixed-point condition*) and $f_1(0) = f_x(0,0) = 1$ (*fold bifurcation condition*). Since $f_1(0) = 1$, we may write

$$f(x, \alpha) = f_0(\alpha) + [1 + g(\alpha)]x + f_2(\alpha)x^2 + O(x^3),$$

where g is smooth and $g(0) = 0$.

Perform a coordinate shift by introducing a new variable ξ:

$$x = \xi + \delta, \tag{5.51}$$

where $\delta = \delta(\alpha)$ is to be chosen suitably later on. The transformation (5.51) yields

$$\tilde{\xi} = \tilde{x} - \delta = f(x, \alpha) - \delta = f(\xi + \delta, \alpha) - \delta.$$

Therefore,

$$\begin{aligned}\tilde{\xi} \;&=\; [f_0(\alpha) + g(\alpha)\delta + f_2(\alpha)\delta^2 + O(\delta^3)] \\ &\quad + \xi + [g(\alpha) + 2f_2(\alpha)\delta + O(\delta^2)]\xi \\ &\quad + [f_2(\alpha) + O(\delta)]\xi^2 + O(\xi^3).\end{aligned}$$

According to (A.1),

$$f_2(0) = \tfrac{1}{2} f_{xx}(0, 0) \neq 0.$$

Hence there is a smooth function $\delta(\alpha)$, which annihilates the parameter-dependent linear term in the above map for all sufficiently small $|\alpha|$. Indeed, the condition for that term to vanish can be written as

$$F(\alpha, \delta) = g(\alpha) + 2f_2(\alpha)\delta + \delta^2\varphi(\alpha, \delta) = 0$$

for some smooth function φ. We have

$$F(0, 0) = 0, \;\; F_\delta(0, 0) = 2f_2(0) \neq 0,$$

which implies the (local) existence and uniqueness of a smooth function $\delta = \delta(\alpha)$ such that $\delta(0) = 0$ and $F(\alpha, \delta(\alpha)) \equiv 0$ (use the Implicit Function Theorem). It follows that

$$\delta(\alpha) = -\frac{g'(0)}{2f_2(0)}\alpha + O(\alpha^2).$$

The map written in terms of ξ is now given by

$$\tilde{\xi} \;=\; \left[f_0'(0)\alpha + \alpha^2\psi(\alpha)\right] \;+\; \xi \;+\; [f_2(0) + O(\alpha)]\,\xi^2 \;+\; O(\xi^3), \qquad (5.52)$$

where ψ is some smooth function.

Consider as a new parameter $\mu = \mu(\alpha)$ the constant (ξ-independent) term of (5.52):

$$\mu = f_0'(0)\alpha + \alpha^2\psi(\alpha).$$

We have

(a) $\mu(0) = 0$;
(b) $\mu'(0) = f_0'(0) = f_\alpha(0, 0)$.

Since

$$f_\alpha(0, 0) \neq 0$$

due to (A.2), the Inverse Function Theorem implies the local existence and uniqueness of a smooth inverse function $\alpha = \alpha(\mu)$ with $\alpha(0) = 0$. Therefore, equation (5.52) now reads

$$\tilde{\xi} = \mu + \xi + a(\mu)\xi^2 + O(\xi^3),$$

where $a(\mu)$ is a smooth function with $a(0) = f_2(0) \neq 0$ due to the first assumption (A.1).

Let $y = |a(\mu)|\xi$ and $\beta = |a(\mu)|\mu$. Then we get

$$\tilde{y} = \beta + y + sy^2 + O(y^3),$$

where $s = \text{sign } a(0) = \pm 1$. □

Example 5.24 (Fold bifurcation in a simple population model)

Consider the following simple population model

$$x(k+1) = \frac{\alpha x^2(k)}{1 + x^2(k)},$$

where $x(k)$ is the population density in year k, and $\alpha > 0$ is a parameter determining both the growth rate for small population size and the maximum population level. This recurrence relation corresponds to iterating the map

$$x \mapsto f(x, \alpha) = \frac{\alpha x^2}{1 + x^2} \tag{5.53}$$

which has a trivial fixed point $x = 0$ for all values of the parameter α. At $\alpha_0 = 2$, however, two nontrivial positive fixed points

$$x_{1,2}(\alpha) = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

appear at $x = 1$ via a fold bifurcation. We leave it to the reader to check the genericity conditions (A.1) and (A.2) of Theorem 5.23. ◇

5.3.2 Flip bifurcation of scalar maps

Example 5.25 (Normal form of the flip bifurcation)

Consider the one-parameter family of scalar dynamical system generated by the following map:

$$x \mapsto f_{(\alpha)}(x) = -(1 + \alpha)x + x^3. \tag{5.54}$$

Let $f(x, \alpha) = -(1+\alpha)x + x^3$. For small $|\alpha|$, the map $f_{(\alpha)}$ is invertible in a neighbourhood of the origin. System (5.54) has for all α the fixed point $x_0 = 0$ with multiplier $\mu = -(1+\alpha)$. The point is linearly stable for small $\alpha < 0$ and is linearly unstable for $\alpha > 0$. At $\alpha = 0$ the point is not hyperbolic, since the multiplier $\mu = f_x(0, 0) = -1$, but is nevertheless (nonlinearly) stable. There are no other fixed points near the origin for small $|\alpha|$.

Consider now the *second iterate* $f^2_{(\alpha)}(x)$ of the map (5.54):

$$f^2_{(\alpha)}(x) = -(1 + \alpha)[-(1 + \alpha)x + x^3] + [-(1 + \alpha)x + x^3]^3.$$

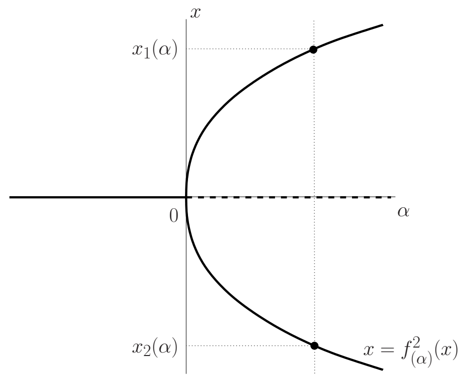

Figure 5.21: Fixed points of f^2_α near a flip bifurcation.

The map $f^2_{(\alpha)}$ obviously has the trivial fixed point $x_0 = 0$. It also has *two* nontrivial fixed points for small $\alpha > 0$:

$$x_{1,2} = f^2_{(\alpha)}(x_{1,2}),$$

where $x_{1,2} = \pm\sqrt{\alpha}$ (see Figure 5.21). These two points are stable and constitute a *cycle of period two* for the original map $f_{(\alpha)}$, since

$$x_2 = f_{(\alpha)}(x_1),\ x_1 = f_{(\alpha)}(x_2),$$

with $x_1 \neq x_2$. Figure 5.22 shows the complete bifurcation diagram of the map (5.54)

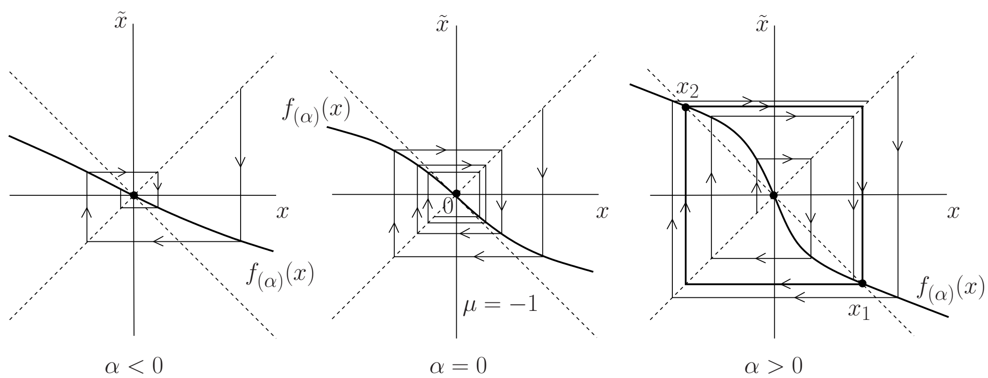

Figure 5.22: Supercritical flip bifurcation.

with the help of a staircase diagram. As α approaches zero from above, the period-two cycle "shrinks" and disappears. This is a *supercritical flip* (or *period-doubling*) *bifurcation.*

The map $x \mapsto -(1+\alpha)x - x^3$ exhibits a *subcritical flip bifurcation*, as one can easily verify by a similar analysis. ◇

Theorem 5.26 *The smooth map*

$$y \mapsto -(1+\alpha)y + y^3 + O(y^4)$$

is locally topologically equivalent near the origin to the map

$$x \mapsto -(1+\alpha)x + x^3. \ \square$$

Proof:

Step 1 (*Fixed point analysis*) Write the first map as

$$y \mapsto g_{(\alpha)}(y) = -(1+\alpha)y + y^3 + y^4\psi(y,\alpha), \tag{5.55}$$

where ψ is a smooth function. Its second iterate has the form

$$g^2_{(\alpha)}(y) = y + (2\alpha + \alpha^2)y - (1+\alpha)(2 + 2\alpha + \alpha^2)y^3 + y^4\Psi(y,\alpha)$$

for some smooth function Ψ. It has a trivial fixed point $y_0 = 0$, as well as two nontrivial fixed points satisfying the equation

$$\Phi(\alpha, y) = 2\alpha + \alpha^2 - (1+\alpha)(2 + 2\alpha + \alpha^2)y^2 + y^3\Psi(y,\alpha) = 0. \tag{5.56}$$

Since $\Phi(0,0) = 0$ and $\Phi_\alpha(0,0) = 2 \neq 0$, the Implicit Function Theorem gives the existence of a smooth solution to (5.56):

$$\alpha = \Lambda(y), \quad \Lambda(0) = 0.$$

Moreover, $\Lambda(y) = y^2 + O(y^3)$, and therefore g^2_α has two fixed points

$$y_{1,2}(\alpha) = \mp\sqrt{\alpha} + O(\alpha) = x_{1,2}(\alpha) + O(\alpha),$$

for small $\alpha > 0$, corresponding to a period-2 cycle of $g_{(\alpha)}$. Thus, the number and (is easy to check) stability of the fixed points and period-2 cycles is the same for the map (5.55) and the normal form (5.54).

Step 2 (*Construction of the homeomorphism*) We use the same method of fundamental domains as in the fold case. Namely, for any sufficiently small $|\alpha|$, we construct two domains D_f and D_g, such that any generic point x in a neighbourhood of the origin $x = 0$ is mapped by $f^K_{(\alpha)}$ with some integer $K = K(x)$ into D_f. Both D_f and D_g will be the union of two closed intervals. Then we can identify D_f and D_g with a (piecewise-linear) homeomorphism h, and set as usual

$$y = g^{-K}_{(\alpha)}(h(f^K_{(\alpha)}(x)).$$

This gives a parameter-dependent map $x \mapsto y$ that can be extended by continuity to the entire neighbourhood of the origin. This map is a local homeomorphism mapping orbits of $f_{(\alpha)}$ onto orbits of $g_{(\alpha)}$.

For $\alpha \leq 0$ with small $|\alpha|$, take a sufficiently small $\varepsilon > 0$ and set

$$D_f = [f_{(\alpha)}(-\varepsilon), \varepsilon] \cup [-\varepsilon, f_{(\alpha)}(\varepsilon)]$$

and

$$D_g = [g_{(\alpha)}(-\varepsilon), \varepsilon] \cup [-\varepsilon, g_{(\alpha)}(\varepsilon)].$$

Now consider the case $\alpha > 0$. First, we perform parameter-dependent scalings of the half-axes $y > 0$ and $y < 0$ in (5.55) such that $y_1(\alpha) = -y_2(\alpha)$ for all sufficiently small $\alpha > 0$. When $x \in (x_1(\alpha), x_2(\alpha))$, set

$$D_f = [\alpha, f_{(\alpha)}(-\alpha)] \cup [f_{(\alpha)}(\alpha), -\alpha], \quad D_g = [\alpha, g_{(\alpha)}(-\alpha)] \cup [g_{(\alpha)}(\alpha), -\alpha].$$

When $x \notin [x_1(\alpha), x_2(\alpha)]$ take a sufficiently small $\varepsilon > 0$ and set

$$\begin{aligned} D_f &= [f_{(\alpha)}(x_1-\varepsilon), x_2+\varepsilon] \cup [x_1-\varepsilon, f_{(\alpha)}(x_2+\varepsilon)], \\ D_g &= [g_{(\alpha)}(y_1-\varepsilon), y_2+\varepsilon] \cup [y_1-\varepsilon, g_{(\alpha)}(y_2+\varepsilon)]. \end{aligned}$$

□

Theorem 5.27 *Suppose that a one-dimensional smooth map*

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}, \ \alpha \in \mathbb{R},$$

has at $\alpha = 0$ the fixed point $x_0 = 0$. Assume that $\mu = f_x(0,0) = -1$. Let that the following generic conditions be satisfied:

(B.1) $\frac{1}{2}[f_{xx}(0,0)]^2 + \frac{1}{3}f_{xxx}(0,0) \neq 0$;
(B.2) $f_{x\alpha}(0,0) + \frac{1}{2}f_\alpha(0,0)f_{xx}(0,0) \neq 0$.

Then there are smooth invertible coordinate and parameter changes transforming the family of maps into

$$y \mapsto -(1+\beta)y \pm y^3 + O(y^4).$$

Proof: Since $f_x(0,0) \neq 1$, the Implicit Function Theorem guarantees that the map has a unique fixed point $x_0(\alpha)$ in some neighbourhood of the origin for all sufficiently small $|\alpha|$. Moreover,

$$x_0(0) = 0, \quad x_0'(0) = \frac{f_\alpha(0,0)}{1 - f_x(0,0)} = \frac{1}{2}f_\alpha(0,0).$$

We can perform a parameter-dependent coordinate shift,

$$x = x_0(\alpha) + \xi,$$

placing the fixed point at $\xi = 0$ for $|\alpha|$ sufficiently small.

The map can then be expressed as

$$\xi \mapsto \tilde{\xi} = g(\xi,\alpha),$$

where $g(\xi,\alpha) = f(x_0(\alpha)+\xi,\alpha) - x_0(\alpha)$ so that

$$g(\xi,\alpha) = g_1(\alpha)\xi + g_2(\alpha)\xi^2 + g_3(\alpha)\xi^3 + O(\xi^4), \tag{5.57}$$

with smooth functions $g_k, k = 1,2,3$ given by

$$g_1(\alpha) = f_x(x_0(\alpha),\alpha), \quad g_2(\alpha) = \tfrac{1}{2}f_{xx}(x_0(\alpha),\alpha), \quad g_3(\alpha) = \tfrac{1}{6}f_{xxx}(x_0(\alpha),\alpha).$$

The function g_1 can be represented as $g_1(\alpha) = -[1+g(\alpha)]$ for some smooth function g. Since $g(0) = 0$ and

$$g'(0) = -f_{xx}(0,0)x_0'(0) - f_{x\alpha}(0,0) = -\left[f_{x\alpha}(0,0) + \tfrac{1}{2}f_\alpha(0,0)f_{xx}(0,0)\right] \neq 0$$

according to assumption (B.2), the function g is locally invertible and can be used to introduce a new parameter:

$$\beta = g(\alpha).$$

Our map (5.57) now takes the form

$$\tilde{\xi} = \mu(\beta)\xi + a(\beta)\xi^2 + b(\beta)\xi^3 + O(\xi^4), \tag{5.58}$$

where $\mu(\beta) = -(1+\beta)$, and the functions $a(\beta)$ and $b(\beta)$ are smooth. We have

$$a(0) = g_2(0) = \tfrac{1}{2} f_{xx}(0,0), \quad b(0) = g_3(0) = \tfrac{1}{6} f_{xxx}(0,0).$$

Let us perform a polynomial change of coordinate:

$$\xi = \eta + \delta\eta^2, \tag{5.59}$$

where $\delta = \delta(\beta)$ is a smooth function to be defined. The transformation (5.59) is close to the identity in some neighbourhood of the origin. Written in the new coordinate η, the map (5.58) takes the form

$$\tilde{\eta} = \mu(\beta)\eta + d(\beta)\eta^2 + c(\beta)\eta^3 + O(\eta^4), \tag{5.60}$$

where d and c are smooth functions of β. Equations (5.59) and (5.60) imply

$$\begin{aligned} \tilde{\xi} &= \tilde{\eta} + \delta\tilde{\eta}^2 = \mu\eta + d\eta^2 + c\eta^3 + \delta(\mu\eta + d\eta^2 + \cdots)^2 + \cdots \\ &= \mu\eta + (d + \delta\mu^2)\eta^2 + (c + 2\delta\mu d)\eta^3 + \cdots \end{aligned}$$

Substituting (5.59) into (5.58), we get

$$\begin{aligned} \tilde{\xi} &= \mu\xi + a\xi^2 + b\xi^3 + \cdots \\ &= \mu(\eta + \delta\eta^2) + a(\eta + \delta\eta^2)^2 + b(\eta + \cdots)^3 + \cdots \\ &= \mu\eta + (a + \delta\mu)\eta^2 + (b + 2a\delta)\eta^3 + \cdots \end{aligned}$$

Therefore

$$d(\beta) = a(\beta) + \delta(\beta)\mu(\beta) - \delta(\beta)\mu^2(\beta), \quad c(\beta) = b(\beta) + 2a(\beta)\delta(\beta) - 2\delta(\beta)\mu(\beta)d(\beta).$$

Thus, the quadratic term in (5.60) can be "killed" for all sufficiently small $|\beta|$ by setting

$$\delta(\beta) = \frac{a(\beta)}{\mu^2(\beta) - \mu(\beta)},$$

since $\mu^2(0) - \mu(0) = 2 \neq 0$. With $\delta(\beta)$ thus selected, $d = 0$ and the cubic coefficient in (5.60) becomes

$$c(\beta) = b(\beta) + 2a(\beta)\delta(\beta) = b(\beta) + \frac{2a^2(\beta)}{\mu^2(\beta) - \mu(\beta)},$$

so that the map (5.60) takes the form

$$\tilde{\eta} = -(1+\beta)\eta + c(\beta)\eta^3 + O(\eta^4).$$

Here the function $c = c(\beta)$ is smooth and

$$c(0) = a^2(0) + b(0) = \tfrac{1}{4}[f_{xx}(0,0)]^2 + \tfrac{1}{6}f_{xxx}(0,0). \tag{5.61}$$

Notice that $c(0) \neq 0$ by assumption (B.1).

Apply the rescaling

$$\eta = \frac{y}{\sqrt{|c(\beta)|}}.$$

In the y-coordinate the map takes the desired form:

$$\tilde{y} = -(1+\beta)y + sy^3 + O(y^4),$$

where $s = \text{sign } c(0) = \pm 1$. $\square$

Example 5.28 (Flip bifurcation in Ricker's map)

Consider the following simple population model due to the fishery biologist Ricker:

$$x(k+1) = \alpha x(k)\mathrm{e}^{-x(k)}.$$

Here $x(k)$ is the population density in year k, and $\alpha > 0$ is the per capita growth rate at very low density. The factor e^{-x} on the right-hand side takes into account the negative effect of competition at high population densities (originally interpreted by Ricker in terms of cannibalism). The above recurrence relation corresponds to the map

$$x \mapsto f(x,\alpha) = \alpha x \mathrm{e}^{-x}. \tag{5.62}$$

This map has the trivial fixed point $x_0 = 0$ for all values of the parameter α. At $\alpha_0 = 1$, however, a nontrivial positive fixed point

$$x_1(\alpha) = \ln \alpha$$

appears via a transcritical bifurcation. The eigenvalue of this point is given by the expression

$$\mu(\alpha) = 1 - \ln \alpha.$$

Thus, x_1 is stable for $1 < \alpha < \alpha_1$ and unstable for $\alpha > \alpha_1$, where $\alpha_1 = \mathrm{e}^2 = 7.38907\ldots$. At the critical parameter value $\alpha = \alpha_1$, the fixed point has multiplier $\mu(\alpha_1) = -1$. Therefore, a flip bifurcation takes place. To apply Theorem 5.27, we need to verify the genericity conditions (B.1) and (B.2) in which all the derivatives must be computed at the fixed point $x_1(\alpha_1) = 2$ and at the critical parameter value $\alpha_1 = \mathrm{e}^2$.

One can check that

$$\tfrac{1}{4}[f_{xx}(2,\mathrm{e}^2)]^2 + \tfrac{1}{6}f_{xxx}(2,\mathrm{e}^2) = \tfrac{1}{6} > 0$$

and

$$f_{x\alpha}(2,\mathrm{e}^2) + \tfrac{1}{2}f_\alpha(2,\mathrm{e}^2)f_{xx}(2,\mathrm{e}^2) = -\tfrac{1}{\mathrm{e}^2} < 0.$$

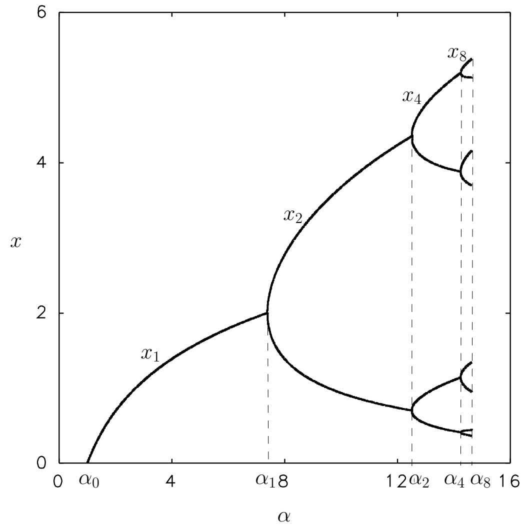

Figure 5.23: Period-doubling (flip) bifurcations in Ricker's map.

Thus, Theorem 5.26 implies that a unique and stable period-two cycle bifurcates from x_1 for $\alpha > \alpha_1$.

The fate of this period-two cycle can be traced further. It can be verified numerically that this cycle loses stability at $\alpha_2 = 12.50925\ldots$ via a supercritical flip bifurcation, giving rise to a stable period-four cycle. This period-four cycle bifurcates again at $\alpha_4 = 14.24425\ldots$, generating a stable period-eight cycle that loses its stability at $\alpha_8 = 14.65267\ldots$. The next period doubling takes place at $\alpha_{16} = 14.74212\ldots$ (see Figure 5.23, where several doublings are presented).

In view of Sharkovsky's Theorem (see Chapter 7), it is natural to expect that there is an *infinite* sequence of bifurcation values: $\alpha_{m(k)},\ m(k) = 2^k,\ k = 1, 2, \ldots$ ($m(k)$ is the period of the cycle before the kth doubling). Moreover, one can check that at least the first few elements of this sequence closely resemble a *geometric progression*. In fact, the quotient

$$\frac{\alpha_{m(k)} - \alpha_{m(k-1)}}{\alpha_{m(k+1)} - \alpha_{m(k)}}$$

tends to $\mu_F = 4.6692\ldots$ as k increases. This phenomenon is called *Feigenbaum's cascade* of period doublings, and the constant μ_F is referred to as the *Feigenbaum constant*. The most surprising fact is that this constant is the same for many different systems exhibiting a cascade of flip bifurcations. This universality has a deep underlying structure, which we will also discuss in Chapter 7. $\diamond$

Remark: The critical normal form coefficient $c(0)$ can be expressed in terms of the second iterate of the map $x \mapsto f_{(\alpha)}(x) = f(x, \alpha)$ by the formula

$$c(0) = -\frac{1}{12}\frac{\partial^3}{\partial x^3} f(f(x,\alpha),\alpha)\bigg|_{(x,\alpha)=(x_0,\alpha_0)},$$

where x_0 is the critical fixed point at the bifurcation parameter value α_0.

5.3.3 Planar Neimark-Sacker bifurcation

In continuous time, we can scale the time variable. In the context of Andronov-Hopf bifurcation, this allowed us to normalize the frequency ω to 1. In discrete time, this is impossible and the position on the unit circle where an eigenvalue crosses the unit circle naturally becomes an extra parameter in the normal form. Moreover, nonlinear terms contribute in a more complicated manner to the dynamics. As a result, the Neimark-Sacker bifurcation is essentially more involved than Andronov-Hopf bifurcation.

Example 5.29 (A model map for the Neimark-Sacker bifurcation)

Consider the two-dimensional discrete-time system generated by the following map depending on the parameter α and its smooth functions $\theta(\alpha), a(\alpha)$, and $b(\alpha)$:

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\mapsto (1+\alpha)\begin{pmatrix} \cos\theta(\alpha) & -\sin\theta(\alpha) \\ \sin\theta(\alpha) & \cos\theta(\alpha) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &+ (x_1^2+x_2^2)\begin{pmatrix} \cos\theta(\alpha) & -\sin\theta(\alpha) \\ \sin\theta(\alpha) & \cos\theta(\alpha) \end{pmatrix}\begin{pmatrix} a(\alpha) & -b(\alpha) \\ b(\alpha) & a(\alpha) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\end{aligned} \tag{5.63}$$

where $a(0) \neq 0$ and $0 < \theta(0) < \pi$.

This system has the fixed point $x_1 = x_2 = 0$ for all α with Jacobian matrix

$$A = (1+\alpha)\begin{pmatrix} \cos\theta(\alpha) & -\sin\theta(\alpha) \\ \sin\theta(\alpha) & \cos\theta(\alpha) \end{pmatrix}.$$

The matrix has eigenvalues $\mu_{1,2} = (1+\alpha)e^{\pm i\theta(\alpha)}$, which makes the map (5.63) invertible near the origin for all small $|\alpha|$. As can be seen, the fixed point at the origin is nonhyperbolic at $\alpha = 0$ due to a complex-conjugate pair of eigenvalues on the unit circle. To analyze the corresponding bifurcation, introduce the complex variable $z = x_1 + ix_2, \bar{z} = x_1 - ix_2, |z|^2 = z\bar{z} = x_1^2 + x_2^2$. The map (5.63) in the z-coordinate reads as

$$z \mapsto e^{i\theta(\alpha)} z(1+\alpha+d(\alpha)|z|^2), \tag{5.64}$$

where $d(\alpha) = a(\alpha) + ib(\alpha)$ is a smooth complex function of the parameter α.

Using the representation $z = \rho e^{i\varphi}$, we obtain for $\rho = |z|$

$$\rho \mapsto \rho|1+\alpha+d(\alpha)\rho^2|.$$

Since

$$\left|1+\alpha+d(\alpha)\rho^2\right| = \sqrt{(1+\alpha+a(\alpha)\rho^2)^2 + b^2(\alpha)\rho^4} = 1+\alpha+a(\alpha)\rho^2 + O(\rho^4)$$

and

$$\arg(1+\alpha+d(\alpha)\rho^2) = \arctan\left(\frac{b(\alpha)\rho^2}{1+\alpha+a(\alpha)\rho^2}\right) = \frac{b(\alpha)}{1+\alpha}\rho^2 + O(\rho^4),$$

we arrive at the following *polar* form of map (5.63):

$$\begin{cases} \rho & \mapsto & \rho(1+\alpha+a(\alpha)\rho^2)+\rho^4 R(\rho,\alpha), \\ \varphi & \mapsto & \varphi+\theta(\alpha)+\rho^2 Q(\rho,\alpha), \end{cases} \tag{5.65}$$

for functions R and Q, which are smooth functions of (ρ, α) near $(\rho, \alpha) = (0, 0)$.[4] Bifurcations of the phase portrait of (5.63) as α passes through zero can easily be analyzed using the latter form, since the mapping for ρ is *independent* of φ. The first equation in (5.65) defines a one-dimensional dynamical system that has the fixed point $\rho = 0$ for all values of α. The point is linearly stable if $\alpha < 0$; for $\alpha > 0$ the point becomes linearly unstable. The stability of the fixed point at $\alpha = 0$ is determined by the sign of the coefficient $a(0)$. Suppose that $a(0) < 0$; then the origin is (nonlinearly) stable at $\alpha = 0$. Moreover, the ρ-map of (5.65) has one additional stable fixed point

$$\rho_0(\alpha) = \sqrt{-\frac{\alpha}{a(\alpha)}} + O(\alpha)$$

for small $\alpha > 0$. The φ-map of (5.65) describes a rotation by an angle depending on ρ and α; it is approximately equal to $\theta(\alpha)$. Thus, by superposition of the mappings defined by (5.65), we obtain the bifurcation diagram for the original two-dimensional system (5.63) (see Figure 5.24).

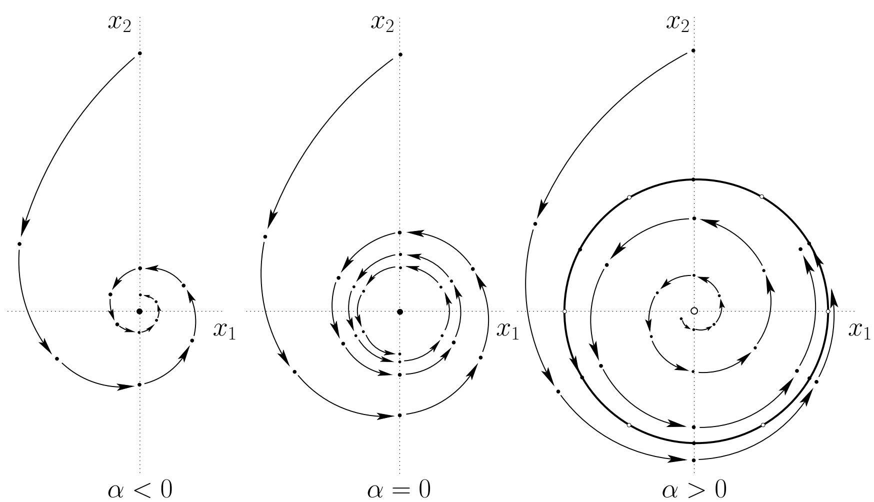

Figure 5.24: Supercritical Neimark-Sacker bifurcation.

The system always has a fixed point at the origin. This point is stable for $\alpha < 0$ and unstable for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the point is nonlinearly stable. The fixed point is surrounded for small $\alpha > 0$ by an isolated *closed invariant curve* that is unique and stable. The curve is a circle of radius $\rho_0(\alpha)$. All orbits starting outside or inside the closed invariant curve, but not at the origin, tend to the curve under iterations of (5.65). This is a *supercritical* Neimark-Sacker bifurcation.

[4]Although $R(\rho, \alpha) = O(\rho)$ for the map (5.63), we factor out only ρ^4, to prepare for a more general case ahead.

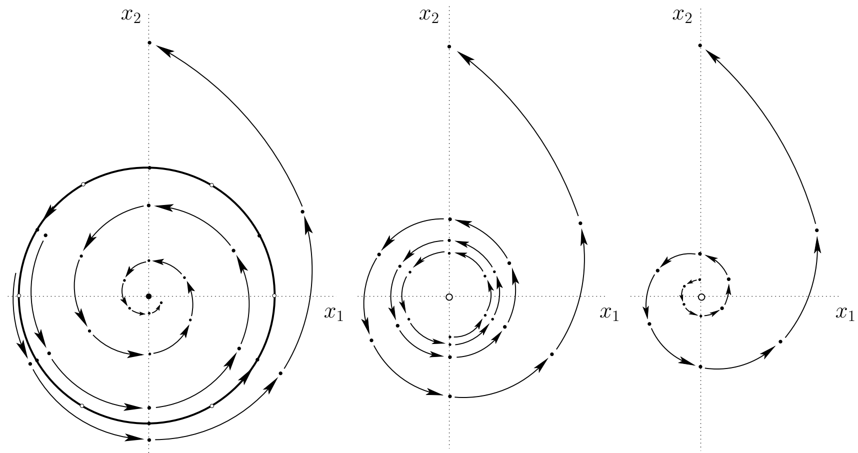

Figure 5.25: Subcritical Neimark-Sacker bifurcation.

When $a(0) > 0$, we get a *subcritical* Neimark-Sacker bifurcation, where an unstable closed invariant curve exists for $\alpha < 0$ with small $|\alpha|$ (see Figure 5.25).

The structure of orbits of (5.63) on the invariant circle depends on whether the ratio between the rotation angle and 2π is rational or irrational. Define

$$\theta_0(\alpha) = \frac{\theta(\alpha) + \rho_0^2(\alpha) Q(\rho_0(\alpha), \alpha)}{2\pi}.$$

If $\theta_0(\alpha)$ is rational, i.e. $\theta_0(\alpha) = \frac{n}{m}$ with $\mathrm{dcd}(n, m) = 1$, then every point on the circle is m-periodic and after m steps it winds n times around the origin. In this case, all orbits on the invariant curve are periodic. If $\theta_0(\alpha)$ is irrational, there are no periodic orbits and all orbits are dense in the circle. $\diamond$

Theorem 5.30 *Consider the map*

$$z \mapsto \tilde{z} = g(z, \bar{z}, \alpha) = \mathrm{e}^{i\theta(\alpha)} z(1 + \alpha + d(\alpha)|z|^2) + G(z, \bar{z}, \alpha), \tag{5.66}$$

where $z = x_1 + ix_2$, $d(\alpha) = a(\alpha) + ib(\alpha)$; $a(\alpha)$, $b(\alpha)$, *and* $\theta(\alpha)$ *are smooth real-valued functions;* $a(0) < 0, 0 < \theta(0) < \pi$, *and* G *is a smooth complex-valued function of* $(z, \bar{z})$ *and* α *such that* $G = O(|z|^4)$ *for small* z.

The real planar map $x \mapsto \tilde{x} = f(x, \alpha)$ *corresponding to the map* (5.66) *has in a neigbourhood of the origin a unique stable closed invariant curve for sufficiently small* $\alpha > 0$. *Moreover, any nontrivial orbit starting in this neigbourhood converges to the closed invariant curve under iteration of this map.*

Proof:
Step 1 (*Construct an attracting annulus*). The point with polar coordinates (ρ, φ) is mapped by (5.66) to the point with polar coordinates $(\tilde{\rho}, \tilde{\varphi})$ with

$$\begin{cases} \tilde{\rho} &= \rho(1 + \alpha + a(\alpha)\rho^2) + \rho^4 R(\rho, \varphi, \alpha), \\ \tilde{\varphi} &= \varphi + \theta(\alpha) + \rho^2 Q(\rho, \varphi, \alpha), \end{cases} \tag{5.67}$$

where R and Q are smooth functions of (ρ, φ, α) near $(\rho, \alpha) = (0, 0)$ and

$$Q(\rho, \varphi, \alpha) = \frac{b(\alpha)}{1+\alpha} + O(\rho^2)$$

(cf. (5.65) but note the difference: now R and Q depend also on φ and, in particular, the ρ-equation is not decoupled from the φ-equation).

Let $1 > \varepsilon > 0$. Introduce an annulus $A_{\alpha,\varepsilon}$ in the plane by the formula

$$A_{\alpha,\varepsilon} = \left\{ (\rho, \varphi) : \sqrt{-\frac{\alpha}{a(\alpha)}}(1-\varepsilon) \le \rho \le \sqrt{-\frac{\alpha}{a(\alpha)}}(1+\varepsilon),\ \varphi \in [0, 2\pi] \right\}.$$

Consider a point (ρ, φ) and its image $(\tilde{\rho}, \tilde{\varphi})$ under the map (5.67). Then

$$\Delta\rho = \tilde{\rho} - \rho = \rho(\alpha + a(\alpha)\rho^2 + \rho^3 R(\rho, \varphi, \alpha))$$

and (recall that $a(\alpha) < 0$)

$$\Delta\rho \ge \rho(\alpha(2\varepsilon - \varepsilon^2) + O(\alpha^{3/2})) \quad \text{if} \quad 0 \le \rho \le \sqrt{-\frac{\alpha}{a(\alpha)}}(1-\varepsilon),$$

$$\Delta\rho \le \rho(\alpha(-2\varepsilon - \varepsilon^2) + O(\alpha^{3/2})) \quad \text{if} \quad \varepsilon_0 \ge \rho \ge \sqrt{-\frac{\alpha}{a(\alpha)}}(1+\varepsilon),$$

where the $O(\alpha^{3/2})$-terms are smooth functions of ε and $\varepsilon_0 > 0$ is a constant independent of α. Now take

$$\varepsilon = \alpha^{1/4}. \tag{5.68}$$

The above inequalities then imply

$$\Delta\rho > 0 \quad \text{if} \quad 0 < \rho \le \sqrt{-\frac{\alpha}{a(\alpha)}}(1 - \alpha^{1/4}),$$

$$\Delta\rho < 0 \quad \text{if} \quad \varepsilon_0 \ge \rho \ge \sqrt{-\frac{\alpha}{a(\alpha)}}(1 + \alpha^{1/4}),$$

for all sufficiently small $\alpha > 0$. (Indeed, for any constant C, $\alpha^{5/4} > C\alpha^{3/2}$ provided $\alpha > 0$ is sufficiently small, so that the $\pm 2\alpha^{5/4}$-terms dominate.)

This proves that any nontrivial orbit of (5.66) starting in a small neighbourhood of the origin containing the annulus $A_{\alpha,\alpha^{1/4}}$ enters this annulus after a finite number of iterations. Notice that $A_{\alpha,\alpha^{1/4}}$ has width of order $O(\alpha^{3/4})$ and contains the circle

$$S_0(\alpha) = \left\{ (\rho, \varphi) : \rho = \sqrt{-\frac{\alpha}{a(\alpha)}} \right\}$$

having radius $O(\alpha^{1/2})$.

Step 2 (*Rescaling and shifting*). To analyse the dynamics inside the annulus $A_{\alpha,\alpha^{1/4}}$, introduce first a new radial variable s by the formula

$$\rho = \sqrt{-\frac{\alpha}{a(\alpha)}}(1+s). \tag{5.69}$$

Substitution of (5.69) into (5.67) gives

$$\left\{ \begin{array}{rcl} \tilde{s} & = & (1-2\alpha)s - \alpha(3s^2 + s^3) + \alpha^{3/2} r(s, \varphi, \alpha), \\ \tilde{\varphi} & = & \varphi + \theta(\alpha) + \alpha\nu(\alpha)(1+s)^2 + \alpha^2 q(s, \varphi, \alpha), \end{array} \right. \tag{5.70}$$

where

$$\nu(\alpha) = -\frac{b(\alpha)}{a(\alpha)},$$

while r and q are smooth real-valued functions of $(s, \varphi, \alpha^{1/2})$.

Next define u by

$$s = \alpha^{1/4} u \tag{5.71}$$

Notice that the band $\{(u, \varphi) : |u| \leq 1\}$ corresponds exactly to the annulus $A_{\alpha, \alpha^{1/4}}$ (see (5.68)). After rescaling according to (5.71), the map (5.70) takes the form

$$\begin{pmatrix} u \\ \varphi \end{pmatrix} \mapsto \begin{pmatrix} \tilde{u} \\ \tilde{\varphi} \end{pmatrix} = F \begin{pmatrix} u \\ \varphi \end{pmatrix}$$

where

$$F : \left\{ \begin{array}{rcl} \tilde{u} & = & (1-2\alpha)u + \alpha^{5/4} H_\alpha(u, \varphi), \\ \tilde{\varphi} & = & \varphi + \omega(\alpha) + \alpha^{5/4} K_\alpha(u, \varphi). \end{array} \right. \tag{5.72}$$

Here

$$\omega(\alpha) = \theta(\alpha) + \alpha\nu(\alpha)$$

is smooth and

$$\begin{array}{rcl} H_\alpha(u, \varphi) & = & -(3u^2 + \alpha^{1/4} u^3) + r(\alpha^{1/4} u, \varphi, \alpha), \\ K_\alpha(u, \varphi) & = & \nu(\alpha)(2u + \alpha^{1/4} u^2) + \alpha^{3/4} q(\alpha^{1/4} u, \varphi, \alpha), \end{array}$$

are smooth functions of $(u, \varphi, \alpha^{1/4})$ that are 2π-periodic in φ.

It is convenient to introduce

$$\lambda = \lambda(\alpha) = \sup_{|u| \leq 1, \varphi \in [0, 2\pi]} \left\{ |H_\alpha|, |K_\alpha|, \left|\frac{\partial H_\alpha}{\partial u}\right|, \left|\frac{\partial K_\alpha}{\partial u}\right|, \left|\frac{\partial H_\alpha}{\partial \varphi}\right|, \left|\frac{\partial K_\alpha}{\partial \varphi}\right| \right\}. \tag{5.73}$$

and note that $\lambda(\alpha)$ remains bounded as $\alpha \to 0$.

Step 3 (Definition of the function space). We will represent closed curves by elements of a function space U. By definition, $u \in U$ is a 2π-periodic continuous function $u = u(\varphi)$ satisfying the following two conditions:

(U.1) $|u(\varphi)| \leq 1$ for all φ;
(U.2) $|u(\varphi_1) - u(\varphi_2)| \leq |\varphi_1 - \varphi_2|$ for all φ_1, φ_2.

The first property means that $u(\varphi)$ is *absolutely bounded* by unity, while the second means that $u(\varphi)$ is *Lipschitz continuous* with Lipschitz constant less than or equal to one. The space U is a complete metric space with respect to the distance induced by the norm

$$\|u\| = \sup_{\varphi \in [0, 2\pi]} |u(\varphi)|.$$

Recall that a map $\mathcal{F} : U \to U$ (transforming a function $u(\varphi) \in U$ into some other function $\tilde{u}(\varphi) = (\mathcal{F}u)(\varphi) \in U$) is a *contraction* if there is a number ϵ, $0 < \epsilon < 1$, such that

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\| \leq \epsilon \|u_1 - u_2\|$$

for all $u_{1,2} \in U$. According to Theorem 3.19, a contraction in a complete metric space has a unique fixed point $u^{(\infty)} \in U$:

$$\mathcal{F}(u^{(\infty)}) = u^{(\infty)}.$$

Moreover, the fixed point $u^{(\infty)}$ is globally asymptotically stable as a fixed point of the infinite-dimensional dynamical system $\{\mathbb{N}, U, \mathcal{F}^k\}$, i.e.

$$\lim_{k \to +\infty} \|\mathcal{F}^k(u) - u^{(\infty)}\| = 0,$$

for all $u \in U$ (see Figure 5.26).

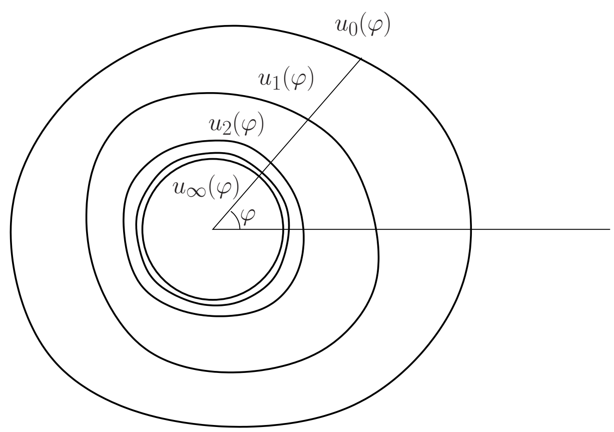

Figure 5.26: Accumulating closed curves.

Step 4 (*Construction of the map* $\mathcal{F}$). We will consider a map $\mathcal{F}$ on U *induced* by F. This means that if u represents a closed curve, then $\tilde{u} = \mathcal{F}(u)$ represents its image under the map F defined by (5.72). Such a map is called *Hadamard's Graph Transform*. It is clear that the continuous map F transforms a closed curve onto a closed curve, but it need not necessarily be true that the image can be represented by a function of φ. We will now verify that for α small this actually takes place.

Suppose that a function $u = u(\varphi)$ from U is given. To construct the map $\mathcal{F}$, we have to specify a procedure that for each given φ finds the corresponding $\tilde{u}(\varphi) = (\mathcal{F}u)(\varphi)$. Notice that F is nearly a *rotation* by the angle $\omega(\alpha)$ in φ. So a point $(\tilde{u}(\varphi), \varphi)$ in the resulting curve is the image of a point $(u(\hat{\varphi}), \hat{\varphi})$ in the original curve with a *different* angle coordinate $\hat{\varphi}$ (see Figure 5.27).

To show that $\hat{\varphi}$ is *uniquely* defined, we have to prove that the equation

$$\varphi = \hat{\varphi} + \omega(\alpha) + \alpha^{5/4} K_\alpha(u(\hat{\varphi}), \hat{\varphi}) \tag{5.74}$$

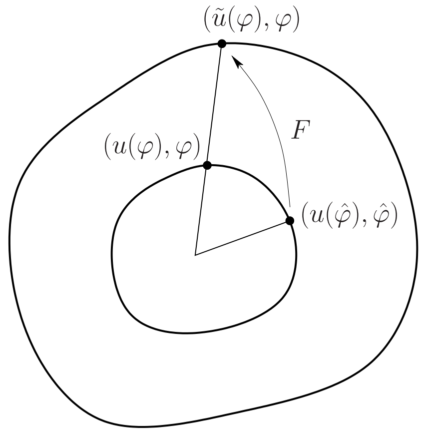

Figure 5.27: Definition of the map $\mathcal{F}$.

has a unique solution $\hat{\varphi} = \hat{\varphi}(\varphi)$ for any given $u \in U$. This is the case, since the right-hand side of (5.74) is a strictly increasing function of $\hat{\varphi}$. Indeed, let $\varphi_2 > \varphi_1$ then, according to (5.72),

$$\begin{aligned}\tilde{\varphi}_2 - \tilde{\varphi}_1 &= \varphi_2 - \varphi_1 + \alpha^{5/4}\left[K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha(u(\varphi_1), \varphi_1)\right] \\ &\geq \varphi_2 - \varphi_1 - \alpha^{5/4}\left|K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha(u(\varphi_1), \varphi_1)\right|.\end{aligned}$$

Combining (U.2) with the definition (5.73) we deduce that

$$\begin{aligned}|K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha(u(\varphi_1), \varphi_1)| &\leq \lambda[|u(\varphi_2) - u(\varphi_1)| + |\varphi_2 - \varphi_1|] \\ &\leq 2\lambda|\varphi_2 - \varphi_1| \ = \ 2\lambda(\varphi_2 - \varphi_1).\end{aligned}$$

This last estimate can also be written as

$$-\left|K_\alpha(u(\varphi_2), \varphi_2) - K_\alpha(u(\varphi_1), \varphi_1)\right| \geq -2\lambda(\varphi_2 - \varphi_1),$$

which implies

$$\tilde{\varphi}_2 - \tilde{\varphi}_1 \geq (1 - 2\lambda\alpha^{5/4})(\varphi_2 - \varphi_1).$$

It follows that the right-hand side of (5.74) is a strictly increasing function, provided α is small enough. Hence (5.74) has a unique solution $\hat{\varphi}(\varphi) \approx \varphi - \omega(\alpha)$. From the above estimates, it directly follows that $\hat{\varphi}(\varphi)$ satisfies the estimate

$$|\hat{\varphi}(\varphi_1) - \hat{\varphi}(\varphi_2)| \leq (1 - 2\lambda\alpha^{5/4})^{-1}|\varphi_1 - \varphi_2|. \tag{5.75}$$

We now define the map $\tilde{u} = \mathcal{F}(u)$ by the formula

$$\tilde{u}(\varphi) = (1 - 2\alpha)u(\hat{\varphi}) + \alpha^{5/4}H_\alpha(u(\hat{\varphi}), \hat{\varphi}), \tag{5.76}$$

where $\hat{\varphi} = \hat{\varphi}(\varphi)$ is the solution of (5.74). The mere definition, of course, is not enough and we have to verify that $\mathcal{F}(u) \in U$ if $u \in U$, i.e. we have to check (U.1) and (U.2) for $\tilde{u} = \mathcal{F}(u)$.

Condition (U.1) for $\tilde{u}$ follows from the estimate

$$|\tilde{u}(\varphi)| \leq (1 - 2\alpha)|u(\hat{\varphi})| + \alpha^{5/4}|H_\alpha(u(\hat{\varphi}), \hat{\varphi})| \leq 1 - 2\alpha + \lambda\alpha^{5/4},$$

where we have used (U.1) for u and the definition (5.73) of λ. Thus, $|\tilde{u}| \leq 1$ if α is small enough and positive.

Let $\hat{\varphi}_1$ and $\hat{\varphi}_2$ be the unique solutions of

$$\varphi = \hat{\varphi}_1 + \omega(\alpha) + \alpha^{5/4} K_\alpha(u_1(\hat{\varphi}_1), \hat{\varphi}_1) \tag{5.77}$$

and

$$\varphi = \hat{\varphi}_2 + \omega(\alpha) + \alpha^{5/4} K_\alpha(u_2(\hat{\varphi}_2), \hat{\varphi}_2), \tag{5.78}$$

respectively. Condition (U.2) for $\tilde{u}$ is then checked by the sequence of estimates:

$$\begin{aligned} |\tilde{u}(\varphi_1) - \tilde{u}(\varphi_2)| &\leq (1-2\alpha)|u(\hat{\varphi}_1) - u(\hat{\varphi}_2)| \\ &\quad + \alpha^{5/4}|H_\alpha(u(\hat{\varphi}_1), \hat{\varphi}_1) - H_\alpha(u(\hat{\varphi}_2), \hat{\varphi}_2)| \\ &\leq (1-2\alpha)|u(\hat{\varphi}_1) - u(\hat{\varphi}_2)| \\ &\quad + \alpha^{5/4}\lambda\big[|u(\hat{\varphi}_1) - u(\hat{\varphi}_2)| + |\hat{\varphi}_1 - \hat{\varphi}_2|\big] \\ &\leq (1 - 2\alpha + 2\lambda\alpha^{5/4})|\hat{\varphi}_1 - \hat{\varphi}_2|, \end{aligned}$$

where the final inequality holds due to the Lipschitz continuity of u. Inserting the estimate (5.75), we get

$$|\tilde{u}(\varphi_1) - \tilde{u}(\varphi_2)| \leq (1 - 2\alpha + 2\lambda\alpha^{5/4})(1 - 2\lambda\alpha^{5/4})^{-1}|\varphi_1 - \varphi_2|.$$

Thus, (U.2) also holds for $\tilde{u}$ for all sufficiently small positive α. Therefore, the map $\tilde{u} = \mathcal{F}(u)$ is well defined.

Step 5 (Verification of the contraction property). Now suppose two functions $u_1, u_2 \in U$ are given. We need to estimate $\|\tilde{u}_1 - \tilde{u}_2\| = \|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|$ in terms of $\|u_1 - u_2\|$. By the definition (5.76) of $\tilde{u} = \mathcal{F}(u)$,

$$\begin{aligned} |\tilde{u}_1(\varphi) - \tilde{u}_2(\varphi))| &\leq (1-2\alpha)|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| \\ &\quad + \alpha^{5/4}|H_\alpha(u_1(\hat{\varphi}_1), \hat{\varphi}_1) - H_\alpha(u_2(\hat{\varphi}_2), \hat{\varphi}_2)| \\ &\leq (1-2\alpha)|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| \\ &\quad + \alpha^{5/4}\lambda\big[|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| + |\hat{\varphi}_1 - \hat{\varphi}_2|\big]. \end{aligned} \tag{5.79}$$

The estimates (5.79) have not solved the problem yet, since we have to use only $\|u_1 - u_2\|$ at the right-hand side. First, express $|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)|$ in terms of $\|u_1 - u_2\|$ *and* $|\hat{\varphi}_1 - \hat{\varphi}_2|$:

$$\begin{aligned} |u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| &= |u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_1) + u_2(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| \\ &\leq |u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_1)| + |u_2(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| \\ &\leq \|u_1 - u_2\| + |\hat{\varphi}_1 - \hat{\varphi}_2|. \end{aligned} \tag{5.80}$$

The last inequality has been obtained using the definition of the norm and the Lipschitz continuity of u_2. To complete the estimates, we need to express $|\hat{\varphi}_1 - \hat{\varphi}_2|$ in terms of $\|u_1 - u_2\|$. Subtracting (5.78) from (5.77), rearranging, and taking absolute values we find

$$\begin{aligned} |\hat{\varphi}_1 - \hat{\varphi}_2| &\leq \alpha^{5/4}|K_\alpha(u_1(\hat{\varphi}_1), \hat{\varphi}_1) - K_\alpha(u_2(\hat{\varphi}_2), \hat{\varphi}_2)| \\ &\leq \alpha^{5/4}\lambda[|u_1(\hat{\varphi}_1) - u_2(\hat{\varphi}_2)| + |\hat{\varphi}_1 - \hat{\varphi}_2|]. \end{aligned}$$

Inserting (5.80) into this inequality, collecting all the terms involving $|\hat{\varphi}_1 - \hat{\varphi}_2|$ on the left, and dividing by $(1 - 2\alpha^{5/4}\lambda)$ we obtain

$$|\hat{\varphi}_1 - \hat{\varphi}_2| \leq (1 - 2\alpha^{5/4}\lambda)^{-1}\alpha^{5/4}\lambda\|u_1 - u_2\|. \tag{5.81}$$

Using the estimates (5.80) and (5.81), we can complete (5.79) as follows:

$$\|\tilde{u}_1 - \tilde{u}_2\| \leq \epsilon\|u_1 - u_2\|,$$

where

$$\epsilon = (1 - 2\alpha)\left[1 + \alpha^{5/4}\lambda(1 - 2\alpha^{5/4}\lambda)^{-1}\right] + \alpha^{5/4}\lambda\left[1 + 2\alpha^{5/4}\lambda(1 - 2\alpha^{5/4}\lambda)^{-1}\right].$$

Since

$$\epsilon = 1 - 2\alpha + O(\alpha^{5/4}),$$

the map $\mathcal{F}$ is a contraction in U for small positive α. Therefore, it has a unique fixed point $u^{(\infty)} \in U$ such that

$$\lim_{k\to\infty} \mathcal{F}^k(u) = u^{(\infty)}$$

for any $u \in U$.

Step 6 (*Asymptotic stability of the invariant curve*). The established contraction of $\mathcal{F}$ implies that the closed invariant curve corresponding to $u^{(\infty)}$ is a *stable invariant set* for the map F defined by (5.72). Indeed, images of any band located in $A_{\alpha,\alpha^{1/4}}$ and contaning $u^{(\infty)}$ will contain this curve and lie inside the band, for all suffucuently high iterates of F.

Now take a point (u_0, φ_0) within the band $\{(u, \varphi) : |u| \leq 1\}$. If the point belongs to the curve given by $u^{(\infty)}$, it remains on this curve under iteration of F, since the map F maps this curve into itself. If the point does not lie on the invariant curve, take some (noninvariant) closed curve passing through it represented by $u^{(0)} \in U$, say $u^{(0)}(\varphi) = u_0 \cos(\varphi - \varphi_0)$. Let us apply the iterations of the map F to this point. We get a sequence of points

$$\{(u_k, \varphi_k)\}_{k=0}^{\infty}.$$

It is clear that each point from this sequence belongs to the corresponding iterate of the curve $u^{(0)}$ under the map $\mathcal{F}$. We have just shown that the iterations of the curve converge to the invariant curve given by $u^{(\infty)}$. Therefore, the point sequence must also converge to the curve. This proves asymptotic stability of the closed invariant curve as the only nontrivial invariant set of the map (5.66) in the annulus $A_{\alpha,\alpha^{1/4}}$. Recalling *Step 1* completes the proof. □

Remarks:

(1) The orbit structure on the closed invariant curve and the variation of this structure when the parameter changes are generically different in the maps (5.66) and (5.64). In (5.66), there exist generically only a finite number of periodic orbits on the closed invariant curve. When the parameter changes, these orbits collide and disappear/appear via fold bifurcations.

(2) The bifurcating invariant curve in (5.66) generically has only finite smoothness, which increases as $\alpha \to 0$. ◇

We now shall prove that any generic two-dimensional system undergoing a Neimark-Sacker bifurcation can be transformed into the form (5.66).

Consider a system

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R},$$

with a smooth function f, which has at $\alpha = 0$ the fixed point $x = 0$ with simple eigenvalues $\mu_{1,2} = \mathrm{e}^{\pm i\theta_0}$, $0 < \theta_0 < \pi$. By the Implicit Function Theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighbourhood of the origin for all sufficiently small $|\alpha|$, since $\mu = 1$ is not an eigenvalue of the Jacobian matrix. We can perform a parameter-dependent coordinate shift, placing this fixed point at the origin. Therefore, we may assume without loss of generality that $x = 0$ is the fixed point of the system for $|\alpha|$ sufficiently small. Thus, the system can be written as

$$x \mapsto A(\alpha)x + F(x,\alpha), \tag{5.82}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with quadratic (or higher order) terms, $F(0,\alpha) = 0$ for all sufficiently small $|\alpha|$. The Jacobian matrix $A(\alpha)$ has two eigenvalues

$$\mu_{1,2}(\alpha) = r(\alpha)\mathrm{e}^{\pm i\varphi(\alpha)},$$

where $r(0) = 1, \varphi(0) = \theta_0$. Thus, $r(\alpha) = 1 + \beta(\alpha)$ for some smooth function $\beta(\alpha), \beta(0) = 0$. Suppose that $\beta'(0) \neq 0$. Then, we can use β as a new parameter and express the multipliers in terms of $\beta : \mu_1(\beta) = \mu(\beta), \mu_2(\beta) = \bar{\mu}(\beta)$, where

$$\mu(\beta) = (1+\beta)\mathrm{e}^{i\theta(\beta)}$$

with a smooth function $\theta(\beta)$ such that $\theta(0) = \theta_0$.

Lemma 5.31 *By the introduction of a complex variable and a new parameter, the system* (5.82) *can be transformed for all sufficiently small* $|\alpha|$ *into the following form:*

$$z \mapsto \mu(\beta)z + g(z,\bar{z},\beta), \tag{5.83}$$

where $\beta \in \mathbb{R}, z \in \mathbb{C}, \mu(\beta) = (1+\beta)\mathrm{e}^{i\theta(\beta)}$, *and* g *is a complex-valued smooth function of* $z, \bar{z}$, *and* β *whose Taylor expansion with respect to* $(z, \bar{z})$ *contains quadratic and higher-order terms*:

$$g(z,\bar{z},\beta) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(\beta) z^k \bar{z}^l,$$

with $k, l = 0, 1, \ldots$. □

The proof of the lemma is completely analogous to that from the Andronov-Hopf bifurcation analysis and is omitted. In particular, we have

$$g(z,\bar{z},0) = \langle p, F(zq + \bar{z}\bar{q}, 0)\rangle,$$

where the vectors $q, p \in \mathbb{C}^2$ satisfy

$$A(0)q = e^{i\theta_0}q, \;\; A^T(0)p = e^{-i\theta_0}p, \;\; \langle p, q\rangle = 1.$$

As in the Andronov-Hopf case, we start by making *nonlinear* (complex) coordinate transformations that will simplify the map (5.83). First, we remove all the quadratic terms.

Lemma 5.32 *The map*

$$z \mapsto \mu z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + O(|z|^3), \tag{5.84}$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$, *can be transformed by an invertible parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2,$$

for all sufficiently small $|\beta|$, *into a map*

$$w \mapsto \mu w + O(|w|^3)$$

without quadratic terms, provided that

$$e^{i\theta_0} \neq 1 \;\; \textit{and} \;\; e^{3i\theta_0} \neq 1.$$

Proof: The inverse change of variables is given by

$$w = z - \frac{h_{20}}{2}z^2 - h_{11}z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + O(|z|^3).$$

Therefore, in the new coordinate w, the map (5.84) takes the form

$$\begin{aligned} w \mapsto \mu w \;\; &+ \;\; \frac{1}{2}(g_{20} + (\mu - \mu^2)h_{20})w^2 \\ &+ \;\; (g_{11} + (\mu - |\mu|^2)h_{11})w\bar{w} \\ &+ \;\; \frac{1}{2}(g_{02} + (\mu - \bar{\mu}^2)h_{02})\bar{w}^2 \\ &+ \;\; O(|w|^3). \end{aligned}$$

Thus, by setting

$$h_{20} = \frac{g_{20}}{\mu^2 - \mu}, \;\; h_{11} = \frac{g_{11}}{|\mu|^2 - \mu}, \;\; h_{02} = \frac{g_{02}}{\bar{\mu}^2 - \mu},$$

we "kill" all the quadratic terms in (5.84). These substitutions are feasible if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$. This is indeed the case, since

$$\begin{aligned} \mu^2(0) - \mu(0) &= e^{i\theta_0}(e^{i\theta_0} - 1) \neq 0, \\ |\mu(0)|^2 - \mu(0) &= 1 - e^{i\theta_0} \neq 0, \\ \bar{\mu}(0)^2 - \mu(0) &= e^{i\theta_0}(e^{-3i\theta_0} - 1) \neq 0, \end{aligned}$$

due to our restrictions on θ_0. □

Remarks:

(1) Let $\mu_0 = \mu(0)$. Then, the conditions on θ_0 used in the lemma can be written as

$$\mu_0 \neq 1,\ \mu_0^3 \neq 1.$$

Notice that the first condition holds automatically due to our initial assumptions on θ_0.

(2) The resulting coordinate transformation is polynomial with coefficients that are smoothly dependent on β. In a small neighbourhood of the origin the transformation is *nearly the identity.*

(3) Notice that the transformation *changes* the coefficients of the cubic terms of (5.84). ♢

Lemma 5.33 *The map*

$$z \mapsto \mu z + \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 + O(|z|^4), \tag{5.85}$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}$, $g_{ij} = g_{ij}(\beta)$, can be transformed by an invertible parameter-dependent change of coordinates

$$z = w + \frac{h_{30}}{6}w^3 + \frac{h_{21}}{2}w^2\bar{w} + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3,$$

for all sufficiently small $|\beta|$, into a map with only one cubic term:

$$w \mapsto \mu w + c_1 w^2\bar{w} + O(|w|^4),$$

provided that

$$e^{2i\theta_0} \neq 1 \quad and \quad e^{4i\theta_0} \neq 1.$$

Proof: The inverse transformation is

$$w = z - \frac{h_{30}}{6}z^3 - \frac{h_{21}}{2}z^2\bar{z} - \frac{h_{12}}{2}z\bar{z}^2 - \frac{h_{03}}{6}\bar{z}^3 + O(|z|^4).$$

Therefore, in the new coordinate w, the map (5.85) takes the form

$$\begin{aligned} w \mapsto \lambda w \quad &+ \quad \frac{1}{6}(g_{30} + (\mu - \mu^3)h_{30})w^3 + \frac{1}{2}(g_{21} + (\mu - \mu|\mu|^2)h_{21})w^2\bar{w} \\ &+ \quad \frac{1}{2}(g_{12} + (\mu - \bar{\mu}|\mu|^2)h_{12})w\bar{w}^2 + \frac{1}{6}(g_{03} + (\mu - \bar{\mu}^3)h_{03})\bar{w}^3 + O(|w|^4). \end{aligned}$$

By setting

$$h_{30} = \frac{g_{30}}{\mu^3 - \mu},\ h_{12} = \frac{g_{12}}{\bar{\mu}|\mu|^2 - \mu},\ h_{03} = \frac{g_{03}}{\bar{\mu}^3 - \mu},$$

we can annihilate all cubic terms in the resulting map except the $w^2\bar{w}$-term, which we consider below. The substitutions are valid since all the involved denominators are nonzero for all sufficiently small $|\beta|$ due to the assumptions concerning θ_0.

One can also try to eliminate the $w^2\bar{w}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\mu(1-|\mu|^2)}.$$

This is possible for small $\beta \neq 0$, but the denominator vanishes at $\beta = 0$ *for all* θ_0. Thus, no extra conditions on θ_0 would help. To obtain a transformation that is smoothly dependent on β, set $h_{21} = 0$, resulting in

$$c_1 = \frac{g_{21}}{2}.$$

□

Remarks:

(1) The conditions imposed on θ_0 in the lemma can also be formulated as

$$\mu_0^2 \neq 1, \ \mu_0^4 \neq 1,$$

and therefore, in particular, $\mu_0 \neq -1$ and $\mu_0 \neq i$. The first condition holds automatically due to our initial assumptions on θ_0.

(2) The remaining cubic $w^2\bar{w}$-term is called a *resonant term*. Note that its coefficient is the *same* as the coefficient of the cubic term $z^2\bar{z}$ in the original map (5.85). ◇

We now combine the two previous lemmas.

Lemma 5.34 (Normal form for the Neimark-Sacker bifurcation)
The map

$$\begin{aligned} z \mapsto \mu z \quad &+ \quad \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 \\ &+ \quad \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 \\ &+ \quad O(|z|^4), \end{aligned}$$

where $\mu = \mu(\beta) = (1+\beta)\mathrm{e}^{i\theta(\beta)}$, $g_{ij} = g_{ij}(\beta)$, *and* $\theta_0 = \theta(0)$ *is such that* $\mathrm{e}^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4$, *can be transformed, for all sufficiently small* $|\beta|$, *by an invertible change of complex coordinate*

$$\begin{aligned} z = w \quad &+ \quad \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2 \\ &+ \quad \frac{h_{30}}{6}w^3 + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3, \end{aligned}$$

depending smoothly on the parameter, into a map

$$w \mapsto \mu w + c_1 w^2\bar{w} + O(|w|^4),$$

with only the resonant cubic term where $c_1 = c_1(\beta)$. □

The composition of the transformations defined in the two previous lemmas gives the required coordinate change. First, annihilate all the quadratic terms. This will also change the coefficients of the cubic terms. The coefficient of $w^2\bar{w}$ will be $\frac{1}{2}\tilde{g}_{21}$, say, instead of $\frac{1}{2}g_{21}$. Then, eliminate all the cubic terms except the resonant one. The coefficient of this term remains $\frac{1}{2}\tilde{g}_{21}$. Thus, all we need to compute to get the coefficient of c_1 in terms of the given equation is a new coefficient $\frac{1}{2}\tilde{g}_{21}$ of the $w^2\bar{w}$-term after the *quadratic* transformation. The computations result in the following expression for $c_1(\alpha)$:

$$c_1 = \frac{g_{20}g_{11}(\bar{\mu} - 3 + 2\mu)}{2(\mu^2 - \mu)(\bar{\mu} - 1)} + \frac{|g_{11}|^2}{1 - \bar{\mu}} + \frac{|g_{02}|^2}{2(\mu^2 - \bar{\mu})} + \frac{g_{21}}{2},$$

which gives, for the critical value of c_1,

$$c_1(0) = \frac{g_{20}(0)g_{11}(0)(1 - 2\mu_0)}{2(\mu_0^2 - \mu_0)} + \frac{|g_{11}(0)|^2}{1 - \bar{\mu}_0} + \frac{|g_{02}(0)|^2}{2(\mu_0^2 - \bar{\mu}_0)} + \frac{g_{21}(0)}{2}, \tag{5.86}$$

where $\mu_0 = e^{i\theta_0}$.

We now summarize the obtained results in the following theorem.

Theorem 5.35 *Suppose a two-dimensional discrete-time system generated by the map*

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^2, \ \alpha \in \mathbb{R},$$

with smooth f, has, for all sufficiently small $|\alpha|$, the fixed point $x = 0$ with multipliers

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)},$$

where $r(0) = 1, \varphi(0) = \theta_0$.

Let the following generic conditions be satisfied:

(C.1) $r'(0) \neq 0$;
(C.2) $e^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4$.

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto (1+\beta)\begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (y_1^2 + y_2^2)\begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix}\begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\|y\|^4) \tag{5.87}$$

with $\theta(0) = \theta_0$ and

$$a(0) = \mathrm{Re}\left(\frac{e^{-i\theta_0}g_{21}}{2}\right) - \mathrm{Re}\left(\frac{(1 - 2e^{i\theta_0})e^{-2i\theta_0}}{2(1 - e^{i\theta_0})}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2, \tag{5.88}$$

where g_{kl} are defined by the expansion

$$\langle p, f(zq + \bar{z}\bar{q}, 0)\rangle = e^{i\theta_0}z + \sum_{2 \le k+l \le 3} \frac{1}{k!l!}g_{kl}z^k\bar{z}^l + O(|z|^4),$$

in which $q, p \in \mathbb{C}^2$ satisfy $f_x(0,0)q = e^{i\theta_0}q$, $f_x^T(0,0)p = e^{-i\theta_0}p$, and $\langle p, q\rangle = 1$.

Proof: The only thing left to verify is the formula (5.88) for $a(0)$. Indeed, by Lemmas 5.31, 5.32, 5.33, and 5.34, the system can be transformed to the complex Poincaré normal form,

$$w \mapsto \mu(\beta)w + c_1(\beta)w|w|^2 + O(|w|^4),$$

for $\mu(\beta) = (1+\beta)e^{i\theta(\beta)}$. This map can be written as

$$w \mapsto e^{i\theta(\beta)}(1+\beta+d(\beta)|w|^2)w + O(|w|^4),$$

where $d(\beta) = a(\beta) + ib(\beta)$ for some real functions $a(\beta)$, $b(\beta)$. A return to the real coordinates $(y_1, y_2), w = y_1 + iy_2$, gives system (5.87). Finally,

$$a(\beta) = \text{Re } d(\beta) = \text{Re}(e^{-i\theta(\beta)}c_1(\beta)).$$

Thus $a(0) = \text{Re}(e^{-i\theta_0}c_1(0))$ and, taking into account (5.86), we get (5.88). □

Example 5.36 (Neimark-Sacker bifurcation of the delayed logistic map) Consider the following recurrence equation:

$$u(k+1) = ru(k)(1-u(k-1)).$$

This is yet another simple population dynamics model, where $u(k)$ stands for the density of a population in year k, and r is the growth rate at low densities. It is assumed that the growth is determined not only by the current population density but also by its density a year ago.

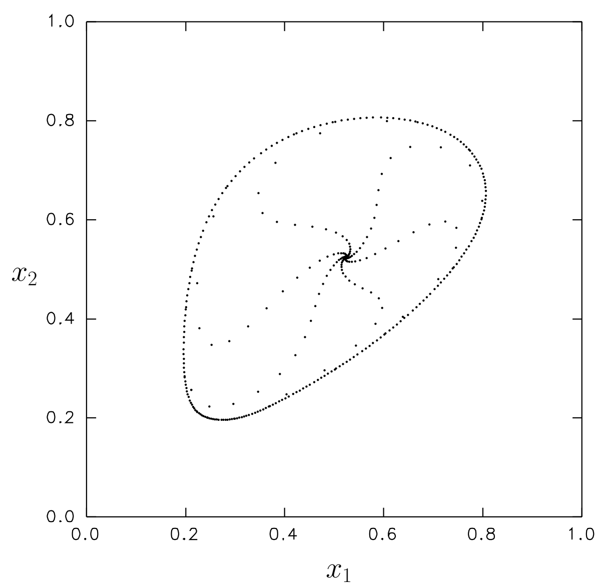

Figure 5.28: Stable invariant curve in the delayed logistic map.

If we introduce $v(k) = u(k-1)$, the equation can be rewritten as

$$\begin{cases} u(k+1) &= ru(k)(1-v(k)), \\ v(k+1) &= v(k), \end{cases}$$

which, in turn, defines the two-dimensional map,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} rx_1(1-x_2) \\ x_1 \end{pmatrix} \equiv \begin{pmatrix} F_1(x,r) \\ F_2(x,r) \end{pmatrix}, \tag{5.89}$$

where $x = (x_1, x_2)^T$. The map (5.89) has the fixed point $(0,0)^T$ for all values of r. For $r > 1$, a nontrivial positive fixed point x^0 appears, with the coordinates

$$x_1^0(r) = x_2^0(r) = 1 - \frac{1}{r}.$$

The Jacobian matrix of the map (5.89) evaluated at the nontrivial fixed point is given by

$$A(r) = \begin{pmatrix} 1 & 1-r \\ 1 & 0 \end{pmatrix}$$

and has eigenvalues

$$\mu_{1,2}(r) = \frac{1}{2} \pm \sqrt{\frac{5}{4} - r}.$$

If $r > \frac{5}{4}$, the eigenvalues are complex and $|\mu_{1,2}|^2 = \mu_1\mu_2 = r - 1$. Therefore, at $r = r_0 = 2$ the nontrivial fixed point loses stability and we have a Neimark-Sacker bifurcation: The critical multipliers are

$$\mu_{1,2} = \mathrm{e}^{\pm i\theta_0}, \ \theta_0 = \frac{\pi}{3} = 60^0.$$

It is clear that conditions (C.1) and (C.2) are satisfied.

We have to verify that the nondegeneracy condition $a(0) \neq 0$ also holds. The critical Jacobian matrix $A_0 = A(r_0)$ has the eigenvectors

$$A_0 q = \mathrm{e}^{i\theta_0} q, \ A_0^T p = \mathrm{e}^{-i\theta_0} p,$$

where

$$q \sim \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ 1\right)^T, \quad p \sim \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ 1\right)^T.$$

To achieve the normalization $\langle p, q \rangle = 1$, we can take, for example,

$$q = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}, \ 1\right)^T, \quad p = \left(i\frac{\sqrt{3}}{3}, \ \frac{1}{2} - i\frac{\sqrt{3}}{6}\right)^T.$$

Now we compose

$$x = x^0 + zq + \bar{z}\bar{q}$$

and evaluate the function

$$H(z, \bar{z}) = \langle p, F(x^0 + zq + \bar{z}\bar{q}, r_0) - x^0 \rangle.$$

Computing its Taylor expansion at $(z, \bar{z}) = (0, 0)$,

$$H(z, \bar{z}) = e^{i\theta_0} z + \sum_{2 \le j+k \le 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4),$$

gives

$$g_{20} = -2 + i\frac{2\sqrt{3}}{3}, \quad g_{11} = i\frac{2\sqrt{3}}{3}, \quad g_{02} = 2 + i\frac{2\sqrt{3}}{3}, \quad g_{21} = 0,$$

which allows us to evaluate $a(0)$ via (5.88) and see that

$$a(0) = -2 < 0.$$

Therefore, by Theorems 5.35 and 5.30, a unique and stable closed invariant curve bifurcates from the nontrivial fixed point for $r > 2$ (see Figure 5.28). ◇

5.4 References

Bifurcations of stationary points and periodic orbits in one- and two-parameter families of 1D and 2D ODEs and maps are treated in many textbooks, including [Arnol'd 1983, Guckenheimer & Holmes 1983, Arrowsmith & Place 1990, Shilnikov, Shilnikov, Turaev & Chua 2001, Wiggins 2003, Kuznetsov 2004]. A useful summary is given in [Arnol'd et al. 1994], while many technical issues are clarified in [Iooss 1979, Vanderbauwhede 1989, Iooss & Adelmeyer 1992].

5.5 Exercises

E 5.5.1 (Fold bifurcation in a simple ecological model)

Consider the following differential equation, which models a single population under a constant harvest:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \alpha,$$

where x is the population number; r and K are the *intrinsic growth rate* and the *carrying capacity* of the population, respectively, and α is the *harvest rate*, which is a control parameter. Find a parameter value α_0 at which the system has a fold bifurcation, and check the genericity conditions of Theorem 5.7. Based on the analysis, explain what might be a result of overharvesting on the ecosystem dynamics.

E 5.5.2 (Andronov-Hopf bifurcation in planar systems)

Check that each of the following systems has an equilibrium that exhibits an Andronov-Hopf bifurcation at some value of α, and compute the first Lyapunov coefficient:

(a) *Rayleigh's equation*:

$$\ddot{x} + \dot{x}^3 - 2\alpha\dot{x} + x = 0;$$

(*Hint:* Introduce $y = \dot{x}$ and rewrite the equation as a system of two differential equations.)

(b) *Van der Pol's oscillator*:

$$\ddot{y} - (\alpha - y^2)\dot{y} + y = 0;$$

(c) *Bautin example*[5]:

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + \alpha y + x^2 + xy + y^2; \end{cases}$$

(d) *Feichtinger model*[6]:

$$\begin{cases} \dot{x}_1 &= \alpha[1 - x_1 x_2^2 + A(x_2 - 1)], \\ \dot{x}_2 &= x_1 x_2^2 - x_2. \end{cases}$$

E 5.5.3 (Sub- and supercritical Hopf bifurcation in a prey-predator model)

Analyze Andronov-Hopf bifurcations in the predator-prey model[7]:

$$\begin{cases} \dot{x} &= \dfrac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -\gamma y(m - x), \end{cases} \tag{5.90}$$

where $\gamma > 0$ is fixed, while $n \geq 0$ and $0 \leq m < 1$ are control parameters.

(a) Find a realationship between parameters m and n corresponding to a Hopf bifurcation of a positive equilibrium in the system.

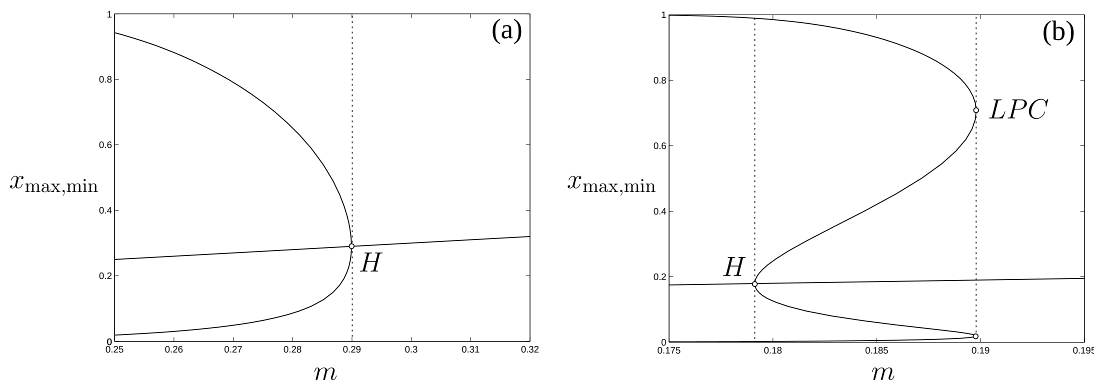

Figure 5.29: Maximal and minimal values of the x-varaible along equilibrium and cycle branches near the Hopf bifurcation of (5.90) for $\gamma = 1$ and (a) $n = 0.2$; (b) $n = 0.05$.

(b) Compute the first Lyapunov coefficient l_1 along the Hopf bifurcation curve H in the (m, n)-plane found at step (a) and show that it vanishes at

$$B = (m, n) = \left(\frac{1}{4}, \frac{1}{8} \right).$$

(c) Fix $\gamma = 1$ and compute numerically amilis of limit cycles bifurcating at the Hopf bifurcation from for $n = 0.2$ and $n = 0.05$ and find a fold bifurcation of limit cycles (LPC) in one of them. (*Hint*: Figure 5.29.)

[5]Bautin, N.N. and Leontovich, E.A. *Methods and Rules for the Qualitative Study of Dynamical Systems on the Plane*, Nauka, Moscow, 1976. In Russian.

[6]Feichtinger, G. 'Hopf bifurcation in an advertising diffusion model', *J. Econom. Behavior Organization* **17** (1992), 401-411.

[7]Bazykin, A.D. and Khibnik, A.I. 'On sharp excitation of self-oscillations in a Volterra-type model', *Biophysika* **26** (1981), 851-853. In Russian.

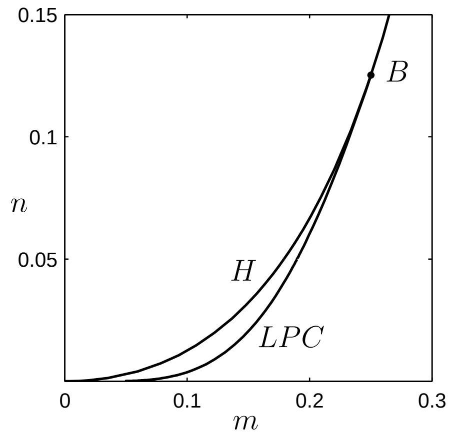

Figure 5.30: Bifurcation curves of (5.90) for $\gamma = 1$: H - Hopf bifurcation, LPC - fold bifurcation of limit cycles.

(d) Continue numerically the fold bifurcation curve LPC in two control parameters and convince yourself that it connects the origin of the (m, n)-plane with the point B found at step (b). (*Hint*: Figure 5.30.)

(e) Produce all typical phase portraits of the system.

E 5.5.4 (Flip bifurcation in scalar population models)

Prove that the generisity conditions (B.1) and (B.2) for the flip bifurcation at $(x, \alpha) = (0, 0)$ of the map

$$x \mapsto xg(x, \alpha)$$

with $g(0, 0) = -1$ are equivalent to

(D.1) $\frac{1}{2} g_{xx}(0, 0) + [g_x(0, 0)]^2 \neq 0$;
(D.2) $g_\alpha(0, 0) \neq 0$.

E 5.5.5 (Fixed points and periodic orbits of the 2-Ricker population model)

Consider the map[8]

$$x \mapsto rxe^{-\nu(x,p)} \quad \text{with} \quad \nu(x, p) = x(1 + pe^{-x}), \tag{5.91}$$

where $r > 1$ and $p > 0$.

(a) Find fixed points of (5.91) and study their fold and period-doubling bifurcations analytically.

(b) Study numerically bifurcations of period-two, -three, and -four cycles of (5.91) within the region $1 \leq r \leq 500, 0 \leq p \leq 40$. *Hint*: Introduce new variable and parameters: $y = \ln x, R = \ln r, P = \ln p$.

(c) Present a combined bifurcation diagram using the original (r, p)-parameters. Where could multiple attractors and chaos be expected?

[8]Davydova, N.V. 'Dynamics and bifurcations of single year class maps', Chapter 5 of *Old ang Yong. Can they coexist?* PhD Thesis, Utrecht University, 2004.

E 5.5.6 (Discrete-time prey-predator model by Maynard Smith)

Consider the following recurrence equations [9]:

$$\begin{cases} x_{k+1} &= \alpha x_k(1-x_k) - x_k y_k, \\ y_{k+1} &= \dfrac{1}{\beta} x_k y_k, \end{cases}$$

which is a discrete-time version of the Lotka-Volterra model. Here x_k and y_k are the prey and predator numbers, respectively, in year (generation) k, and it is assumed that in the absence of prey the predators become extinct in one generation. Parameters α and β are positive.

(a) Introduce the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x(1-x) - xy \\ \dfrac{1}{\beta} xy \end{pmatrix}, \tag{5.92}$$

and prove that a nontrivial fixed point of this map undergoes a Neimark-Sacker bifurcation on a curve in the (α, β)-plane.

(b) Determine the direction of the closed invariant-curve bifurcation by computing the corresponding normal form coefficient.

(c) Iterate the map (5.92) numerically for various parameter values to see what happens to the closed invariant curve away from the Neimark-Sacker bifurcation.

[9] Maynard Smith, J. *Mathematical Ideas in Biology*, Cambridge University Press, London, 1968.