NLDV COMPUTER SESSION XIV:
Using CONTENT to continue bifurcations of equilibria in two parameters

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1 Bifurcations of equilibria in a chemical model

In this session we will use CONTENT to continue equilibria and their bifurcations in the following ODE system with three phase variables and seven parameters by Bykov, Yablonskii, & Kim (1978):

\[
\begin{align*}
\dot{x} &= 2k_1 z^2 - 2k_{-1} x^2 - k_3 xy \\
\dot{y} &= k_2 z - k_{-2} y - k_3 xy \\
\dot{s} &= k_4 z - k_{-4} s,
\end{align*}
\]

where \( z = 1 - x - y - s \). The model describes CO oxidation on platinum:

1) \( \text{O}_2 + 2\text{Pt} \rightarrow 2\text{PtO} \)
2) \( \text{CO} + \text{Pt} \rightarrow \text{PtCO} \)
3) \( \text{PtCO} + \text{PtO} \rightarrow 2\text{Pt} + \text{CO}_2 \)
4) \( \text{CO} + \text{Pt} \rightarrow (\text{PtCO}) \)

In the ODEs, \( z, x, y, s \) are scaled concentrations of Pt, PtO, PtCO, and a non-reactable form (PtCO), respectively, while \( k_j \) stand for the corresponding reaction rates.

The ratio

\[ K = \frac{k_{-4}}{k_4} \]

and the parameter \( q_2 \) will be used as bifurcation parameters.

1.1 Specify the system in CONTENT

Specify a new ODE system in CONTENT with the coordinates \((x,y,s)\) and time \((t)\):

```c
double z;
z=1-x-y-s;
x'=2*Q1*z*z-2*Q5*x*x-Q3*x*y;
y'=Q2*z-Q6*y-Q3*x*y;
s'=Q4*z-K*Q4*s;
```

The reaction rates \( k_1, k_2, k_3, k_4, k_{-1}, k_{-2} \) are denoted by \( Q1, Q2, Q3, Q4, Q5, Q6 \), respectively. Generate the derivatives of order 1,2, and 3 symbolically.

Click Ok to compile the system and the derivatives.

1.2 Equilibrium continuation

Select Type|Initial point|Point and Type|Curve|Equilibrium. In the Starter window input the following numerical values, corresponding to a stable equilibrium in the system:
Activate the parameter $Q_2$.

In the **Continuer** window, decrease the maximal step size along the curve to $\text{MaxStepsize} = 0.025$.

Open a **2D Graphic** window with $(Q_2, x)$ as the axes and input the following visibility limits:

- **Abscissa:** $0.5$ to $2.0$
- **Ordinate:** $0.0$ to $0.16$

Also open a **Numeric** window and select all **Eigenvalues** to be visible using **Window|Layout**. Clear all windows.

Start the continuation of the equilibrium curve with the **Compute|Backward** command. There will be four bifurcation points detected: two Hopf points (labeled by $H$) and two fold points (labeled by LP). Resume computations at the bifurcation points by pressing the **Space** bar, and abort them when the curve leaves the window. You should get the central S-shaped curve presented in Fig. 1. In a narrow interval of $Q_2$-values (between two LP-points) the system has three equilibria.

Rename the computed equilibrium curve via **Select|Curve...** and **Actions|Rename** into $K=0.4$ Equilibrium(-)

Change the value of the parameter $K$ in the **Starter** window:

$K = 0.15$

and repeat the computations with **Compute|Backward**. You will get a monotone curve without bifurcation points (the left curve in Fig. 1). There is only one equilibrium for all parameter values $Q_2$. Finally, change the value of $K$ to

$K = 2.0$
and click on Compute|Backward again. Another equilibrium curve will come out with the order of the singularities reversed (the right curve in Fig. 1). There is now a big interval of the parameter values, in which there are three equilibria in the system.

Store the last two computed curves permanently by renaming them into

K=2.0 Equilibrium(+)
K=0.15 Equilibrium(+)

via Select|Curve... and Actions|Rename....

1.3 Continuation of fold and Hopf bifurcations

This section explains how to continue fold and Hopf bifurcations with respect to two control parameters. While doing so, you will see how CONTENT detects some codim 2 bifurcations and computes their normal form coefficients.

Redraw the computed equilibrium curves in the graphics window with the Window |Redraw diagram command.

1.3.1 Fold continuation

Click at the upper limit point labeled LP of the middle curve. The Starter and Continuer windows will be updated. Activate the second parameter, namely K. The new Starter window corresponding to the continuation of a limit point curve is appeared.
Compute the limit point curve in both directions to get Fig. 2. Notice that both parameters ($Q_2, K$) are varied along the curve. Each point in the curve is a limit point the equilibrium curve at the corresponding value of $K$.

There were detected three points, corresponding to the so called codim 2 bifurcations: two Bogdanov-Takens (BT) and one cusp (CP). At each BT point the system has an equilibrium with a double zero eigenvalue, while at the CP point there is an equilibrium with a simple zero eigenvalue but zero coefficient of the fold normal form.

Rename the obtained limit point branches as following:

- Fold(-)(standard)
- Fold(+)(standard)

### 1.3.2 Hopf continuation

The Bogdanov-Takens points are common points for the limit point curves and curves corresponding to equilibria with eigenvalues $\lambda_1 + \lambda_2 = 0, \lambda_3 \neq 0$. Actually, at each BT point, the Hopf bifurcation curve (with $\lambda_{1,2} = \pm i\omega_0, \omega_0 > 0$) turns into the neutral saddle curve (with real $\lambda_1 = -\lambda_2$). Thus, we can start a Hopf curve from a Bogdanov-Takens curve.

Select by the mouse the upper Bogdanov-Takens point BT as initial for the continuation of the Hopf curve. The new Starter window suitable for the continuation of a Hopf curve will appear automatically. Check that the type of the

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**Figure 2:** Limit point curve in Bykov’s model: BT - Bogdanov-Takens points; CP - cusp
initial point and that of the curve to compute have also been changed. Activate the parameter $K$.

**Compute|Backward** with several **Compute|Extend** produces a *closed* Hopf curve as in Fig. 3. Actually, only the part of it to the left of two BT points corresponds to a Hopf bifurcation; the other part represents a neutral saddle. As it has been mentioned, this transition happens at the Bogdanov-Takens points. There are two more codim 2 bifurcation points at the Hopf point: Two *generalized Hopf* (GH) points, where the first Lyapunov coefficient $l_1$ vanishes.

![Figure 3: Hopf curve in Bykov’s model: GH - generalized Hopf points](image)

**1.3.3 Two-parameter bifurcation diagram**

Rename the obtained Hopf curve into

Hopf (+)

and comment out using `%` all the equilibrium curves.

Open a new **2D Graphic** window to plot the bifurcation diagram in the $(Q2,K)$-plane with the visibility limits

<table>
<thead>
<tr>
<th>Q2</th>
<th>0.7</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>0.1</td>
<td>1.1</td>
</tr>
</tbody>
</table>

and activate **Window|Redraw diagram** in the **Main window**. The codim 1 curves will be redrawn in all the windows. In the $(Q2,k)$-window, you will get a diagram similar to that in Fig. 4.
Figure 4: Fold and Hopf curves in Bykov's model: BT- Bogdanov-Takens, CP -cusp, GH - generalized Hopf points
The edge between two fold curves is a parameter region where the system has three equilibria. Crossing a Hopf curve results in the appearance of either stable or unstable periodic orbit. There are other (global) bifurcations in the model.

2 Additional problem

- Compute fold and Hopf bifurcation curves in the \((\alpha, \delta)\)-plane for the predator-prey model:

\[
\begin{align*}
\dot{x} &= x - \frac{xy}{1 + \alpha x}, \\
\dot{y} &= -2y + \frac{xy}{1 + \alpha x} - \delta y^2,
\end{align*}
\]

where \(x, y, \alpha, \delta > 0\). Sketch (or compute) all typical phase portraits of the model.