

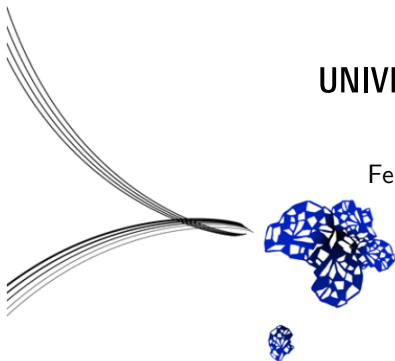
Numerical Bifurcation Analysis of Maps

H.G.E. Meijer

AAMP (EEMCS)

UNIVERSITEIT TWENTE.

February 16/18 2010



Aim of the course

codim 2 bifurcations are organizing centers in bifurcation diagrams

Part 1 **Analysis of codim 2 bifurcations:**

Normal forms, Center Manifolds, Unfoldings

→ Get a feeling of dynamical behaviour.

Part 2 **Bifurcations of invariant tori:**

KAM Resonance tongues, Bubble analysis, homoclinic bifurcations.

→ Get a feeling of fine details near torus bifurcations

Infinite sequences of bifurcations

Setting

Consider a map

$$F : x \mapsto F(x, \alpha) \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.$$

Study dynamics near a fixed point of the k -th iterate of the map.
Fixed points satisfy $F(x^0, \alpha^0)^k - x^0 = 0$ and have multipliers

$$\{\mu_1, \mu_2, \dots, \mu_n\} = \sigma(A),$$

where $A = F_x(x^0, \alpha^0)$.

k is the period of the fixed point.

W.l.o.g. $k = 1$, $x_0 = 0$, $\alpha_0 = 0$.

Notation

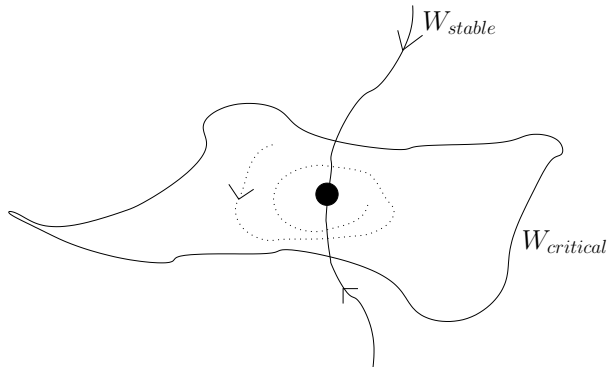
- ▶ **variables** $x \in \mathbb{R}$ and $z \in \mathbb{C}$
- ▶ **multi-index** For a multi-index ν we have $\nu = (\nu_1, \nu_2, \dots, \nu_n), \nu_i \in \mathbb{Z}_{\geq 0}$
 $\nu! = \nu_1! \nu_2! \dots \nu_n!$,
 $|\nu| = \nu_1 + \nu_2 + \dots + \nu_n$ and
 $\tilde{\nu} \leq \nu$ if $\tilde{\nu}_i \leq \nu_i$ for all $i = 1, \dots, n$.
- ▶ $\langle u, v \rangle = \bar{u}^T v$ is the standard scalar product in \mathbb{C}^n (or \mathbb{R}^n).

Center Manifold: 1

Decompose phase space(W) near steady solution:

$$W = W_u \oplus W_s \oplus W_c$$

Manifolds W_i invariant under the mapping F .



Center Manifold: 2

Center Manifold W_c (mod $(\mu) = 1$)

Bifurcations occur on W_c .

Normal form determines locally properties of the solutions.

Check:

1. Nondegenerate : Coefficients nonzero?

Predict the presence of heteroclinic/homoclinic structures and invariant circles.

2. Transversal : Depends on parameters

Transversality allows to switch to new branches.

Center Manifold: Invariance

$$\begin{array}{ccc}
 \mathbb{R}^{n_0} \ni w & \xrightarrow{H} & u \in \mathbb{R}^n \\
 \downarrow G & & \downarrow F \\
 \dot{w} & \xrightarrow{H} & \dot{u}
 \end{array}$$

HOMOLOGICAL EQUATION:

$$F(H(w)) = H(G(w))$$

where F Critical Map, G Normal Form
Center Manifold $x = H(w)$

Center Manifold Reduction: Ansatz

Let

$$\begin{aligned} F(x) &= Ax + \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) \\ &\quad + \frac{1}{24}D(x, x, x, x) + \frac{1}{120}E(x, x, x, x, x) + \dots \end{aligned}$$

and expand the functions G, H into Taylor series with unknown coefficients,

$$G(w) = \sum_{|\nu| \geq 1} \frac{1}{\nu!} g_\nu w^\nu, \quad H(w) = \sum_{|\nu| \geq 1} \frac{1}{\nu!} h_\nu w^\nu,$$

Center Manifold Reduction: Equations

Insert this into the homological equation and collect the coefficients of the w^ν -terms in the homological equation. This gives a linear system for h_ν :

$$L_\nu h_\nu = R_\nu.$$

where $L_\nu = (A - \mu^\nu I)$ with the multipliers μ .

Singular if $\mu^\nu = 1$. Interpretation: These terms are needed in the normal form.

- ▶ Iterative solutions for higher order terms.
- ▶ Critical coefficients come from singular systems.
- ▶ If necessary singular systems are solved by bordered systems.
- ▶ Parameters can be included in this reduction process
- ▶ Method by Elphick et.al.(1987)

Center Manifold Reduction: ODE's

$$\begin{array}{ccc}
 \mathbb{R}^{n_0} \ni w & \xrightarrow{H} & u \in \mathbb{R}^n \\
 \downarrow G & & \downarrow F \\
 \dot{w} & \xrightarrow{H} & \dot{u}
 \end{array}$$

Center Manifold W_c ($\Re(\lambda) = 0$)

Homological equation:

$$F(H(w)) = (D_w H)G(w)$$

$$L_\nu = (A - \langle \nu, \lambda \rangle I)$$

Vectorfield Approximation

Observation: composition $A \circ F$ is close to the identity.

Theorem (Takens, Neimark): Suppose $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism and $D\Phi(0)$ has all eigenvalues on the unit circle. Denote by S the semi-simple part of $D\Phi(0)$. Then there exists a diffeomorphism Ψ and a vectorfield X such that

$$\Psi \circ \Phi \circ \Psi^{-1} = \phi_X(t=1) \circ S$$

in the sense of Taylor series.

Proof: Global Analysis of Dynamical Systems: Festschrift dedicated to Floris Takens for his 60th birthday. Eds. H.W Broer, B. Krauskopf G. Vegter, see Thm 4.6 p20.

Remark:

- ▶ Φ is the time-1 map of the flow of the vectorfield X .
- ▶ parameters can be included.

Reduced ODEs for codim 2 bifurcations

- ▶ Cusp $\dot{x} = \beta_1 + \beta_2 x + x^3$
- ▶ Bautin $\dot{x} = x(\beta_1 + \beta_2 x^2 + x^4)$
- ▶ Bogdanov-Takens $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \begin{pmatrix} \dot{x}_2 \\ \beta_1 + \beta_2 x_1 + x_1^2 - x_1 x_2 \end{pmatrix}$
- ▶ Zero-Hopf $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \begin{pmatrix} \beta_1 + x_1^2 + s x_2^2 \\ x_2(\beta_2 + \theta x_1 + x_2^2) \end{pmatrix}$
- ▶ Double Hopf $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \begin{pmatrix} x_1(\beta_1 - x_1^2 - \theta x_2^2) \\ x_2(\beta_2 - \delta x_1^2 \pm x_2^2) \end{pmatrix}$

Fold-Hopf: Normal form

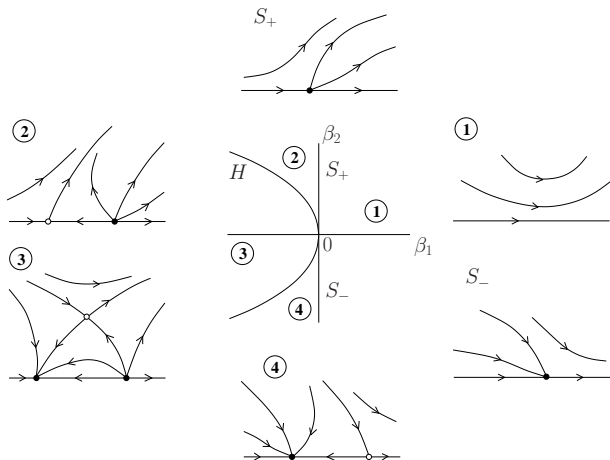
$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} \begin{pmatrix} \beta_1 + x^2 + s|z|^2 \\ (\beta_2 + i\omega)z + (\theta + i\vartheta)xz + x^2z \end{pmatrix}$$

Introduce cylindrical coordinates $(x, z) = (x_1, x_2 e^{i\phi})$, scalings then give amplitude

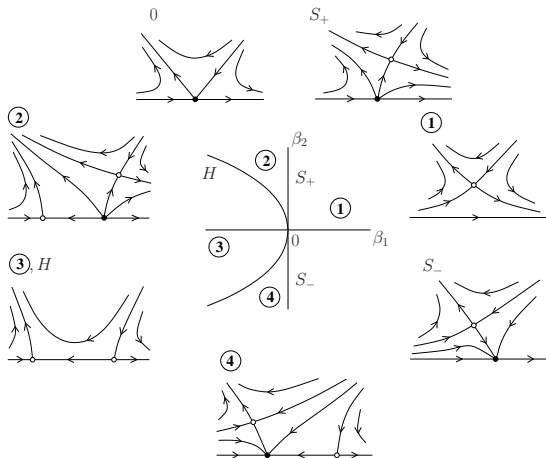
system $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \begin{pmatrix} \beta_1 + x_1^2 + sx_2^2 \\ x_2(\beta_2 + \theta x_1 + x_2^2) \end{pmatrix}$ Bifurcation curves:

- ▶ **fold** $\beta_1 = 0$
- ▶ **Hopf** $\beta_1 = -\left(\frac{\beta_2}{\theta}\right)^2$
- ▶ **Torus** If $s\theta < 0$, $\beta_2 = 0$, $\theta\beta_1 < 0$.
- ▶ **Heteroclinic** If $s < 0 < \theta$, $\beta_2 = \frac{\theta}{3\theta-2}\beta_1$, $\beta_1 < 0$.

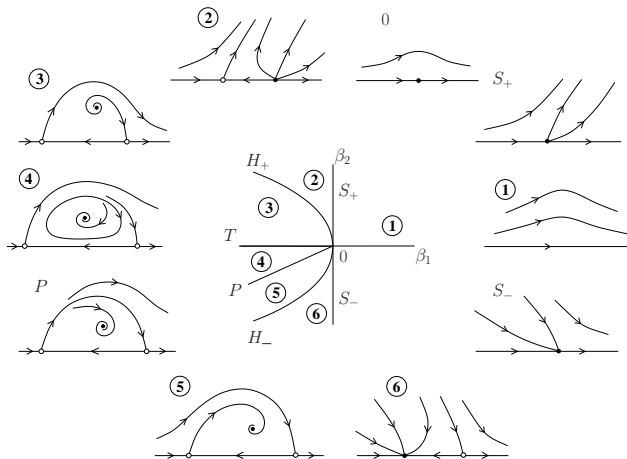
Fold-Hopf Unfolding $s = 1, \theta > 0$



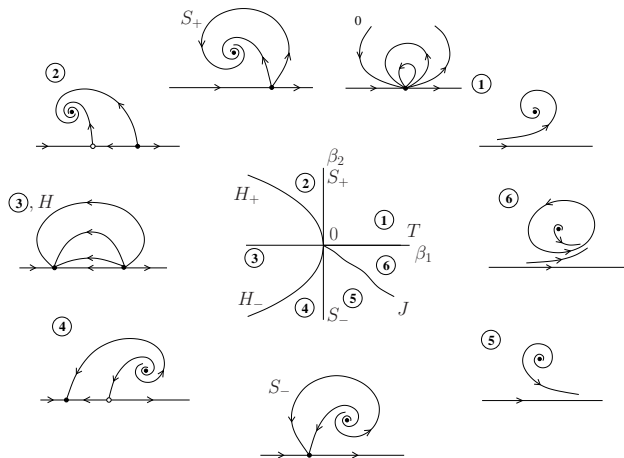
Fold-Hopf Unfolding $s = -1, \theta < 0$



Fold-Hopf Unfolding $s = -1, \theta > 0$



Fold-Hopf Unfolding $s = 1, \theta < 0$

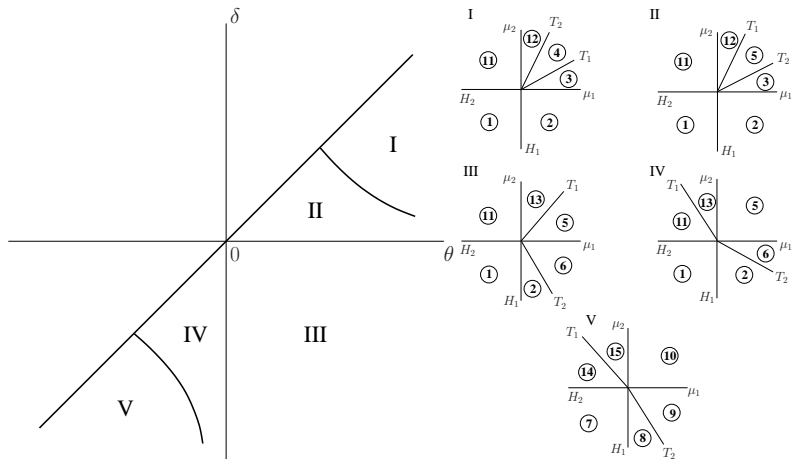


Double Hopf: Normal form

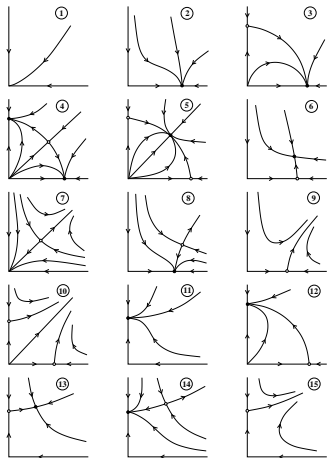
$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \end{pmatrix} = \begin{pmatrix} (i\omega_1(\beta) + \beta_1)w_1 + f_{2100}w_1|w_1|^2 + f_{1011}w_1|w_2|^2 \\ (i\omega_2(\beta) + \beta_2)w_2 + g_{1110}w_2|w_1|^2 + g_{0021}w_2|w_2|^2 \end{pmatrix} + O(\|(w_1, w_2)\|^4) \quad (1)$$

There are always two curves of Neimark-Sacker bifurcations.

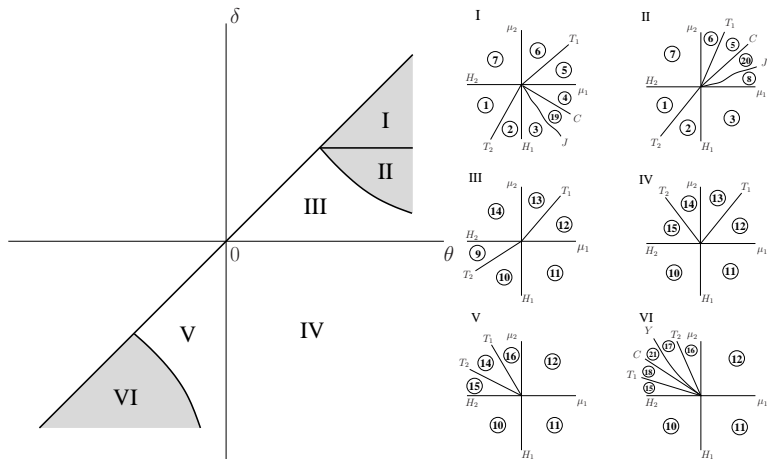
Simple case: parameter diagrams



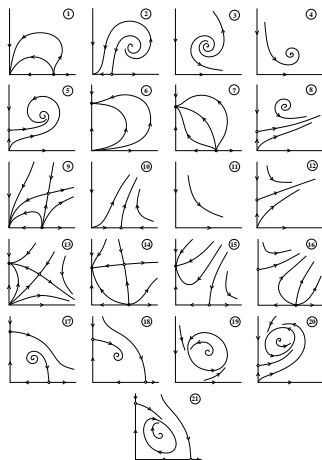
Simple case: phase portraits



Difficult case: parameter diagrams



Difficult case: phase portraits



Fold and Period-doubling

1. *Fold*: The fixed point has a simple eigenvalue $\lambda_1 = 1$ and no other eigenvalues on the unit circle, while the restriction of F to a one-dimensional center manifold at the critical parameter value has the form

$$\xi \mapsto \xi + \frac{1}{2}a\xi^2 + O(\xi^3), \quad (2)$$

where $a \neq 0$. When the parameter crosses the critical value, two fixed points coalesce and disappear. If $Av = F_x v$ and $B(u, v) = F_{xx}[u, v]$ are evaluated at the critical fixed point, then

$$a = \langle q^*, B(q, q) \rangle, \quad (3)$$

where $Aq = q$, $A^T q^* = q^*$, and $\langle q^*, q \rangle = 1$.

2. *Flip*: The fixed point has a simple eigenvalue $\lambda_1 = -1$ and no other eigenvalues on the unit circle, while the restriction of $(??)$ to a one-dimensional center manifold at the critical parameter value can be transformed to the normal form

$$\xi \mapsto -\xi + \frac{1}{6}b\xi^3 + O(\xi^4), \quad (4)$$

where $b \neq 0$. When the parameter crosses the critical value, a cycle of period

Neimark-Sacker

The fixed point has simple critical eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$ and no other eigenvalues on the unit circle. Assume that

$$e^{iq\theta_0} - 1 \neq 0, \quad q = 1, 2, 3, 4 \quad (\text{no strong resonances}).$$

Then, the restriction of (??) to a two-dimensional center manifold at the critical parameter value can be transformed to the normal form

$$\eta \mapsto \eta e^{i\theta_0} \left(1 + \frac{1}{2} d |\eta|^2 \right) + O(|\eta|^4), \quad (6)$$

where η is a complex variable and d is a complex number. Further assume that

$$c = \operatorname{Re} d \neq 0.$$

Under the above assumptions, a unique *closed invariant curve* around the fixed point appears when the parameter crosses the critical value. One has the following expression for d :

$$d = \frac{1}{2} e^{-i\theta_0} \langle v^*, C(v, v, \bar{v}) + 2B(v, (I_n - A)^{-1} B(v, \bar{v})) + B(\bar{v}, (e^{2i\theta_0} I_n - A)^{-1} B(v, v)) \rangle, \quad (7)$$

where $Av = e^{i\theta_0} v$, $A^T v^* = e^{-i\theta_0} v^*$, and $\langle v^*, v \rangle = 1$.

List of local codim 2 bifurcations for maps

- (1) $\mu_1 = 1, b = 0$ (**cusp**)
- (2) $\mu_1 = -1, c = 0$ (**generalized flip**)
- (3) $\mu_{1,2} = e^{\pm i\theta_0}, \operatorname{Re} [e^{-i\theta_0} c_1] = 0$ (Chenciner bifurcation)
- (4) $\mu_1 = \mu_2 = 1$ (1:1 resonance)
- (5) $\mu_1 = \mu_2 = -1$ (**1:2 resonance**)
- (6) $\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{2\pi}{3}$ (1:3 resonance)
- (7) $\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{\pi}{2}$ (1:4 resonance)
- (8) $\mu_1 = 1, \mu_2 = -1$ (**fold-flip**)
- (9) $\mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0}$ (“fold-Hopf for maps”)
- (10) $\mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0}$ (“flip-Hopf for maps”)
- (11) $\mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2}$ (“Hopf-Hopf for maps”)

Cusp

The critical normal form is

$$w \mapsto G(w) = w + \left(\frac{1}{2}bw^2\right) + \frac{1}{6}cw^3 + \dots$$

on the center manifold

$$H(w) = wh_1 + \frac{w^2}{2}h_2 + \frac{w^3}{6}h_3 + \dots$$

The first three terms of the expansion are given by

$$\begin{aligned}w : & (A - I)h_1 = 0 \\w^2 : & (A - I)h_2 = bh_1 - B(h_1, h_1) \\w^3 : & (A - I)h_3 = ch_1 - C(h_1, h_1, h_1) - 3B(h_1, h_2)\end{aligned}$$

Cusp

So we first obtain the eigenvectors such that

$$Aq = q, A^T p = p, \langle p, q \rangle = 1,$$

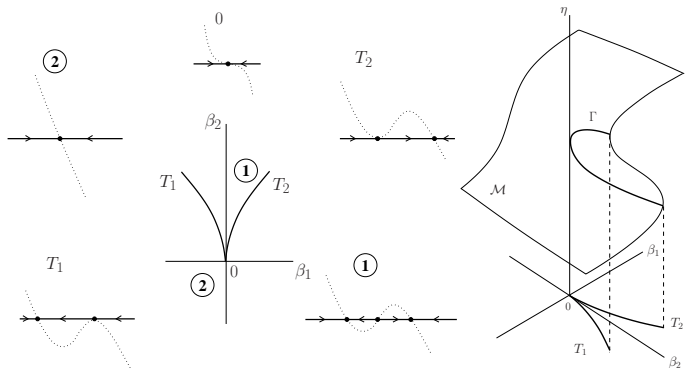
Then higher order terms give

$$b = \langle p, B(q, q) \rangle = 0, \\ h_2 = -(A - I_n)^{INV} B(q, q),$$

and finally the critical normal form coefficient

$$c = \langle p, C(q, q, q) + 3B(q, h_2) \rangle$$

Cusp: Unfolding



Degenerate Period-Doubling

$$Aq = -q, A^T p = -p, \langle p, q \rangle = 1, \quad c = 0$$

The critical normal form

$$w \mapsto G(w) = -w + \left(\frac{1}{6} cw^3 \right) + \frac{1}{120} gw^5 + \dots$$

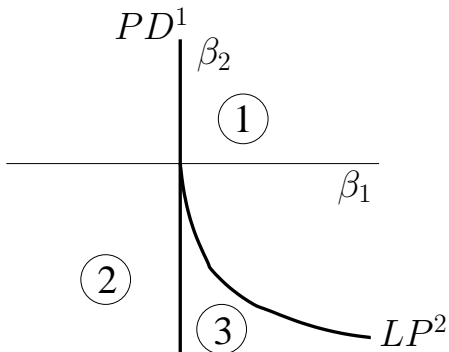
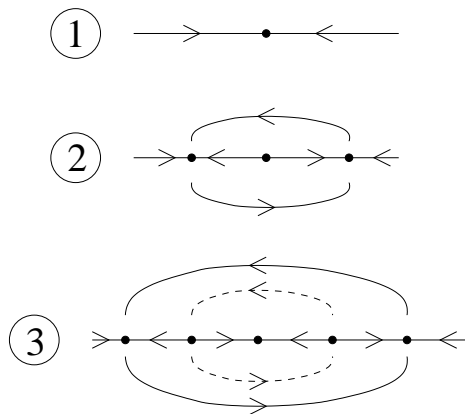
$$H(w) = wq + \frac{w^2}{2} h_2 + \frac{w^3}{6} h_3 + \frac{w^4}{24} h_4 + \frac{w^5}{120} h_5 + \dots$$

where

$$\begin{aligned} h_2 &= -(A - I_n)^{-1} B(q, q) \\ h_3 &= -(A + I_n)^{INV} [C(q, q, q) + 3B(q, h_2)] \\ h_4 &= -(A - I_n)^{-1} [4B(q, h_3) + 3B(h_2, h_2) + \\ &\quad 6C(q, q, h_2) + D(q, q, q, q)] \end{aligned}$$

$$\begin{aligned} g &= \langle p, 5B(q, h_4) + 10B(h_2, h_3) + \\ &\quad 10C(q, q, h_3) + 15C(q, h_2, h_2) + \\ &\quad 10D(q, q, q, h_2) + E(q, q, q, q, q) \rangle \end{aligned}$$

Degenerate Period-Doubling: Unfolding



1:2 Resonance: normalization

The normal form G (including parameters) is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x + y \\ \beta_1 + (-1 + \beta_2)y + c_1x^3 + d_1x^2y \end{pmatrix} + \dots$$

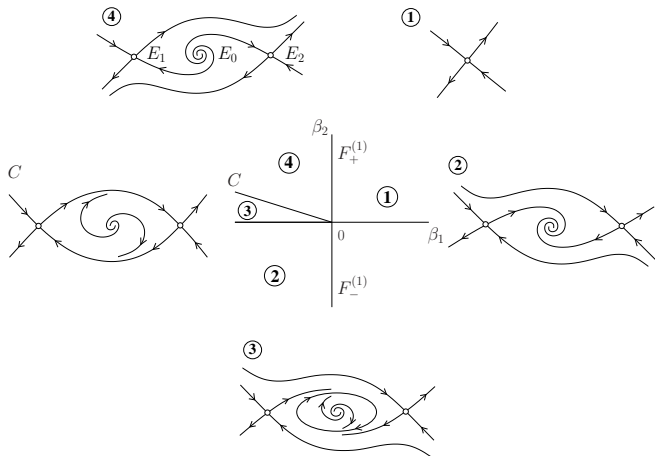
If $c_1 < 0$ a codim 1 branch of Neimark-Sacker bifurcation of double period emanates.

Asymptotic expression of the new branch

$$H^2 : (x^2, y, \beta_1, \beta_2) = \left(-\frac{1}{c_1}, 0, 1, \left(2 + \frac{d_1}{c_2} \right) \right) \varepsilon$$

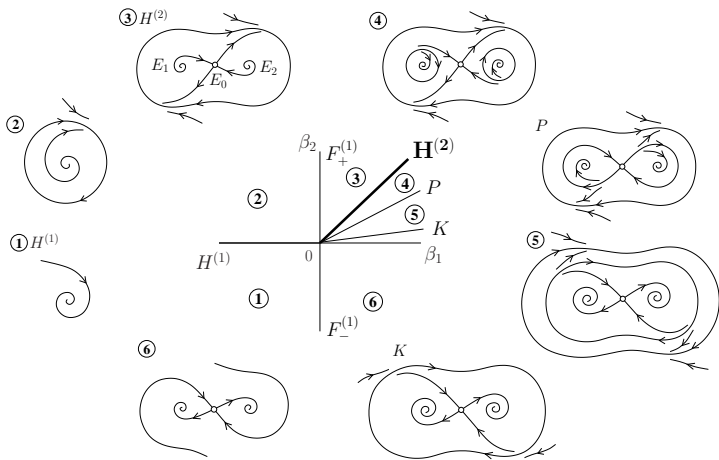
Unfolding $c_1 > 0$

No new local branches



Unfolding $c_1 < 0$:

New codim 1 branch H^2 (local bifurcation)



1:2 Resonance: normalization

Introduce (generalized) eigenvectors:

$$\begin{aligned} Aq_0 &= -q_0, Aq_1 = -q_1 + q_0, \\ A^T p_0 &= -p_0, A^T p_1 = -p_1 + p_0, \\ \langle p_0, q_1 \rangle &= \langle p_1, q_0 \rangle = 1, \quad \langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle = 0. \end{aligned}$$

Collecting the quadratic terms we get

$$\begin{aligned} (A - I_n)h_{20} &= -B(q_0, q_0) \\ (A - I_n)h_{11} &= -B(q_0, q_1) - h_{20} \\ (A - I_n)h_{02} &= -B(q_1, q_1) - 2h_{11} + h_{20} \end{aligned}$$

These are all solvable, since $\lambda = 1$ is not an eigenvalue of A .

1:2 Resonance: Cubic normalization

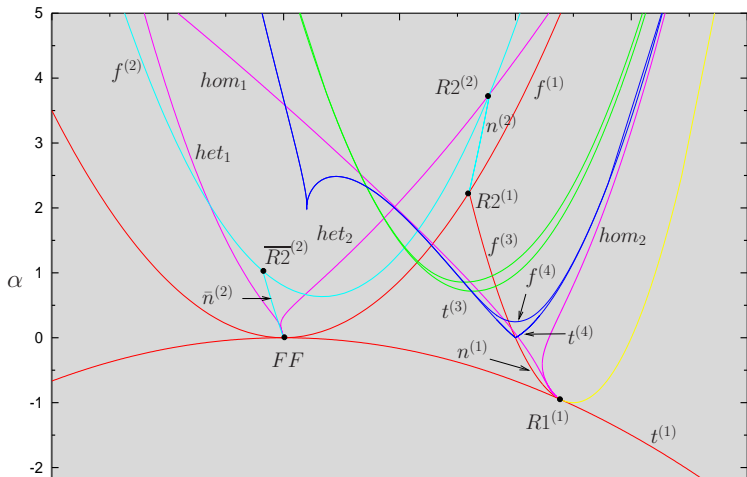
We only need cubic terms to find the coefficients.

$$\begin{aligned}c_1 &= \langle p_0, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle, \\d_1 &= \langle p_0, C(q_0, q_0, q_1) + B(q_1, h_{20}) + 2B(q_0, h_{11}) \rangle \\&\quad + \langle p_1, C(q_0, q_0, q_0) + 3B(q_0, h_{20}) \rangle\end{aligned}$$

Non-degenerate if $c_1 \neq 0$ and $d_1 + c_1 \neq 0$.

Example I: GHM

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ a - b * x - y * y + r * x * y \end{pmatrix}$$



Example II: Adaptive control

Golden&Ydstie(1988):

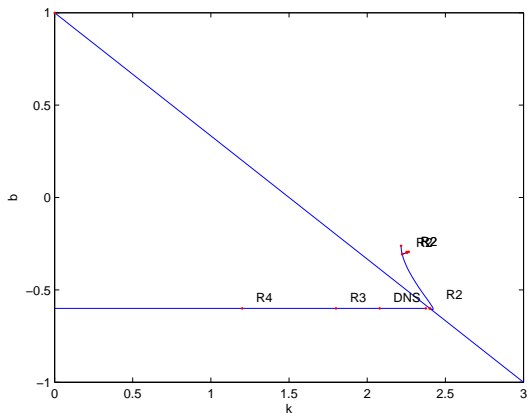
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y \\ bx + k + yz \\ z - \frac{ky}{c+y^2}(bx + k + yz - 1) \end{pmatrix}$$

Unique fixed point

$$x = y = 1, z = 1 - b - k.$$

Loses stability by Period-Doubling or Neimark-Sacker bifurcation.

Example II: Bifurcation Diagram



$c=.5$

Fold-flip

The hypernormal form is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \frac{1}{2}a_0x^2 + \frac{1}{2}b_0y^2 + \frac{1}{6}c_0x^3 + \frac{1}{2}d_0xy^2 \\ -y + xy \end{pmatrix} + \dots$$

Nondegeneracy conditions:

$$a_0 \neq 0, b_0 \neq 0$$

and

$$b_0c_0 - a_0^2b_0 - 3a_0b_0 - a_0d_0 \neq 0.$$

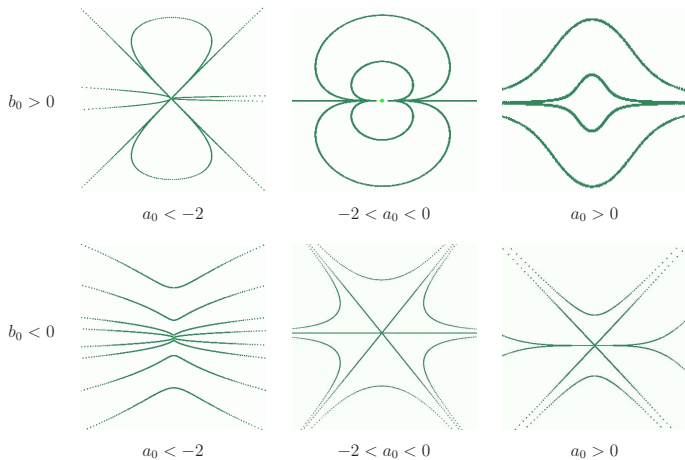
Approximating vectorfield

$$X(x, \mu) = \begin{pmatrix} \mu_1 + \left(-\frac{1}{2}a_0\mu_1 + \mu_2\right)x_1 + \frac{1}{2}a_0x_1^2 + \frac{1}{2}b_0x_2^2 + d_1x_1^3 + d_2x_1x_2^2 \\ \frac{1}{2}\mu_1x_2 - x_1x_2 + d_3x_1x_2^2 + d_4x_2^3 \end{pmatrix} \quad (8)$$

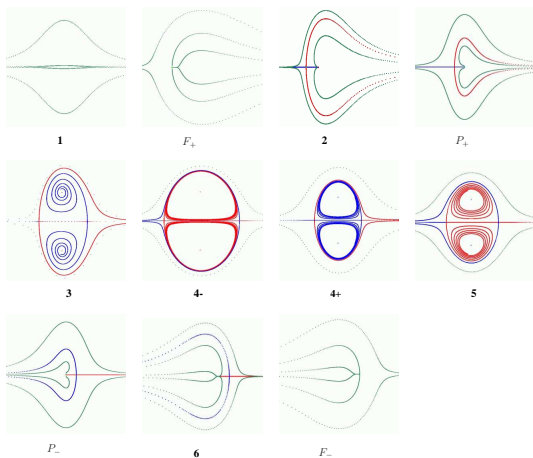
with

$$d_1 = \frac{1}{6} \left(c_0 - \frac{3}{2}a_0^2 \right), \quad d_2 = \frac{1}{2} \left(d_0 + \frac{1}{2}b_0(2 - a_0) \right), \quad d_3 = \frac{1}{4}(a_0 - 2), \quad d_4 = \frac{1}{4}b_0.$$

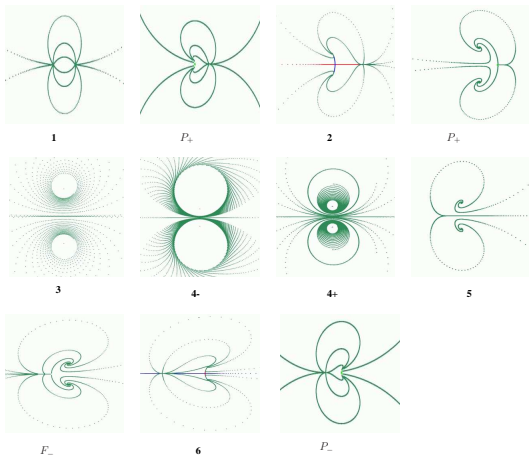
Fold-Flip: Critical Phase portraits



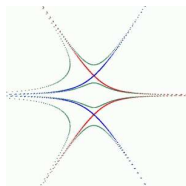
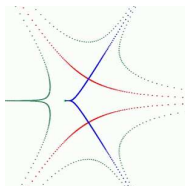
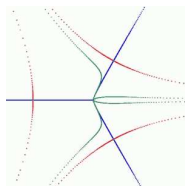
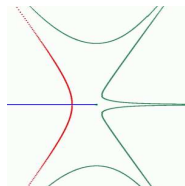
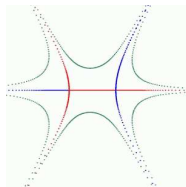
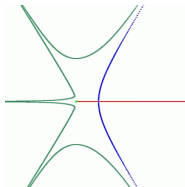
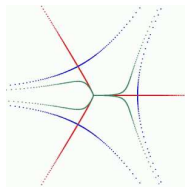
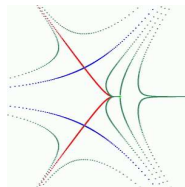
Fold-Flip: Case $a_0, b_0 > 0$



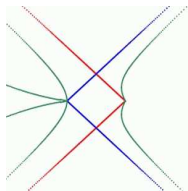
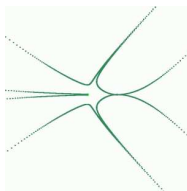
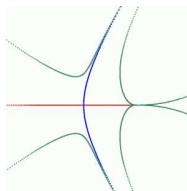
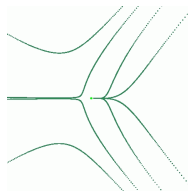
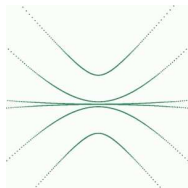
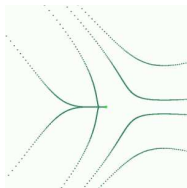
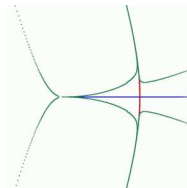
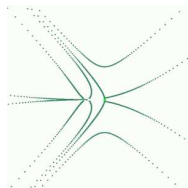
Fold-Flip: Case $a_0 < 0 < b_0, 0$



Fold-Flip: Case $a_0 > 0 > b_0$

**1** F_+ **2** P_+ **3** P_- **4** F_-

Fold-Flip: Case $a_0, b_0 < 0$

**1** P_+ **2** F_+ **3** F_- **4** P_-