The Dynamics of Reaction-Diffusion Patterns

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(Rob Gardner, Tasso Kaper, Yasumasa Nishiura, Keith Promislow, Bjorn Sandstede)
STRUCTURE OF THE TALK

- Motivation
- Topics that won’t be discussed
- Analytical approaches
- Patterns close to equilibrium
- Localized structures
- Periodic patterns & Busse balloons
- Interactions
- Discussion and more ...
MOTIVATION

Reaction-diffusion equations are perhaps the most ‘simple’ PDEs that generate complex patterns

↑

Reaction-diffusion equations serve as (often over-) simplified models in many applications

Examples:

FitzHugh-Nagumo (FH-N) - nerve conduction
Gierer-Meinhardt (GM) - ‘morphogenesi’s’

.........
EXAMPLE: Vegetation patterns

Interaction between plants, soil & (ground) water modelled by 2- or 3-component RDEs.

Some of these are remarkably familiar ...

At the transition to `desertification’ in Niger, Africa.
The **Klausmeier & Gray-Scott (GS) models**

\[
\begin{align*}
W_t &= CW_x - WP^2 + A(1 - W) \quad \text{(Klausmeier)} \\
P_t &= D_p \Delta P + WP^2 - BP \\
\end{align*}
\]

\(W(x, y, t) \leftrightarrow \text{water}, \ P(x, y, t) \leftrightarrow \text{plant biomass}\)

\[
\begin{align*}
U_t &= D_u \Delta U - UV^2 + A(1 - U) \quad \text{(Gray – Scott)} \\
V_t &= D_v \Delta V + UV^2 - BV \\
\end{align*}
\]

\(U(x, y, t), V(x, y, t) \leftrightarrow \text{concentrations}\)

- water flow on hill side \(\leftrightarrow CW_x\)
- horizontal water flow \(\leftrightarrow D_W \Delta W \text{ or } D_W \Delta W^\gamma\)

\(\Rightarrow \text{Klausmeier} \iff \text{GS/GS in porous media: GKGS}\)

[Meran, Rietkerk, Sherratt, ...]
The dynamics of patterns in the GS equation

[J. Pearson (1993), Complex patterns in a simple system]
There is a (very) comparable richness in types of vegetation patterns ...
EXAMPLE: Gas-discharge systems

From: http://www.uni-muenster.de/Physik.AP/Purwins/...

\[ \partial_t u = d_u^2 \Delta u + f(u) - \kappa_3 v - \kappa_4 w + \kappa_1 - \kappa_2 \int u \, d\Omega + \mu (\nabla u)(\nabla u), \]

\[ \tau \partial_t v = d_v^2 \Delta v + u - v - \kappa_1 ' + \kappa_2 ' \int v \, d\Omega , \]

\[ \Theta \partial_t w = d_v^2 \Delta w + u - w , \]

A PARADIGM MODEL \( \uparrow \uparrow \) (Nishiura et al.)

\[ \begin{align*}
U_t &= U_{\xi\xi} + U - U^3 - \varepsilon (\alpha V + \beta W + \gamma), \\
\tau V_t &= \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V, \\
\theta W_t &= \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W ,
\end{align*} \]

In 1D: van Heijster, D, Kaper, Promislow, in 2D: van Heijster, Sandstede
Again from the work (homepage) of the Münster group

Spot interactions in 2 dimensions

As 1D structure: pulse ↔ 2-front
From Peter van Heijster, AD, Tasso Kaper, Keith Promislow

1-dimensional pulses appearing from N-front dynamics.

PDE dynamics reduce to N-dim ODEs for front positions

($\alpha, \beta, \gamma, D, \varepsilon$) = (6, -3, -1, 5, 0.1)

($\alpha, \beta, \gamma, D, \varepsilon$) = (2, -1, -0.25, 5, 0.01)

**SEMI-STRONG INTERACTIONS**
TOPICS THAT WON’T BE DISCUSSED:

- **SCALAR EQUATIONS**

\[ U_t = \Delta U + F(U), \]

\[ U(x, t) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}, \Omega \subset \mathbb{R}^d. \]

‘Tools’:
- Maximum principles
- Gradient structure

‘Waves in random media’ [Berestycki, Hamel, Xin, ...]
• GRADIENT FLOWS, such as

* the Cahn-Hillard equation ($\leftrightarrow$ interface dynamics),

$$U_t = -\Delta((\varepsilon^2)\Delta U + F(U)),$$

$U(x, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}, \; \Omega \subset \mathbb{R}^2$, and

* the real Ginzburg-Landau equation ($\leftrightarrow$ defects),

$$(U_t = ) \Delta U + U - |U|^2 U \; (= 0),$$

$U(x, t) : \Omega \times \mathbb{R}^+ \to \mathbb{C}, \; \Omega \subset \mathbb{R}^2.$

[Fife, Brezis, Nishiura, Sternberg, ...]
• INTERFACE DYNAMICS in 2D (curvature!)

* in gradient systems (↔ Cahn-Hilliard)

* in singularly perturbed ‘excitable’ systems

\[
\begin{align*}
U_t &= \Delta U + F(U, V) \\
V_t &= \delta \Delta V + \varepsilon G(U, V)
\end{align*}
\]

\(U, V : \Omega \times \mathbb{R}^+ \to \mathbb{R}, \ \Omega \subset \mathbb{R}^2, \ 0 < \varepsilon, \delta \ll 1,\)

* in general

• BOUNDARY EFFECTS

[Fife, ‘Japanese school’ (Mimura, Nishiura, ...), Sandstede, Scheel, ...]
ANALYTICAL APPROACHES

Restriction/Condition: ‘We’ want explicit control on the nature/structure of the solutions/patterns

⇒

• Study solutions ‘near’ simple patterns

→ Modulated patterns & modulation equations.

• Study equations ‘close to’ simple equations (??)

→ (Singularly) perturbed equations & near-gradient/near-integrable systems

(nonlinear Schrödinger ↔ complex Ginzburg-Landau)
* **SOLUTIONS NEAR SIMPLE PATTERNS**

* ‘Weakly nonlinear stability theory’

(\iff\ evolution of small patterns near a weakly unstable trivial state)

→ the **complex Ginzburg-Landau equation** (and more).

* **Modulated wave trains**

(\iff\ dynamics of almost spatially periodic patterns)

→ the **Burgers equation**, the **Korteweg-de Vries equation**, the **Kuramoto-Sivashinsky equation**, ...

* **Modulated localized structures.**

[Eckhaus, Newell, Schneider, Kopell, van Harten, D, Sandstede, Scheel, ...]
EQUATIONS NEAR SIMPLE EQUATIONS

* SINGULARLY PERTURBED RDEs

Natural assumption: \((U, V)\) are bounded on \(\mathbb{R}^d\). Then,

\[
\begin{align*}
U_t &= \Delta U + F(U, V) \\
V_t &= \varepsilon^2 \Delta V + G(U, V)
\end{align*}
\]

\[
\begin{align*}
\varepsilon^2 U_t &= \tilde{\Delta} U + \varepsilon^2 F(U, V) \\
V_t &= \tilde{\Delta} V + G(U, V)
\end{align*}
\]

with \(0 < \varepsilon^2 = \frac{D_V}{D_U} \ll 1 \leadsto U \approx U_0\), constant & \(V\) solves

\[
V_t = \tilde{\Delta} V + G(U_0, V)
\]

a scalar equation.

Nevertheless, SP-RDEs exhibit the dynamics of systems.
PATTERNS CLOSE TO EQUILIBRIUM

EXAMPLE: 2-component systems in $\mathbb{R}^1$, 

\[
\begin{align*}
U_t &= U_{xx} + F(U, V) \\
V_t &= D V_{xx} + G(U, V)
\end{align*}
\]

A ‘trivial pattern’ $(U(x, t), V(x, t)) \equiv (U_0, V_0)$ solves 

\[
F(U_0, V_0) = G(U_0, V_0) = 0.
\]

Its linear stability is determined by setting 

\[
(U(x, t), V(x, t)) = (U_0, V_0) + (\alpha, \beta) e^{ikx+\lambda(k^2)t}
\]

with $k \in \mathbb{R}$, $(\alpha, \beta) \in \mathbb{R}^2$. The eigenvalues $\lambda_{1,2}(k^2) \in \mathbb{C}$ can be computed explicitly as functions of $k^2$. 
Two typical pattern-generating bifurcations

Small amplitude patterns at near-criticality are described by a modulation equation for the complex amplitude $A$, where $A = A(\xi, \tau)$ is related to $(U, V)$ by

$$(U(x, t), V(x, t)) = (U_0, V_0) + \varepsilon A e^{ik_c x + \lambda_c t}(\alpha_c, \beta_c) + \text{c.c.} + \text{h.o.t.}$$

[Note. Turing-Hopf: no reversibility (GKGS), $k_c, \lambda_c \neq 0$]
**Turing:** Evolution of $A$ is described by the rGL,

$$A_T = A_{\xi\xi} + A \pm |A|^2 A.$$ 

**(Turing-)Hopf:** Evolution of $A$ is described by the cGL,

$$A_T = (1 + ia)A_{\xi\xi} + A \pm (1 + ib)|A|^2 A.$$ 

[proofs of validity by Schneider]

**Turing:** Dynamics of patterns fully understood (near-criticality).

**(Turing-)Hopf:** Stable periodic patterns for $\pm \rightarrow -$ and 

$$1 + ab > 0 \quad (\text{Benjamin} - \text{Feir/Newell})$$

**Q:** Dynamics small amplitude patterns if $1 + ab < 0$??
cGL analysis in GKGS model

\[
\begin{align*}
    U_t &= U_{xx}^\gamma + CU_x + A(1 - U) - UV^2 \\
    V_t &= \delta^2\sigma V_{xx} - BV + UV^2,
\end{align*}
\]

With
- \( \delta^\sigma \ll 1 \): ratio spreading speed plants:water
- nonlinear diffusion \( \gamma \geq 1 \) (mostly \( \gamma = 1 \) or 2)
- \( A \) main parameter \( \sim \) yearly precipitation
- \( C \sim \) slope, \( B \sim \) mortality plants

For given \( B, C \) a Turing \( (C = 0) / \text{Turing-Hopf} \ (C \neq 0) \) bifurcation takes place at \( A_{T(H)} \) (for decreasing \( A \))

\[ \rightarrow \text{A cGL analysis near } A = A_{T(H)}(B, C) \]

[van der Stelt, D., Hek, Rademacher]
\[ A_\tau = (a_1 + ia_2)A_{\xi\xi} + (b_1 + ib_2)A + (L_1 + iL_2)|A|^2A \]

\[ \rightarrow L_1 = L_1(B,C) \quad \& \quad L_1(B,C) < 0 \leftrightarrow \pm \rightarrow - \text{(patterns)} \]

\[ \gamma = 1 \quad \text{left} \quad \gamma = 2 \quad \text{right} \]

**B-F/N also OK:** always stable patterns at onset (!?)

**Note** \( C = 0: \)

\[ A_\tau = 2\sqrt{2}A_{\xi\xi} + b_1(\gamma)A + L_1(\gamma)|A|^2A \quad \text{with} \]

\[ b_1(\gamma) = [-39 + 27\sqrt{2} + (41 - 29\sqrt{2})\gamma] \left( \frac{g\gamma}{b} \right)^{\frac{1}{1+\gamma}} \frac{1}{b} \]

\[ L_1(\gamma) = -\frac{1}{9}(2 - \sqrt{2}) \left[ 18(3 + 2\sqrt{2}) + 12(2 + \sqrt{2})\gamma + (-8 + 3\sqrt{2})\gamma^2 \right] \left( \frac{g\gamma}{b} \right)^{\frac{2}{1+\gamma}} b^3 \]
Q: NEAR-CRITICAL PATTERN FORMATION IN \( \mathbb{R}^2 \)??

\[ \text{(spatial symm.)} \]

\[ \text{(Turing)} \]

\( \Re(\lambda) > 0 \)

CANNOT BE COVERED BY A 2-D cGL,

\[ A_t = D_{11} A_{\xi\xi} + D_{12} A_{\xi\eta} + D_{22} A_{\eta\eta} + A \pm (1 + ib) |A|^2. \]

NOTE: Even the GL-extension of the system of coupled amplitude equations for hexagonal patterns only covers a small part of the ring of unstable ‘modes’.
LOCALIZED STRUCTURES

Far-from-equilibrium patterns that are ‘close’ to a trivial state, except for a small spatial region.

A (simple) pulse in GS

A 2-pulse or 4-front in a 3-component model
Pulses/fronts correspond to homo-/hetero-clinic orbits.

Prototypical example (that drove the development of ‘geometric singular perturbation theory’ [Fenichel, Jones, ...]):

**FitzHugh-Nagumo**

Construct 1-pulse, or 2-front homoclinic orbit in a 3-D singularly perturbed system
\[ \gamma_{\text{hom}}(\xi) \subset W^u(P) \cap W^s(P) \]

(= 3-D \cap 3-D in 6-D space)
SPECTRAL STABILITY

EXAMPLE: 2-component system on $\mathbb{R}^1$.

$$(U(x, t), V(x, t)) = (U_{\text{hom}}(x), V_{\text{hom}}(x)) + (u(x), v(x))e^{\lambda t}$$

$$\Rightarrow \mathcal{L}(x) \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

Introduce $\Phi(x) = (u, u_x, v, v_x)$, then

$$\Phi_x = \mathcal{A}(x; \lambda) \Phi,$$

with $\mathcal{A}$ a $4 \times 4$ matrix with $\text{Tr} \mathcal{A} = 0$, and

$$\lim_{x \to \pm \infty} \mathcal{A}(x; \lambda) = \mathcal{A}_\infty(\lambda)$$
Let \( \{ \Phi_1(x; \lambda), \Phi_2(x; \lambda), \Phi_3(x; \lambda), \Phi_4(x; \lambda) \} \) be 4 independent solutions so that

\[
\lim_{x \to -\infty} \Phi_{1,2}(x; \lambda) = 0, \quad \lim_{x \to +\infty} \Phi_{3,4}(x; \lambda) = 0
\]

(this is possible for \( \lambda \notin \sigma_{\text{ess}} \)). The Evans function associated to this stability problem is defined by

\[
\mathcal{D}(\lambda) = \det [\Phi_1(x; \lambda), \Phi_2(x; \lambda), \Phi_3(x; \lambda), \Phi_4(x; \lambda)]
\]

- \( \mathcal{D} \) does not depend on \( x \)
- \( \mathcal{D} \) is analytic as function of \( \lambda \) for \( \lambda \notin \sigma_{\text{ess}} \)
- \( \mathcal{D} = 0 \Leftrightarrow \lambda \) is an eigenvalue

[Evans, Alexander, Gardner, Jones, Pego, Weinstein]
If the system is singularly perturbed, $\mathcal{D}(\lambda)$ can be decomposed,

$$\mathcal{D}(\lambda) = \mathcal{D}_{\text{fast}}(\lambda)\mathcal{D}_{\text{slow}}(\lambda)$$

- $\mathcal{D}_{\text{fast}}(\lambda)$ is analytic for $\lambda \notin \sigma_{\text{ess}}$;
- $\mathcal{D}_{\text{slow}}(\lambda)$ is meromorphic.
- the zeroes of $\mathcal{D}_{\text{fast}}(\lambda)$ are given by a scalar problem and can be determined; some of these correspond to poles of $\mathcal{D}_{\text{slow}}(\lambda)$
- the zeroes of $\mathcal{D}_{\text{slow}}(\lambda)$ can be determined by a Melnikov-like approach

[D,Gardner,Kaper, ..,Veerman]
What about localized 2-D patterns?

Spots, stripes, ‘volcanoes’, ..... most (all?) existence and stability analysis done for (or ‘close to’) ‘symmetric’ patterns

(Again) \( \text{PDE} \sim \text{ODE-analysis} \)

Note, however: **polar/spherical** symmetries,

\[
\Delta \rightarrow \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r},
\]

an inhomogeneous term with singularity at \( r = 0 \).

[Ward,Wei,Winter, van Heijster & Sandstede, ....]
The ‘volcano/ring-patterns are (fairly) well-understood

[Pearson, Swinney et al. 1994]

[Morgan & Kaper, 2004]
PERIODIC PATTERNS & BUSSE BALLOONS

A natural connection between periodic patterns near criticality and far-from-equilibrium patterns

Region in (k,R)-space in which STABLE periodic patterns exist

[Busse, 1978] (convection)
A Busse balloon for the GS model

‘Fall of patterns’ at k=0

[D, Rademacher & van der Stelt, ’12]
Periodic patterns near $k=0$: singular localized pulses (of vegetation pattern kind)

Coexisting stable patterns (for the same parameter values)
What do we know analytically?

- **Near onset/the Turing bifurcation**: ‘full analytical control’ through Ginzburg-Landau theory.

- A complete classification of the generic character of the boundary of the Busse balloon [Rademacher & Scheel, ’07].

- **Near the ‘fall of patterns’**: existence and stability of singular patterns [D, Gardner & Kaper, ’01; van der Ploeg & D, ’05; D, Rademacher & van der Stelt ’12].

No further general insight in (the boundary of) the Busse balloon.
A spin-off: the Hopf dance, a novel fine-structure

A ‘dance’ of intertwining Hopf bifurcations.

The homoclinic \((k=0)\) ‘oasis’ pattern is the last to destabilize (Ni’s conjecture)
Two types of Hopf bifurcations?

Why only these two?
Spectral analysis

STABILITY: ‘Solution’ = ‘Pattern’ + ‘Perturbation’

- LINEARIZATION: ‘Perturbation’ = $P(x)e^{\lambda t}$, $\lambda \in \mathbb{C}$.
- INSTABILITY: There is a $\lambda$ s.t. $\text{Re}(\lambda) > 0$.
- FACT: $\lambda = \{\Lambda_i(s), s \in [-1, 1], i = 1, 2, ..., N/\infty\}$.

Note: ±1 endpoints correspond to $\mathcal{H}_{\pm 1}$ Hopf bifurcations.
The long wavelength limit ($k \sim 0$)

- The critical spectral branch $\Lambda_h(s)$ ‘unrolls’.
- The ‘oasis’ state is the last pattern to destabilize.
- $\Lambda_h(s)$ shrinks and ROTATES as $k \to 0$.

By: Evans function for periodic patterns [Gardner, Zumbrun, D & van der Ploeg]
A novel general insight in the ‘fall of patterns’

In a general class – well, … – of reaction-diffusion models:

- The **homoclinic ‘oasis’ pattern** is the last pattern to become unstable (↔Ni’s conjecture).

- The **Hopf dance**: near the destabilization of the homoclinic pattern, the Busse balloon has a ‘**fine structure**’ of two intertwining curves of Hopf bifurcations.

[D, Rademacher & van der Stelt, ’12]
The spectral branch is only to leading order a straight line/an interval.

In general it will be (slightly) bent.

This may yield small regions of ‘internal Hopf destabilizations’ and the corners in the boundary of the BB will disappear ↔ the orientation of the belly.
A more typical Busse balloon?

Or more generic (?): sometimes a co-dimension 2 intersection, sometimes an ‘internal Hopf bridge’?
This is however not the case. In the class of considered model systems, a BELLY DANCE takes place.

The belly always points away from the Im-axis near the ‘corner’ at which +1 and -1 cross at the same time.
WHY??

The theory includes in essence ‘all explicit models in the literature’

(↔ Gray-Scott/Klausmeier, Gierer-Meinhard, Schnakenberg, gas-discharge, ….)

HOWEVER, if one looks carefully it’s clear these models are in fact very special.

All these prototypical systems exhibit patterns that are only ‘locally nonlinear’ (?)
WHAT?

\[(\text{GM}) \quad \begin{cases} U_t &= U_{xx} - \mu U + V^2 \\ V_t &= \varepsilon^2 V_{xx} - V + \frac{V^2}{U} \end{cases}\]

\[(\text{GS}) \quad \begin{cases} U_t &= U_{xx} + A(1 - U) - UV^2 \\ V_t &= \varepsilon^2 V_{xx} - BV + UV^2 \end{cases}\]

These equations share special non-generic features.

⇒ Consider the ‘slow’ and ‘fast’ reduced limits.
THE MOST GENERAL MODEL:

- Reaction-diffusion equation.
- Two-components, \( U(x, t) \) \& \( V(x, t) \).
- On the unbounded domain: \( x \in \mathbb{R}^1 \).
- A stable background state \((U, V) \equiv (0, 0)\).
- Singularity perturbed: \( U(x, t) \) ‘slow’, \( V(x, t) \) ‘fast’.

\[
\begin{align*}
U_t &= U_{xx} + \mu_{11}U + \mu_{12}V + F(U, V; \varepsilon) \\
V_t &= \varepsilon^2 V_{xx} + \mu_{21}U + \mu_{22}V + G(U, V; \varepsilon)
\end{align*}
\]

* with \( \mu_{11} + \mu_{22} < 0 \) and \( \mu_{11}\mu_{22} - \mu_{12}\mu_{21} > 0 \).
* some technical conditions on \( F(U, V) \) and \( G(U, V) \).
THE SLOW REDUCED LIMIT: $\varepsilon = 0$, $V(x, t) \equiv 0$.

The slow fields:

(GM) $U_t = U_{xx} - \mu U$

(GS) $U_t = U_{xx} + A(1 - U)$

GENERAL $U_t = U_{xx} + \mu_{11}U + F(U, 0; 0)$

GM/GS: LINEAR, $F(U, 0; 0) \equiv 0!!$

Crucial for stability analysis & for Hopf/belly dance
Consider existence and stability of pulses in generic singularly perturbed systems (i.e. systems that are also nonlinear outside the localized fast pulses)

→ Significant extension Evans function approach (↔ Frits Veerman)

A Gierer-Meinhardt equation with a ‘slow non-linearity’

A chaotically oscillating standing pulse??
Busse balloons in the GKGS model

\[
\begin{align*}
U_t &= U_{xx}^{\gamma} + C U_x + A(1-U) - UV^2 \\
V_t &= \delta^{2\sigma} V_{xx} - BV + UV^2,
\end{align*}
\]

\(\gamma = 1, \quad B = C = 0.2\)

\(\gamma = 2\)

[van der Stelt, D., Hek, Rademacher]
Some spectral plots near the cusp

Many, many open questions about structure & nature of Busse balloons in RD-systems

→ new project D & Rademacher
What about (almost) periodic 2-D patterns?

DEFECT PATTERNS

Slow modulations of (parallel) stripe patterns + localized defects

Phase-diffusion equations with defects as singularities

[Cross, Newell, Ercolani, ....]
INTERACTIONS (OF LOCALIZED PATTERNS)

A hierarchy of problems

- Existence of stationary (or uniformly traveling) solutions
- The stability of the localized patterns
- The INTERACTIONS

Note: It’s no longer possible to reduce the PDE to an ODE
WEAK INTERACTIONS

General theory for exponentially small tail-tail interactions

[Ei, Promislow, Sandstede]

\[
\frac{d}{dt} \Gamma = C_1 e^{-C_2 \Gamma} \quad \text{at leading order, for } \Gamma \text{ large enough}
\]

Essential: components can be treated as ‘particles’

\[
\vec{U}(x, t) = \vec{U}_h(x + \frac{1}{2} \Gamma) + \vec{U}_h(x - \frac{1}{2} \Gamma)
\]

is solution of the PDE up to exponentially small terms
SEMI-STRONG INTERACTIONS

• Pulses evolve and change in magnitude and shape.

• Only $O(1)$ interactions through one component, the other components have negligible interactions.

‘Gap’ in decay rates $\Leftrightarrow$ PDE is singularly perturbed
Pulses are no ‘particles’ and may ‘push’ each other through a ‘bifurcation’.

Semi-strong dynamics in two (different) modified GM models

finite-time blow-up

a symmetry breaking bifurcation

[D. & Kaper ’03]
Example: Pulse-interactions in (regularized) GM

Existence and Stability

Theorem. [Doelman, Gardner, Kaper]
Let $\varepsilon$ be small enough.
- For $0 < \mu \ll \frac{1}{\varepsilon^4}$ there is a homoclinic pulse solution $(U_h(x), V_h(x)) = \Phi_h(x)$.
- For $\mu > \mu_{\text{Hopf}}$ the pulse is spectrally stable.
The construction of the 2-pulse $\Phi_\Gamma(x)$

- 2 different ODE reductions with (unknown) speeds $\pm c$: one at each 'fast' $V$-pulse;
- outside 'fast' regions $c$ is negligible $O(\varepsilon^6)$ effect: solve the 'slow' $U$-eqs.

- distance between pulses $= 2\Gamma(t) = 2 \int_{t_0}^{t} c(s) \, ds = \text{‘time-of-flight’ } P_1 \rightarrow P_2 = F(c) = F(\frac{d}{dt} \Gamma(t))$
\[ \Rightarrow \frac{d}{dt} \Gamma = \frac{1}{2} \varepsilon^2 \sqrt{\mu} e^{-2\varepsilon^2 \Gamma} \sqrt{\mu} \]

\[ \sup U_\Gamma = A(\Gamma) \]
\[ \sup V_\Gamma = \frac{3}{2} A(\Gamma) \]

\[ A(\Gamma) = \frac{\sqrt{\mu}}{3} \frac{1}{1 + e^{-2\varepsilon^2 \Gamma} \sqrt{\mu}} \]

\textbf{Intrinsically formal result} \cite{Doelman, Kaper, Ward}

\textbf{Note:} \( \Gamma \gg 1/\varepsilon^2 \rightarrow \) the weak interaction limit:

\[ \frac{d}{dt} \Gamma = \frac{1}{2} \varepsilon^2 \sqrt{\mu} e^{-2\varepsilon^2 \Gamma} \sqrt{\mu} \quad \text{and} \quad A(\Gamma) = \frac{\sqrt{\mu}}{3}, \text{ constant} \]

\( (2 \ 'copies' \ of \ the \ stationary \ pulses) \)
Stability of the 2-pulse solution:

Q: What is ‘linearized stability’?

A: ‘Freeze’ solution and determine ‘quasi-steady eigenvalues’

Note: ‘not unrealistic’, since 2-pulse evolves slowly

Two pairs of eigenvalues ‘travel’ through $C$ as function of the distance $\Gamma$ between the pulses, and approach the eigenvalues of the stationary 1-pulse solutions as $\Gamma \to \infty$. 
The **Evans function** approach can be used to **explicitly** determine the paths of the eigenvalues.

\[ \sigma_{\text{ess}} \]

\[ -1 \quad -0.75 \quad -0.5 \]

\[ \Uparrow \quad \Gamma \to \infty \]

\[ \lambda_j(\Gamma(t)) > 0 \quad \text{for} \quad \Gamma(t) < \Gamma^* \]

**Note:** \[ \Gamma^* = \frac{\log 3}{\epsilon^2 \sqrt{\mu}} \quad \text{for} \quad \mu > \mu_t > \mu_{\text{Hopf}} \]
Nonlinear Asymptotic Stability & Validity

**Theorem** [Doelman, Kaper, Promislow]

Define $W(x, t)$ by

$$(U(x, t), V(x, t)) = \Phi_{\Gamma(t)}(x) + W(x, t).$$

Let $\varepsilon > 0$ be sufficiently small, $\mu > \mu_{\text{Hopf}}$, and assume that $(U_0(x), V_0(x))$ is sufficiently close to $\Phi_{\Gamma(0)}(x)$ with $\Gamma(0) > \Gamma^*$. Then there exist $M, \nu > 0$ such that

$$\|W\|_X \leq M(e^{-\nu t}\|W_0\|_X + \varepsilon^3)$$

with $\|W\|_X = \varepsilon\|W_1\|_{L^2} + \frac{1}{\varepsilon}\|\partial_x W_1\|_{L^2} + \|W_2\|_{H^1}$

**Proof:** Renormalization Group Method
THE 3-COMPONENT (gas-discharge) MODEL

\[
\begin{align*}
U_t &= U_{\xi\xi} + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma), \\
\tau V_t &= \frac{1}{\varepsilon^2} V_{\xi\xi} + U - V, \\
\theta W_t &= \frac{D^2}{\varepsilon^2} W_{\xi\xi} + U - W,
\end{align*}
\]

[Peter van Heijster, AD, Tasso Kaper, Keith Promislow ’08,’09,’10]
Simple and explicit results on existence and stability

**Theorem**

*Our system possesses a standing pulse if there exists an* $A \in (0, 1)$ *which solves*

$$\alpha A^2 + \beta A^{\frac{2}{D}} = \gamma.$$  

*Moreover, if* $|\alpha D| > |\beta|$  *and* $\text{sgn}(\alpha) \neq \text{sgn}(\beta)$, *then there is a saddle-node bifurcation of homoclinic orbits at* $\gamma = \gamma_{SN}$.

**Theorem**

*The standing pulse with* $O(1)$-*parameters is stable if and only if*

$$\alpha A^2 + \frac{\beta}{D} A^{\frac{2}{D}} > 0.$$
Sub- and supercritical bifurcations into traveling pulses

$$\tau = \mathcal{O}(1/\varepsilon^2) = \hat{\tau}/\varepsilon^2, \text{ speed } = \mathcal{O}(\varepsilon^2) = \varepsilon^2 c$$

Bifurcation diagrams for two typical parameter combinations

(There is an explicit analytical expression for $\hat{\tau}^*$, etc)
Interaction between Hopf and bifurcation into traveling pulse

\[ \tau = \mathcal{O}(1/\varepsilon^2) \]

Simulations for two typical parameter combinations
Front interactions: similar validity/reduction results

\[ \dot{\Gamma}_i(t) = (-1)^{i+1} \frac{3}{2} \sqrt{2\varepsilon} \left[ \gamma + \alpha \left( -e^{\varepsilon(\Gamma_1-\Gamma_i)} + \ldots + (-1)^{i-1} e^{\varepsilon(\Gamma_{i-1}-\Gamma_i)} \right) + (-1)^i e^{\varepsilon(\Gamma_i-\Gamma_{i+1})} + \ldots + (-1)^{N-1} e^{\varepsilon(\Gamma_i-\Gamma_N)} \right] + \beta \left( -e^{\frac{\varepsilon}{\tau_D}(\Gamma_1-\Gamma_i)} + \ldots + (-1)^{i-1} e^{\frac{\varepsilon}{\tau_D}(\Gamma_{i-1}-\Gamma_i)} + (-1)^i e^{\frac{\varepsilon}{\tau_D}(\Gamma_i-\Gamma_{i+1})} + \ldots + (-1)^{N-1} e^{\frac{\varepsilon}{\tau_D}(\Gamma_i-\Gamma_N)} \right) \] for \( i = 1 \ldots N. \)

The formation of a 5-front traveling wave
\[
\begin{align*}
N &= 4 \\
\dot{\Gamma}_1(t) &= \frac{3}{2} \sqrt{2}\varepsilon \left( \gamma + \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_2) + e^\varepsilon(\Gamma_1-\Gamma_3) - e^\varepsilon(\Gamma_1-\Gamma_4) \right) \\
&\quad + \beta \left( -e^\delta(\Gamma_1-\Gamma_2) + e^\delta(\Gamma_1-\Gamma_3) - e^\delta(\Gamma_1-\Gamma_4) \right) \right), \\
\dot{\Gamma}_2(t) &= -\frac{3}{2} \sqrt{2}\varepsilon \left( \gamma + \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_2) + e^\varepsilon(\Gamma_2-\Gamma_3) - e^\varepsilon(\Gamma_2-\Gamma_4) \right) \\
&\quad + \beta \left( -e^\delta(\Gamma_1-\Gamma_2) + e^\delta(\Gamma_2-\Gamma_3) - e^\delta(\Gamma_2-\Gamma_4) \right) \right), \\
\dot{\Gamma}_3(t) &= \frac{3}{2} \sqrt{2}\varepsilon \left( \gamma + \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_3) + e^\varepsilon(\Gamma_2-\Gamma_3) - e^\varepsilon(\Gamma_3-\Gamma_4) \right) \\
&\quad + \beta \left( -e^\delta(\Gamma_1-\Gamma_3) + e^\delta(\Gamma_2-\Gamma_3) - e^\delta(\Gamma_3-\Gamma_4) \right) \right), \\
\dot{\Gamma}_4(t) &= -\frac{3}{2} \sqrt{2}\varepsilon \left( \gamma + \alpha \left( -e^\varepsilon(\Gamma_1-\Gamma_4) + e^\varepsilon(\Gamma_2-\Gamma_4) - e^\varepsilon(\Gamma_3-\Gamma_4) \right) \\
&\quad + \beta \left( -e^\delta(\Gamma_1-\Gamma_4) + e^\delta(\Gamma_2-\Gamma_4) - e^\delta(\Gamma_3-\Gamma_4) \right) \right). 
\end{align*}
\]
DISCUSSION AND MORE ....

There is a well-developed theory for ‘simple’ patterns (localized, spatially periodic, radially symmetric, ...) in ‘simple’ equations.

In 1 spatial dimension ‘quite some’ analytical insight can be obtained, but more complex dynamics are still beyond our grasp ...

Challenges:

- Defects in 2-dimensional stripe patterns
- Strong pulse interactions (1 D!)
- ....
The GS equation perhaps is one of the most well-studied reaction-diffusion equations of the last decades. It’s mostly famous for exhibiting ‘self-replication dynamics’.
The pulse self-replication mechanism

A generic phenomenon, originally discovered by Pearson et al in ’93 in GS. Studied extensively, but still not understood.

[Pearson, Doelman, Kaper, Nishiura, Muratov, Ward, ....]
And there is more, much more ...

A structurally stable Sierpinsky gasket ...

[Ohta, in GS & other systems]
Various kinds of spot-, front-, stripe-interactions in 2D

[van Heijster, Sandstede]