Stability of Travelling Waves

Waves: Spectrum & Evans Function

Arthur Vromans

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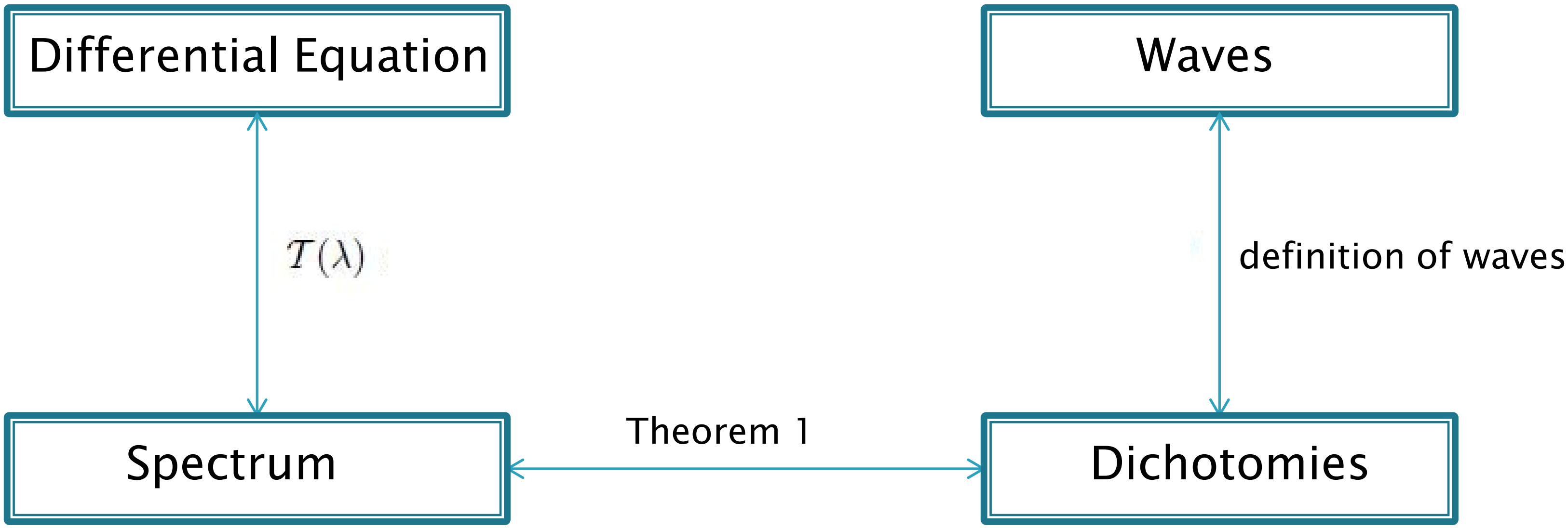
• Author of *Stability of travelling waves (2002)*

Last Lecture



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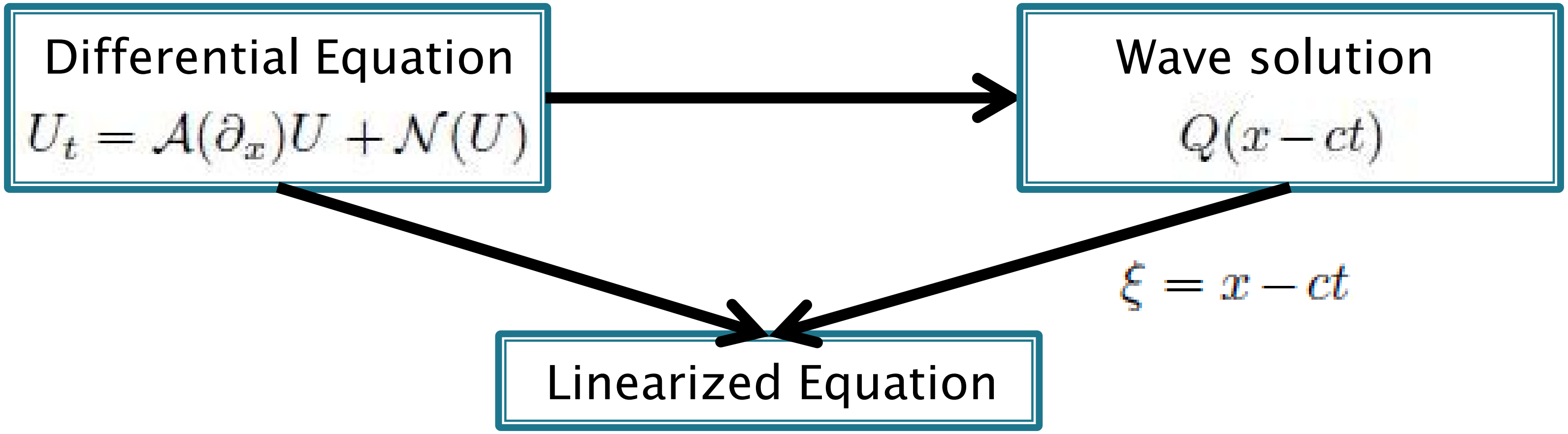
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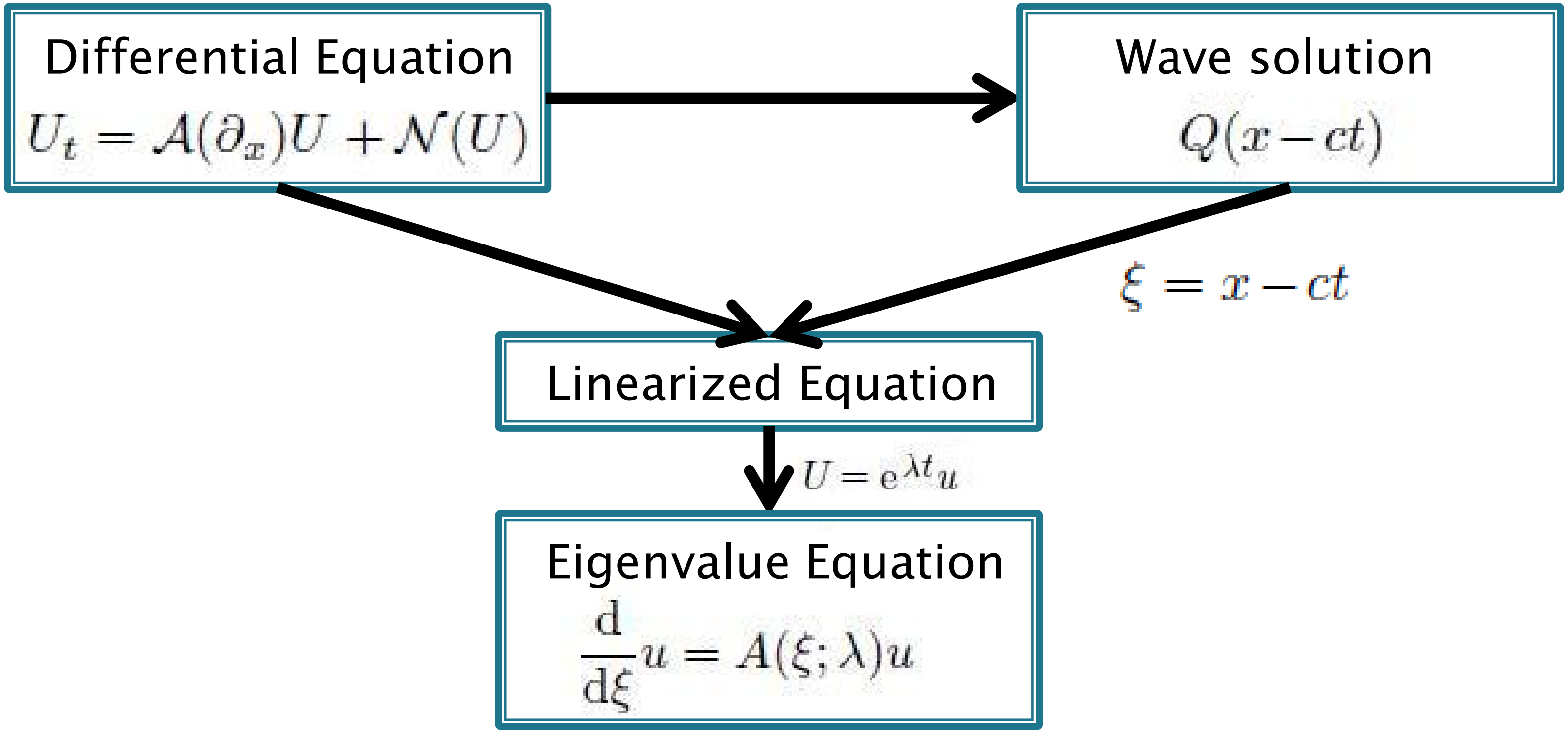
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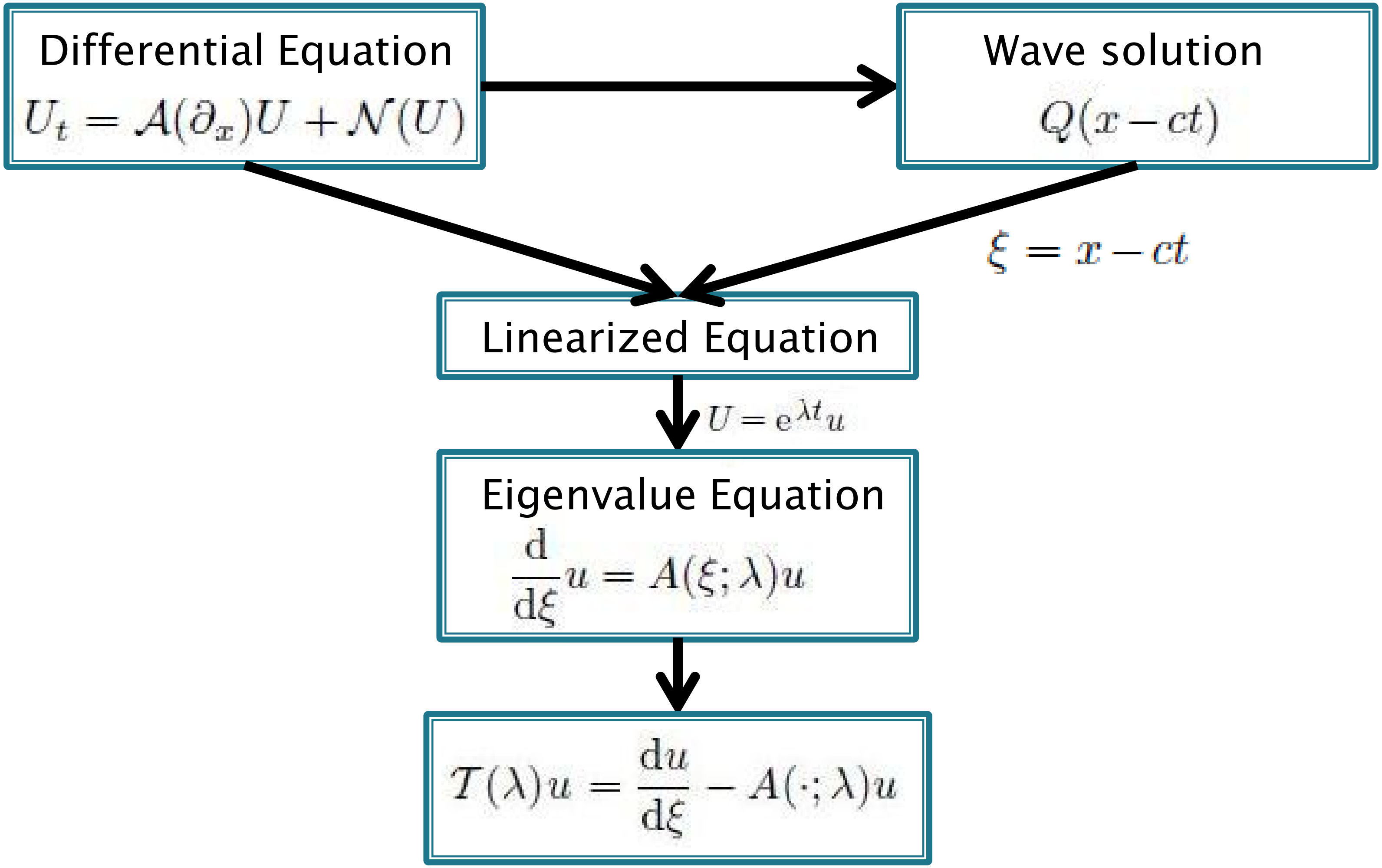
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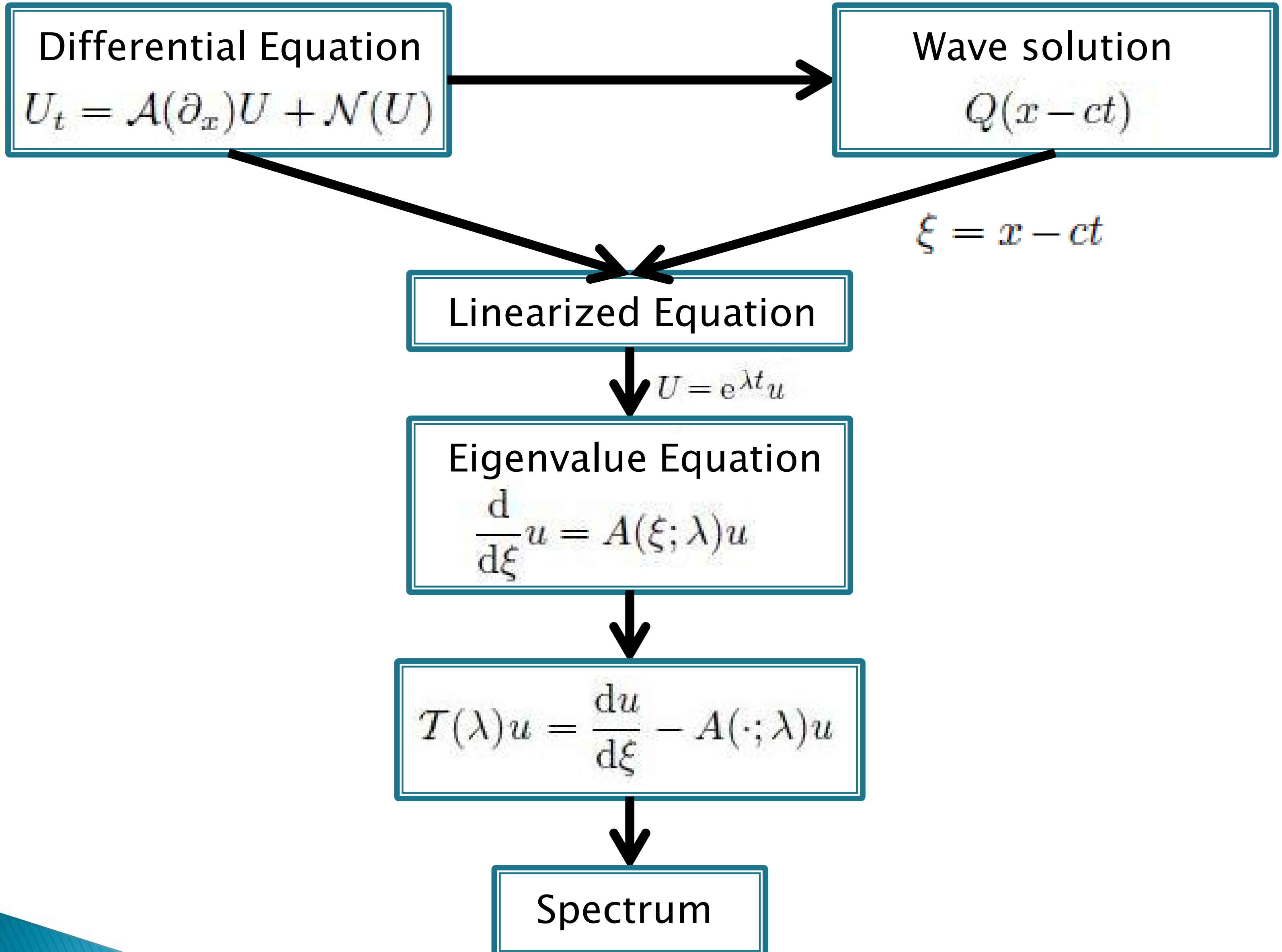
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Theorem 1

$\mathbb{C}$

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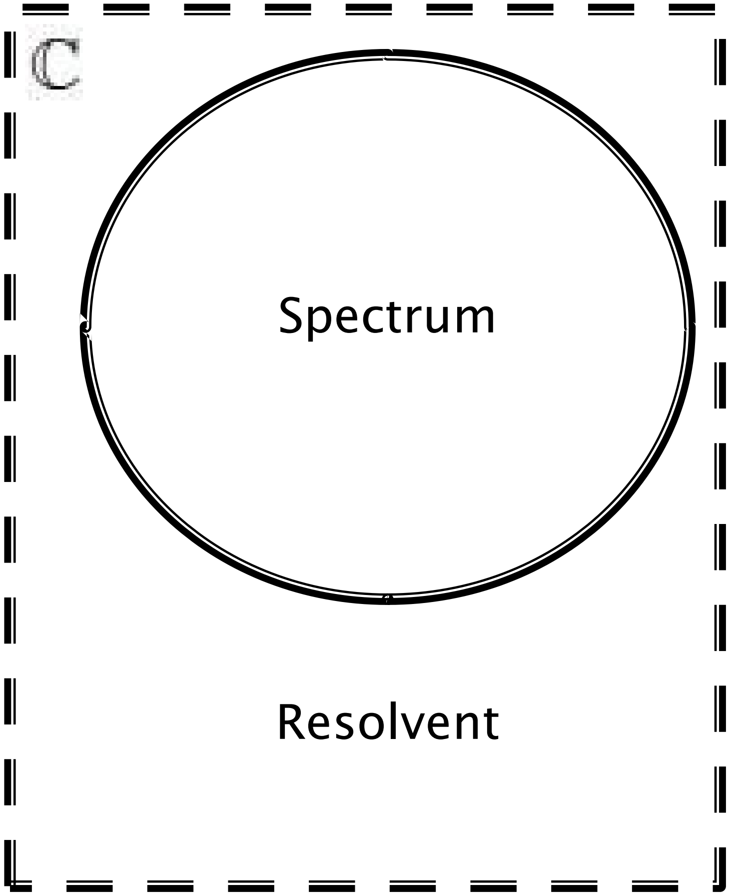

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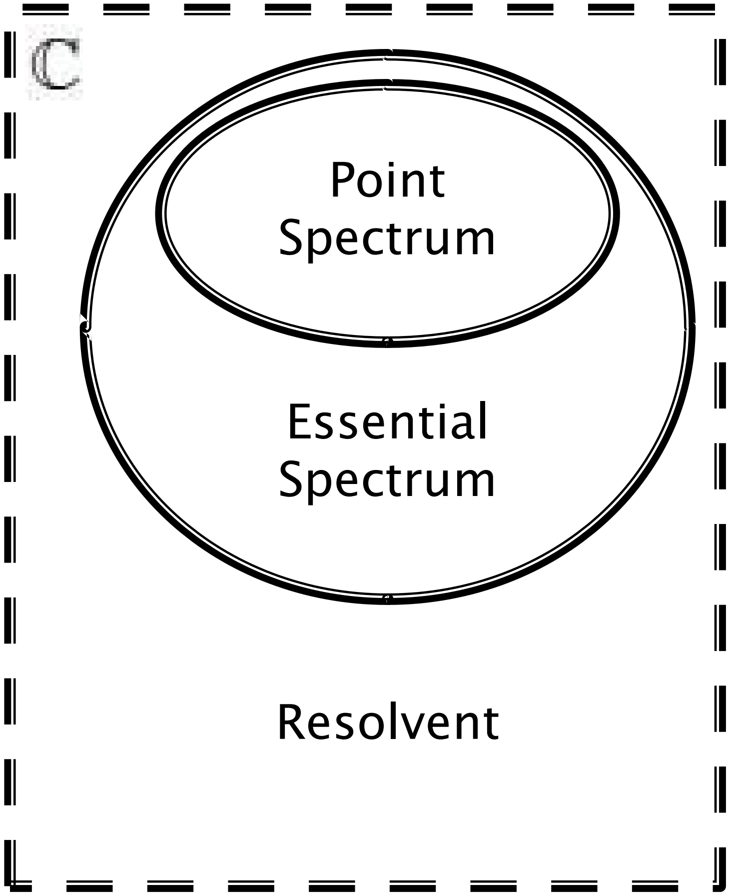

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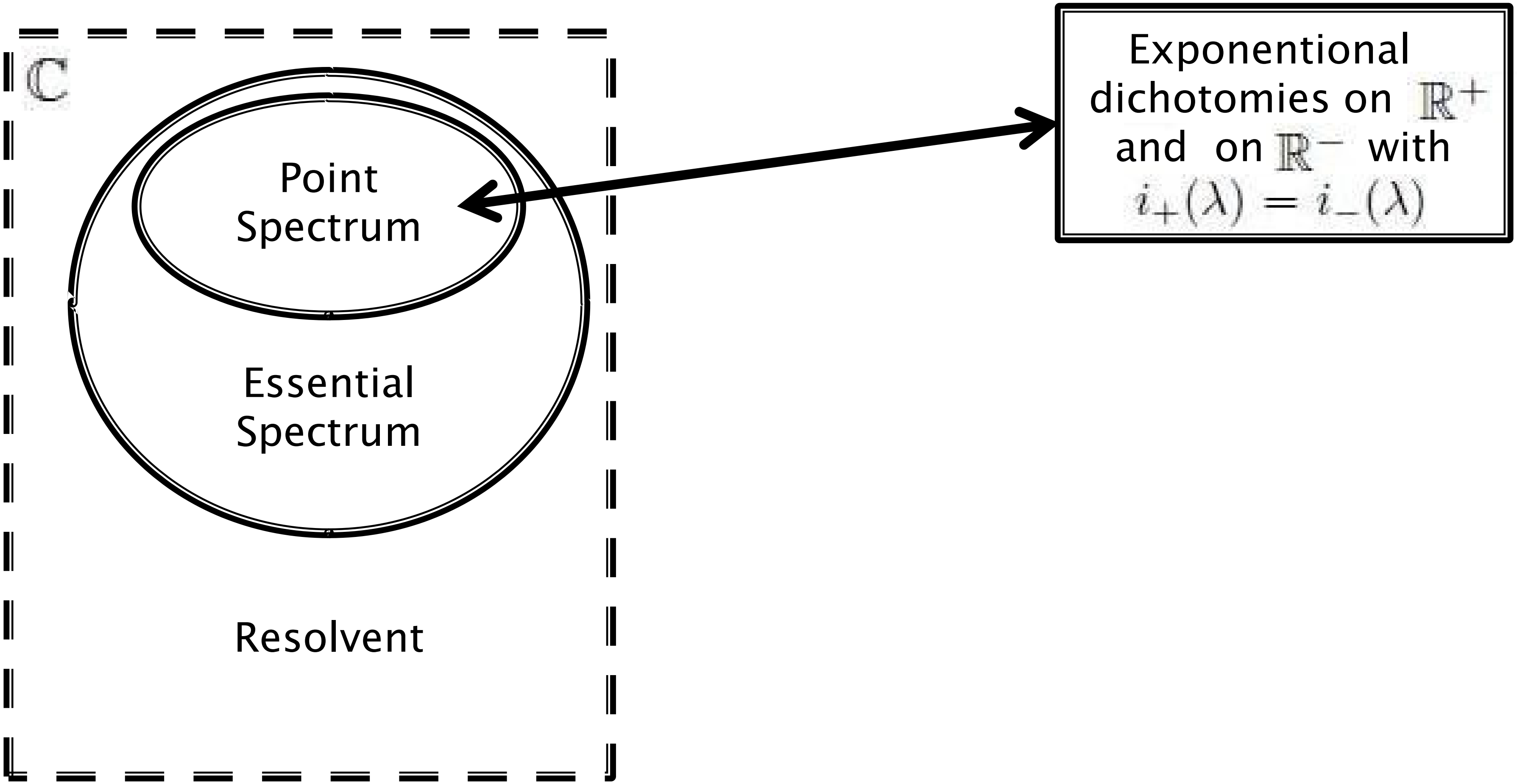

$\mathbb{C}$
Point Spectrum
Essential Spectrum
Resolvent
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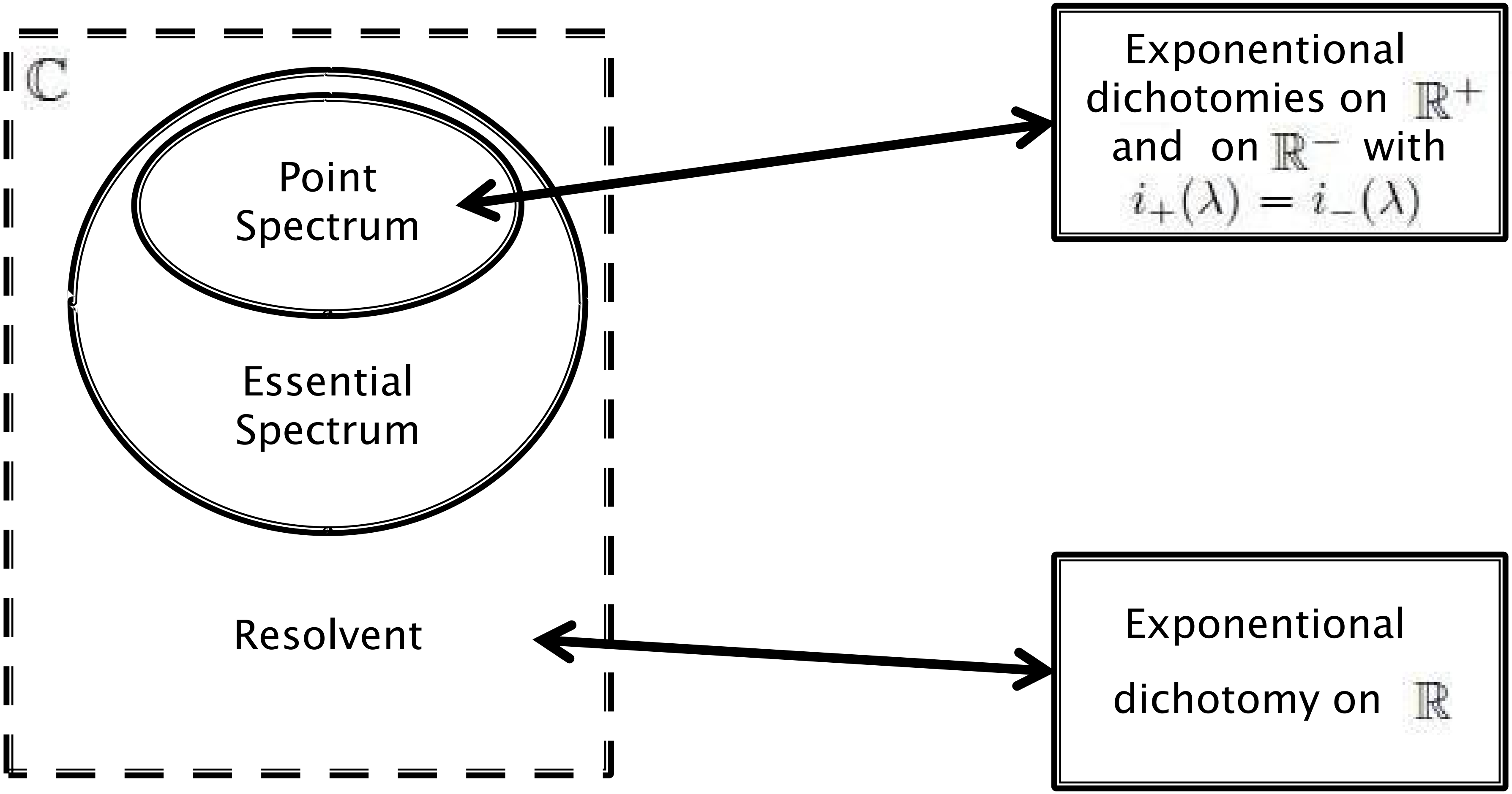
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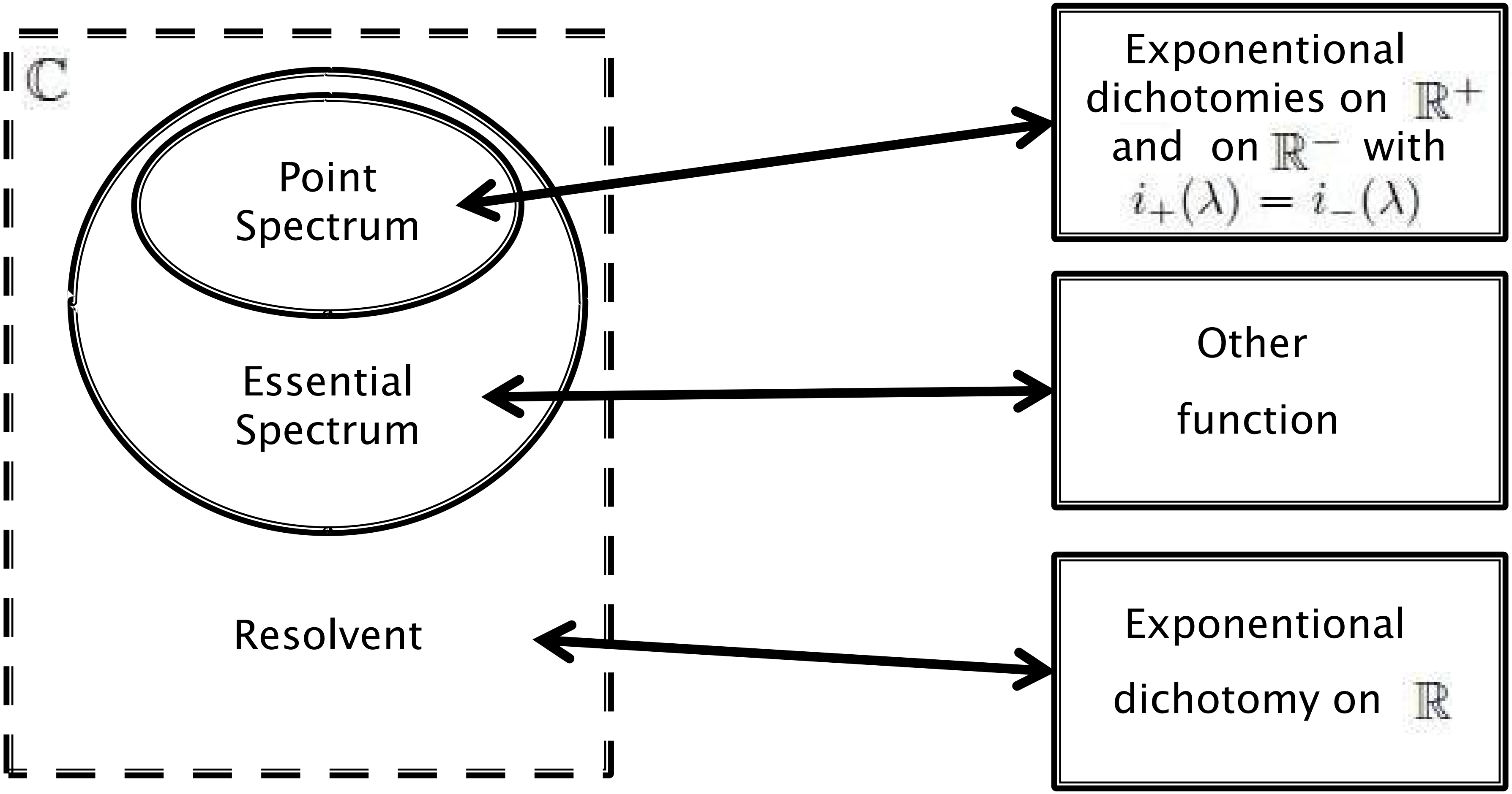
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Other function
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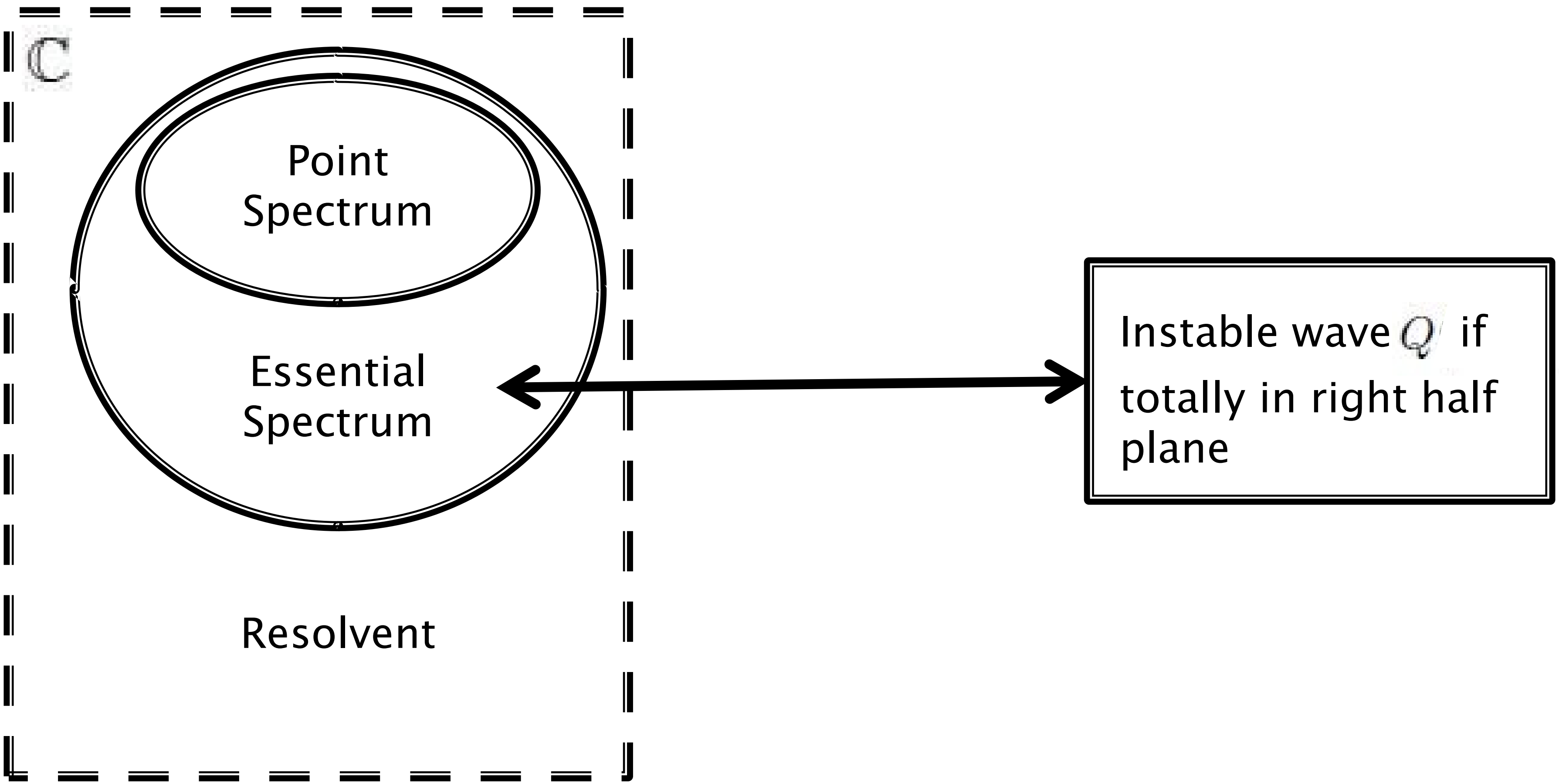
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Waves

Most common types

Homogeneous rest states

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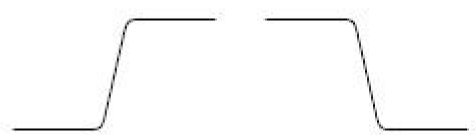

Front and back

Waves

Most common types

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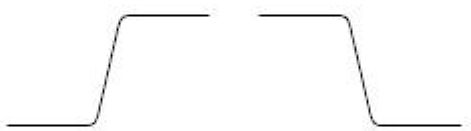

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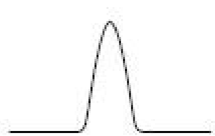

Pulse

Waves

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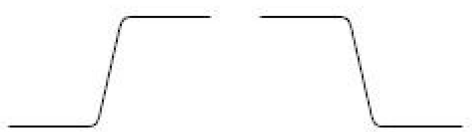

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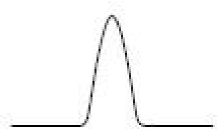

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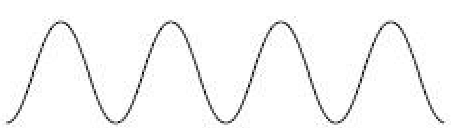

Periodic wave train

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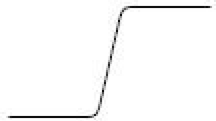
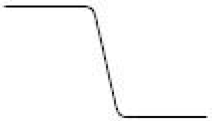
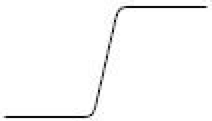


Fronts and Backs



The wave has two asymptotic rest states $\lim_{\xi \to \pm\infty} Q(\xi) = Q_{\pm} \in \mathbb{R}^N$



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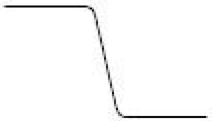

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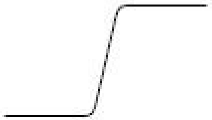

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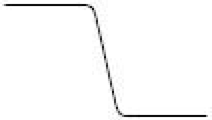

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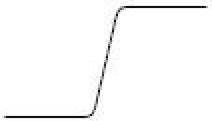

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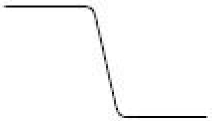

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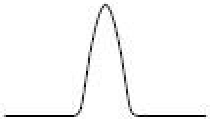
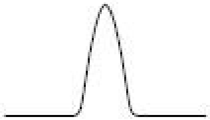


Pulses



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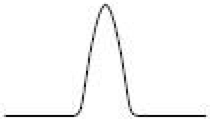

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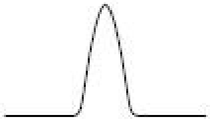

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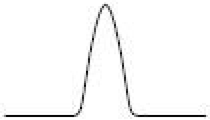

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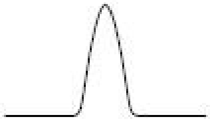

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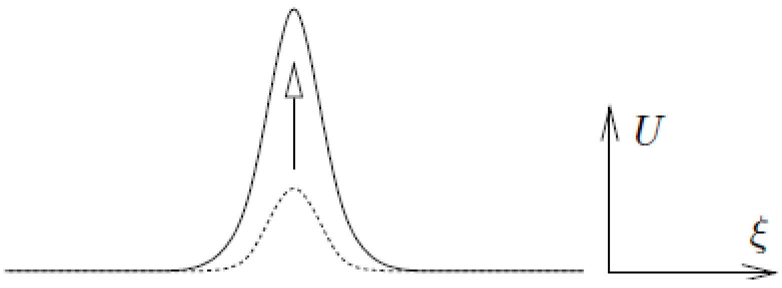
Instability types

Absolute instability



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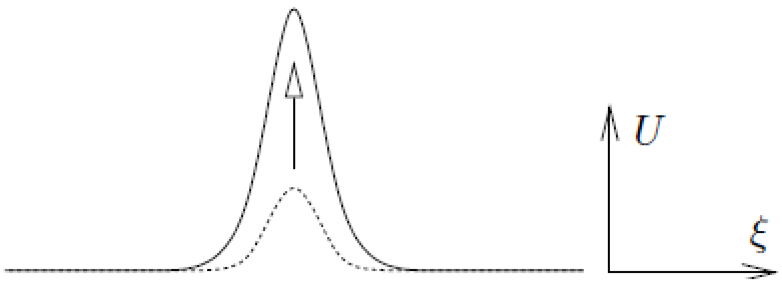
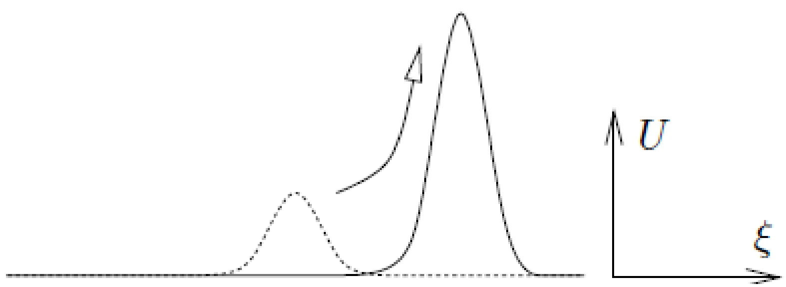


Convective instability



The Evans function

We assume a Morse index k. Than we obtain ordered bases $u_{k+1}(\lambda), \ldots, u_n(\lambda)$ and $u_1(\lambda), \ldots, u_k(\lambda)$ of spaces $\mathrm{R}(P_+(0;\lambda))$ and $\mathrm{N}(P_-(0;\lambda))$.

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$$\boxed{D(\lambda) = \mathrm{N}(P_-(0;\lambda)) \wedge \mathrm{R}(P_+(0;\lambda))}$$

Theorem 4.1 *The Evans function $D(\lambda)$ is analytic in $\lambda \in \Omega$ and has the following properties.*

- $D(\lambda) \in \mathbb{R}$ *whenever* $\lambda \in \mathbb{R} \cap \Omega$.
- $D(\lambda) = 0$ *if, and only if,* λ *is an eigenvalue of* $\mathcal{T}$.
- *The order of* λ_* *as a zero of the Evans function* $D(\lambda)$ *is equal to the algebraic multiplicity of* λ_* *as an eigenvalue of* $\mathcal{T}$.

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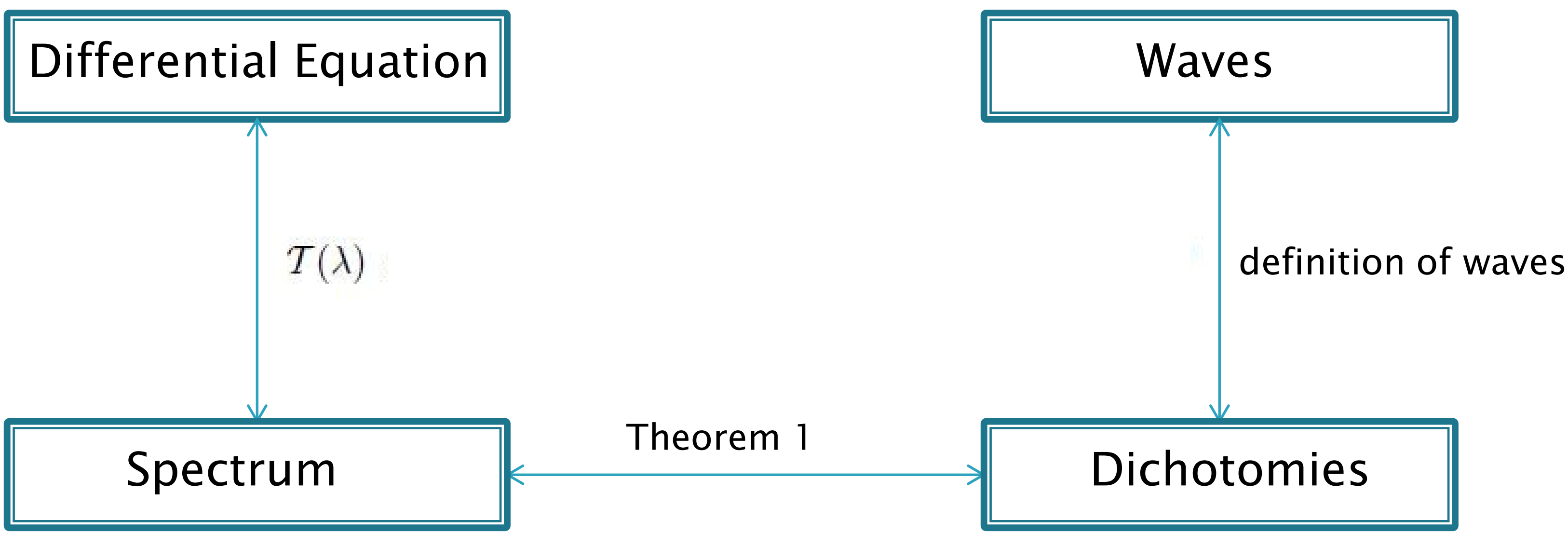
↓



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