On the Top of a Function.

Maximum Principle and Sub-/Supersolutions

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Outline

Progression

Boundary Value Problem

Study boundary value problems: bounded, open $U \subset \mathbb{R}^n$.
To find: $u : \overline{U} \to \mathbb{R}$.
Let $f : U \to \mathbb{R}, g : \partial U \to \mathbb{R}$ given functions.

$$\begin{cases} Lu = f(x) \text{ in } U \\ \;\; u = g(x) \text{ on } \partial U, \end{cases} \tag{1}$$

where L a second-order partial differential operator, given by

$$Lu = \underbrace{\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u}_{Mu} + c(x)u. \tag{2}$$

Elliptic Operators

Definition (Elliptic Operator)

L is an *(uniformly) elliptic operator* if there exists $\theta > 0$ such that for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$,

$$\theta|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j. \tag{3}$$

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L is *pointwise elliptic* if θ depends on $x \in U$.
L is *elliptic degenerate* if $0 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$, and there exists a fixed unit vector ζ such that for all $x \in U$,

$$\theta \leq \sum_{i,j=1}^{n} a_{ij}\zeta_i\zeta_j. \tag{4}$$

Parabolic Operators

Definition (Parabolic Operator)

Let $Q := (0, T) \times U$ for some $T > 0$. A *parabolic operator* is operator of the form

$$P := \partial_t - \sum_{i,j=1}^{n} a_{ij}(t,x)\partial_{ij} - \sum_{i=1}^{n} b_i(t,x)\partial_i - c(t,x), \tag{5}$$

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Can write: $P = \partial_t - L$.

Examples

Example (Elliptic Operators)

- Laplace operator: $\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0$;
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Example (Parabolic Operators)

- Heat operator: $u_t - \Delta u = 0$;
- Kolmogorov's equation: $u_t - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} = 0$;
- Scalar reaction-diffusion equation: $u_t - \Delta u = f(u)$.

Progression

1. Differential Operators
2. The Maximum Principle
   - The Weak Maximum Principle
   - The Strong Maximum Principle
   - Application to Boundary-Value Problems
3. Eigenvalues and Solutions
   - Principal Eigenvalue
   - Sub- and Supersolutions
4. Outlook

The Weak Maximum Principle

Assumptions:

From now on: bounded, open $U \subset \mathbb{R}^n$ with boundary smooth enough.
Furthermore: $u \in C^2(U) \cap C(\overline{U})$.

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Theorem (Weak Maximum Principle)

Let L elliptic (degenerate) operator with $Lu \geq 0$ in U.

1. If $c(x) \equiv 0$ in U then $\max_{\overline{U}} u = \max_{\partial U} u$.
2. If $c(x) \leq 0$ in U and $\max_{\overline{U}} \geq 0$, then $\max_{\overline{U}} u = \max_{\partial U} u$.

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Weaker form:

2. If not assumed $\max_{\overline{U}} u \geq 0$, then $\max_{\overline{U}} u \leq \max_{\partial U} u^+$.

Proof (1).

Consider first $Lu > 0$ in U. Then maximum on boundary: $x_0 \in \partial U$, with

$$Du(x_0) = 0; \quad D^2u(x_0) \leq 0. \tag{6}$$

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General: define $u_\varepsilon := u(x) + \varepsilon e^{\lambda x_1}$, $\varepsilon > 0$, $\lambda > 0$ sufficiently large. This function satisfies:

$$Lu_\varepsilon > 0 \text{ in } U \quad \Rightarrow \quad \max_{\overline{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon. \tag{8}$$

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Letting $\varepsilon \to 0$ gives the result. □

Proof (2).

Let $U^+ := \{u > 0\} \subset U$, then

$$\begin{aligned} Mu = Lu - c(x)u \geq 0 \text{ on } U^+ \\ u = 0 \text{ on } \partial U^+. \end{aligned} \tag{9}$$

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Hence, by part (1), if $U^+ \neq \emptyset$:

$$0 \leq \max_{\overline{U}} u = \max_{\overline{U}^+} u = \max_{\partial U^+} u = \max_{\partial U^+ \cap \partial U} u = \max_{\partial U} u. \tag{10}$$

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Otherwise: $u \leq 0$ everywhere. □

Corollary

Let L an elliptic (degenerate) operator with $c(x) \leq 0$ in U.

If $Lu \geq 0$ in U and $u \leq 0$ on ∂U, then $u \leq 0$ in U.

Weak Maximum Principle for Parabolic Operator

Parabolic boundary of Q: $\partial_p Q := \{\{0\} \times \overline{U}\} \cup \{[0, T] \times \partial U\}$.

Theorem (Weak Maximum Principle for Parabolic Operator)

Let $P = \partial_t - L$ a parabolic degenerate operator, u C^1 wrt. t, such that $Pu \leq 0$ in U.

1. If $c(t, x) \equiv 0$, or
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Proof.

The proof goes like the elliptic case. □

The Strong Maximum Principle

Theorem (Strong Maximum Principle)

Let L be elliptic operator, U connected, u such that $Lu \geq 0$ in U.

1. If $c \equiv 0$ and $\max_{\overline{U}} u = u(x_0)$ at interior point $x_0 \in U$, then u constant in U.
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The proof uses *Hopf's Lemma*.

Hopf's Lemma

Lemma (Hopf)

Let L and u as before. Suppose there exists $p \in \partial U$ such that $u(p) > u(x)$ for all $x \in U$.

1. If $c \equiv 0$ in U, then
$$\frac{\partial u}{\partial \xi}(p) > 0, \tag{11}$$
where ξ the outer unit normal at p.
2. If $c \leq 0$ in U and $u(p) \geq 0$, then same result holds.

Application to Boundary-Value Problems

Dirichlet problem: let $f : U \to \mathbb{R}, g : \partial U \to \mathbb{R}$ given functions.

Theorem

When it exists, the solution of

$$\begin{cases} Lu = f(x) \text{ in } U \\ u = g(x) \text{ on } \partial U, \end{cases} \tag{12}$$

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Similar results for other boundary value problems.

Progression

Class $C^{k,\gamma}$ functions

Definition

Function $u : U \to \mathbb{R}$ is of class $C^{k,\gamma}$, $0 < \gamma < 1$, if the norm

$$\begin{aligned} \|u\|_{C^{k,\gamma}(\overline{U})} &:= \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} \\ &:= \sum_{|\alpha| \leq k} \sup_{x \in U} |D^\alpha u(x)| + \sum_{|\alpha|=k} \sup_{x,y \in U} \left(\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} \right) \end{aligned} \tag{13}$$

is finite.

These functions constitute the *Hölder space* $C^{k,\gamma}(\overline{U})$, which is Banach.

Principal Eigenvalue

Definition

Let U be domain with ∂U of class $C^{2,\gamma}$; L elliptic operator with coefficients of class $C^{0,\gamma}(\overline{U})$.
Suppose $\varphi_1 \geq 0$ is eigenfunction of $-L$, which satisfies

$$\begin{cases} \varphi_1 > 0 \text{ in } U \\ \dfrac{\partial \varphi_1}{\partial \xi} < 0 \text{ on } \partial U. \end{cases} \tag{14}$$

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Eigenvalue λ_1 is called *principal eigenvalue* and eigenfunction φ_1 *principal eigenfunction*.

Existence and Uniqueness

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Proof.

The proof uses Krein-Rutman theory, by taking an order on space $C^1_0(\overline{U})$. □

Sub- and Supersolutions

Want to find C^2-solution of

$$\begin{cases} Lu + f(x,u) = 0 \text{ in } U \\ \qquad\qquad u = 0 \text{ on } \partial U. \end{cases} \tag{16}$$

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Definition (Sub-/Supersolution)

A *subsolution* is function $\underline{u} \in C^2(U)$ satisfying

$$\begin{cases} L\underline{u} + f(x,\underline{u}) \geq 0 \text{ in } U \\ \underline{u} \leq 0 \text{ on } \partial U. \end{cases} \tag{17}$$

Similarly, a *supersolution* is function $\overline{u} \in C^2(U)$ satisfying

$$\begin{cases} L\overline{u} + f(x,\overline{u}) \leq 0 \text{ in } U \\ \overline{u} \geq 0 \text{ on } \partial U. \end{cases} \tag{18}$$

Theorem

Let U of class $C^{2,\gamma}$, L elliptic operator with coefficients of class $C^{0,\gamma}$ and $f : \overline{U} \times \mathbb{R} \to \mathbb{R}$ satisfying:

For any $r > 0$, there exists $C(r) > 0$ such that for all $x, y \in \overline{U}$, $s, t \in [-r, r]$

$$|f(x,s) - f(y,t)| \leq C(|x-y|^{\gamma} + |s-t|). \tag{19}$$

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Assume there exists a subsolution $\underline{u}$ and a supersolution $\overline{u}$, both $C^{0,\gamma}$, such that $\underline{u} \leq \overline{u}$. Then there exist at least one solution u with $\underline{u} \leq u \leq \overline{u}$. Moreover, there is a minimal and a maximal one.

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Can generalize to Sobolev spaces. Then $f(x,s)$ has *Carathéodory conditions*:

$$\begin{cases} x \to f(x,s) \text{ is measurable in } x, \text{for all } s \in \mathbb{R}, \\ s \to f(x,s) \text{ is continuous in } s, \text{for a.e. } x \in U. \end{cases} \tag{20}$$

Proof.

Consider sequences of functions (v_n) and (w_n), solutions for $L - C + f(x,\cdot) + Cs$ and satisfying:

$$-r \leq \underline{u} = v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = \overline{u} \leq r. \tag{21}$$

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(v_n) and (w_n) converge to solutions $v, w \in C^{2,\gamma}$

If $u \in C^2$ is solution with $\underline{u} \leq u \leq \overline{u}$, then $v \leq u \leq w$. □

Solutions in Time

What happens if we start with some sub-/supersolution?

Solutions in Time

What happens if we start with some sub-/supersolution?

- If subsolution $\underline{u}$ blows up, then solution u blows up;
- If subsolution $\underline{u}$ blows up in finite time, then solution u blows up in finite time;
- If $\overline{u}$ is global (in time) supersolution above u, then u global.

Progression

- Further reading

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