Convergence of Solutions of Bistable Nonlinear Diffusion Equations to Travelling Front Solutions

Dirk van Kekem

May 23, 2012
1 Motivation

2 Travelling Fronts

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1. Motivation

2. Travelling Fronts

3. Uniform Convergence to a Front

4. Diverging Fronts

5. Conclusions
Asymptotic behavior as $t \to \infty$ of solutions $u(x, t)$ of the *bistable* nonlinear diffusion equation

$$u_t - u_{xx} - f(u) = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R}_+, \quad u(x, 0) = \varphi(x)$$

where

$$f(0) = f(1) = 0, \quad f'(0) < 0, \ f'(1) < 0.$$  

Moreover, $f \in C^1$ and has only one zero for $u = \alpha \in (0, 1)$. 

![Graph of a function with a peak and zeros at 0 and 1, with a zero labeled as $\alpha$.]
### Motivation

#### Simple Examples

**Example**

1. Fisher’s equation: \( f(u) = u(1 - u) \)
   to describe the spreading of biological populations. (not \( f'(0) < 0 \))

2. Newell-Whitehead-Segel equation: \( f(u) = u(1 - u^2) \)

3. Zeldovich equation: \( f(u) = u(1 - u)(u - \alpha) \) and \( 0 < \alpha < 1 \)
**Simple Examples**

**Example**

1. Fisher’s equation: \( f(u) = u(1 - u) \)
2. Newell-Whitehead-Segel equation: \( f(u) = u(1 - u^2) \) to describe Rayleigh-Benard convection. (not \( f'(0) < 0 \))
3. Zeldovich equation: \( f(u) = u(1 - u)(u - \alpha) \) and \( 0 < \alpha < 1 \)
Simple Examples

Example

1. Fisher’s equation: \( f(u) = u(1 - u) \)
2. Newell-Whitehead-Segel equation: \( f(u) = u(1 - u^2) \)
3. Zeldovich equation: \( f(u) = u(1 - u)(u - \alpha) \) and \( 0 < \alpha < 1 \) that arises in combustion theory.
Result (Uniqueness of Solution)

If \( \varphi \) piecewise continuous and \( 0 \leq \varphi(x) \leq 1 \), then there exists one and only one bounded classical solution \( u(x, t) \) of

\[
\begin{align*}
  u_t - u_{xx} - f(u) &= 0, \quad x \in \mathbb{R}, t \in \mathbb{R}_+, \\
  u(x, 0) &= \varphi(x),
\end{align*}
\]

(3)

with \( 0 \leq u(x, t) \leq 1 \) for all \( x, t \).
If $\varphi$ piecewise continuous and $0 \leq \varphi(x) \leq 1$, then there exists one and only one bounded classical solution $u(x, t)$ of

$$u_t - u_{xx} - f(u) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}_+,$$

$$u(x, 0) = \varphi(x),$$

with $0 \leq u(x, t) \leq 1$ for all $x, t$.

Fix these conditions on $\varphi, f$, so that we are concerned only with this unique bounded solution.
Motivation

Travelling Fronts

Uniform Convergence to a Front

Diverging Fronts

Conclusions
Definition (Travelling Front)

A *travelling front* is a solution $U$ of

$$u_t - u_{xx} - f(u) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}_+,$$

$$u(x, 0) = \varphi(x) \tag{4}$$

of the form

$$u(x, t) = U(x - ct) = U(\xi), \tag{5}$$

with $U(-\infty) = 0, U(\infty) = 1$.

$c$ is speed with opposite sign as $\int_0^1 f(u)du$. 
A *traveling front* is a solution $U$ of

\[ u_t - u_{xx} - f(u) = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R}_+, \]

\[ u(x, 0) = \varphi(x) \]  \hspace{1cm} (4)

of the form

\[ u(x, t) = U(x - ct) = U(\xi), \]  \hspace{1cm} (5)

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c is *speed* with opposite sign as $\int_0^1 f(u) \, du$.

Limits of $U$ when $x \to \infty$ should exist and be unequal.
Connects the homogeneous states.
These solutions move with constant speed without changing their shape.
Let \( U \) be a travelling front. Then \( P := \frac{dU}{d\xi} \) satisfies

\[
P' + \frac{f}{P} = -c
\]

\[
P(0) = P(1) = 0,
\]

(6)
Let $U$ be a travelling front. Then $P := \frac{dU}{d\xi}$ satisfies

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(6)

**Lemma**

Let $\alpha$ be small, and let $P_1(U), P_2(U)$ be solutions of (6) with corresponding speed $c_1, c_2$. Assume $P_1(U), P_2(U) > 0$ for $U \in (0, U_0]$ we have

$$P_1(U) \leq P_2(U) \quad \text{if } c_1 \leq c_2.$$  

(7)

Moreover, with our conditions on $f$, there exists at most one solution which is positive in $(0, 1)$. 
Uniqueness

Theorem

Suppose for $\alpha \in (0, 1)$ that one of the following holds:

(a) $f \leq 0$ in $(0, \alpha)$, $f > 0$ in $(\alpha, 1)$, $\int_{0}^{1} f(u) du > 0$;
(b) $f < 0$ in $(0, \alpha)$, $f \geq 0$ in $(\alpha, 1)$, $\int_{0}^{1} f(u) du < 0$;
(c) $f < 0$ in $(0, \alpha)$, $f > 0$ in $(\alpha, 1)$.

Then there is a unique solution of (6) which is positive in $(0, 1)$.
Theorem

Suppose for $\alpha \in (0, 1)$ that one of the following holds:

(a) $f \leq 0$ in $(0, \alpha)$, $f > 0$ in $(\alpha, 1)$, $\int_0^1 f(u)du > 0$;
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(c) $f < 0$ in $(0, \alpha)$, $f > 0$ in $(\alpha, 1)$.

Then there is a unique solution of (6) which is positive in $(0, 1)$.

Note: we can reconstruct $U$ from a solution $P$ by integrating:
$U'(\xi) = P(U)$, $U(0) = \frac{1}{2}$. 

Can have different types of convergence. Will concentrate only on two possibilities:
1. Convergence to travelling wave front with only one zero-point
2. Starting with a function which has sufficiently large part above $\alpha$.

This yields two diverging travelling fronts.

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Convergence to Travelling Front Solutions
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Uniqueness

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Can have different types of convergence. Will concentrate only on two possibilities:

1. Convergence to travelling wave front with only one zero-point $\alpha$
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Uniform Convergence

Theorem

Let, as before, $f(u) < 0$ for $0 < u < \alpha$; $f(u) > 0$ for $\alpha < u < 1$. Let $U$ be a travelling front solution with speed $c$ and suppose

$$\limsup_{x \to -\infty} \varphi(x) < \alpha, \quad \liminf_{x \to \infty} \varphi(x) > \alpha,$$

then solution $u(x, t)$ of (1) satisfies

$$|u(x, t) - U(x - ct - \xi_0)| < Ke^{-\omega t},$$

for some constants $K, \omega > 0$ and $\xi_0$. 
Lemma

There exists constants $\xi_1, \xi_2$ and $q_0, \mu \geq 0$ such that

$$U(\xi - \xi_1) - q_0 e^{-\mu t} \leq v(\xi, t) \leq U(\xi - \xi_2) + q_0 e^{-\mu t}.$$  \hfill (10)
Proof.

Define \( v(\xi, t) = u(x, t), \xi = x - ct; \) satisfies:

\[
\begin{align*}
  v_t - v_{\xi\xi} - cv_\xi - f(v) &= 0, \\
  v(\xi, 0) &= \varphi(\xi).
\end{align*}
\]  

(11)
Proof.

Define $v(\xi, t) = u(x, t), \xi = x - ct$; satisfies:

$$v_t - v_{\xi\xi} - cv_{\xi} - f(v) = 0, \quad \xi \in \mathbb{R}, \ t \in \mathbb{R}_+, \quad v(\xi, 0) = \varphi(\xi).$$  \hspace{1cm} (11)

Construct subsolution $\underline{v}(\xi, t) := \max(0, U(\xi - z(t)) - q(t))$, for suitably chosen $q(t) \geq 0$ and $z(t)$. 

Proof.

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v_t - v_{\xi\xi} - cv_{\xi} - f(v) &= 0, \\
\xi &\in \mathbb{R}, t \in \mathbb{R}^+, \\
v(\xi, 0) &= \varphi(\xi).
\end{align*}$$

(11)

Construct subsolution $\bar{v}(\xi, t) := \max(0, U(\xi - z(t)) - q(t))$, for suitably chosen $q(t) \geq 0$ and $z(t)$.

If $\bar{v} > 0$ then for $q(t) = q_0 e^{-\mu t}$ and with a clever choice of $z(t), \xi_1, \xi_2$ this results in

$$\begin{align*}
\bar{v}_t - \bar{v}_{\xi\xi} - c\bar{v}_{\xi} - f(\bar{v}) &\leq 0.
\end{align*}$$

(12)
Proof.

Define \( v(\xi, t) = u(x, t), \xi = x - ct \); satisfies:

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Construct subsolution \( \underline{v}(\xi, t) := \max(0, U(\xi - z(t)) - q(t)) \), for suitably chosen \( q(t) \geq 0 \) and \( z(t) \).

If \( \underline{v} > 0 \) then for \( q(t) = q_0 e^{-\mu t} \) and with a clever choice of \( z(t), \xi_1, \xi_2 \) this results in

\[
\underline{v}_t - \underline{v}_{\xi\xi} - c\underline{v}_\xi - f(\underline{v}) \leq 0.
\]

Hence,

\[
U(\xi - \xi_1) - q_0 e^{-\mu t} = U(\xi - \xi_1) - q(t) \leq \underline{v}(\xi, t) \leq v(\xi, t). \hfill (12)
\]
Can take \( q_0 = O(\varepsilon) \) such that \( |v(\xi, t) - U(\xi - \xi_0)| < \omega(\varepsilon) \), for a constant \( \xi_0 \) and a function \( \omega \).
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Moreover, we can estimate for $\pm z < 0$:

$$|1 \pm v(\xi, t)|, \quad |v_\xi(\xi, t)|, \quad |v_{\xi\xi}(\xi, t)|, \quad |v_t(\xi, t)| < C \left( e^{-\frac{c}{2} \pm \sigma} z + e^{-\mu t} \right),$$

(13)

for positive constants $\sigma > \frac{|c|}{2}$, $C$, $\mu$. 
Lemma

There exists a value $\xi_0$ such that

$$\lim_{t \to \infty} |v(\xi, t) - U(\xi - \xi_0)| = 0. \quad (14)$$
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Proof.

Let $\varepsilon > 0$ satisfy $|c|\varepsilon < 2\mu$. Define truncated function $w$ by

$$w(\xi, t) = \begin{cases} 
0 & \text{for } z \leq -\varepsilon t - 1, \\
v(\xi, t) & \text{for } |z| \leq \varepsilon t, \\
1 & \text{for } z \geq \varepsilon t + 1, 
\end{cases}$$

with a smooth connection between the different parts.
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\end{cases}$$  \hspace{1cm} (15)

with a smooth connection between the different parts.

Then $w$ can be used to find a limit function $\tilde{v}(\xi) = \lim_n w(\cdot, t'_n)$, which satisfies $\tilde{v}_{\xi\xi} + c\tilde{v}_\xi + f(\tilde{v}) = 0$. 

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Lemma

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Then $w$ can be used to find a limit function $\tilde{v}(\xi) = \lim_n w(\cdot, t'_n)$, which satisfies $\tilde{v}_{\xi\xi} + c\tilde{v}_\xi + f(\tilde{v}) = 0$.

Moreover, $\tilde{v}(-\infty) = 0$, $\tilde{v}(\infty) = 1$, so by uniqueness of travelling fronts, $\tilde{v}(\xi) = U(\xi - \xi_0)$ for some $\xi_0$. 

$\square$
Define

\[ h(\xi, t) := w(\xi, t) - U(\xi - \xi_0 - \alpha(t)), \]

where \( \alpha(t) \) is chosen so that \( h \) is orthogonal to \( e^{c\xi} \) for large \( t \). Existence of \( \alpha \) follows from the \textit{Implicit Function Theorem}.

The estimates \( |h(\xi, t)|, |\alpha(t)| < Ce^{-\nu t}, \nu > 0 \) imply that \( w \) converges exponentially to \( U(\xi - \xi_0) \).

From (13) and the definition of \( w \) it follows that

\[ |v(\xi, t) - w(\xi, t)| < Ce^{-\tilde{\nu} t}. \]

Hence, \( v(\xi, t) \) converges exponentially to \( U(\xi - \xi_0) \).
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where \( \alpha(t) \) is chosen so that \( h \) is orthogonal to \( e^{c\xi} \) for large \( t \). Existence of \( \alpha \) follows from the *Implicit Function Theorem*. The estimates
\[ |h(\xi, t)|, \quad |\alpha(t)| < Ce^{-\nu t}, \quad \nu > 0 \]
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Hence, \( v(\xi, t) \) converges exponentially to \( U(\xi - \xi_0) \).
Progression

1. Motivation
2. Travelling Fronts
3. Uniform Convergence to a Front
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Simple Example

If $0 \leq \varphi(x) < \alpha$ for all $x$, then

$$\lim_{t \to \infty} u(x, t) = 0.$$  \hspace{1cm} (16)
Simple Example

If \( 0 \leq \varphi(x) < \alpha \) for all \( x \), then

\[
\lim_{t \to \infty} u(x, t) = 0. \tag{16}
\]

Let \( \varphi(x) \leq \alpha - \delta < \alpha \), then \( u \) is bounded by the supersolution \( \overline{u}(t) \) defined by

\[
\begin{cases}
\overline{u}(t) = f'(\overline{u}), \\
\overline{u}(0) = \alpha - \delta,
\end{cases} \tag{17}
\]

and the subsolution \( \underline{u}(t) \) of the same equation, with \( \underline{u}(0) = \inf \varphi(x) \).

Obviously, \( u, \overline{u} \to 0 \) as \( t \to \infty \).
Simple Example

If $0 \leq \varphi(x) < \alpha$ for all $x$, then

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Let $\varphi(x) \leq \alpha - \delta < \alpha$, then $u$ is bounded by the supersolution $\overline{u}(t)$ defined by

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and the subsolution $\underline{u}(t)$ of the same equation, with $\underline{u}(0) = \inf \varphi(x)$. Obviously, $u, \overline{u} \to 0$ as $t \to \infty$.

Similarly, if $\alpha < \varphi(x) \leq 1$ for all $x$, then

$$\lim_{t \to \infty} u(x, t) = 1.$$ \hfill (18)
Theorem

Let $f$ as before, with $\int_0^1 f(u)du > 0$. Let $\varphi$ satisfy

$$\limsup_{|x| \to \infty} \varphi(x) < \alpha \quad \varphi(x) > \alpha + \eta \text{ for } |x| < L,$$

where $\eta, L > 0$.

If $L(\eta, f)$ large enough, then solution $u(x, t)$ of (1) satisfies

$$|u(x, t) - U(x - ct - \xi_0)| < Ke^{-\omega t}, \quad x < 0,$$
$$|u(x, t) - U(-x - ct - \xi_1)| < Ke^{-\omega t}, \quad x > 0,$$

for some constants $K, \omega > 0$ and $\xi_0, \xi_1$.  


Situation

Diverging Fronts

Convergence to Travelling Front Solutions
Situation

\[ \varphi(x) \]

\[ 1 \]

\[ 0 \]

\[ x \]

\[ \eta \]

\[ \alpha + \eta \]

\[ \alpha \]
Proof of the Theorem

Lemma

There exist constants $q_0, \mu > 0$ and $\xi_1, \xi_2$, such that

\[
U(x - ct - \xi_1) + U(-x - ct - \xi_1) - 1 - q_0 e^{-\mu t} \leq u(x, t) \\
\leq U(x - ct - \xi_2) + U(-x - ct - \xi_2) - 1 + q_0 e^{-\mu t}.
\] (21)
Proof of the Theorem

Lemma

There exist constants $q_0, \mu > 0$ and $\xi_1, \xi_2$, such that

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lemma

There exist functions $\omega(\varepsilon), T(\varepsilon)$, defined for small $\varepsilon > 0$ and with $\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0$, such that if

$$|u(x, t_0) - U(x - ct_0 - x_0)| < \varepsilon, \quad (22)$$

for some $t_0 > T(\varepsilon)$, some $x_0$ and all $x < 0$, then

$$|u(x, t) - U(x - ct - x_0)| < \omega(\varepsilon), \quad (23)$$

for all $t > t_0, x < 0$. 

Define the *left truncated function* by

\[
u_l(x, t) = \begin{cases} 
u(x, t) & x < 0, \\ 1 - \zeta(x)(1 - \nu(x, t)) & x \geq 0, \end{cases}
\]  

(24)

with \( \zeta(x) \in C^\infty(\mathbb{R}), \zeta(x) \equiv 1 \) for \( x \leq 0 \) and \( \zeta(x) \equiv 0 \) for \( x \geq 1 \).
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with \( \zeta(x) \in C^\infty(\mathbb{R}) \), \( \zeta(x) \equiv 1 \) for \( x \leq 0 \) and \( \zeta(x) \equiv 0 \) for \( x \geq 1 \).

Moreover, \( v_l(\xi, t) = u_l(x, t) = u_l(\xi + ct, t) \).
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\]  

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with \( \zeta(x) \in C^\infty(\mathbb{R}) \), \( \zeta(x) \equiv 1 \) for \( x \leq 0 \) and \( \zeta(x) \equiv 0 \) for \( x \geq 1 \). Moreover, \( v_l(\xi, t) = u_l(x, t) = u_l(\xi + ct, t) \).

The rest of the proof follows by slightly modifying the proofs of the lemma’s in the uniform convergence case.
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We have the following:
Consider \( f \) with only three zeroes, at \( x = 0, \alpha, 1, \)
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1. If \( \varphi(x) < \alpha \) or \( \varphi(x) > \alpha \) for all \( x \), then \( u(x, t) \) converges to 0 or 1, resp.
We have the following:

Consider $f$ with only three zeroes, at $x = 0, \alpha, 1$,

1. If $\varphi(x) < \alpha$ or $\varphi(x) > \alpha$ for all $x$, then $u(x, t)$ converges to 0 or 1, resp.

2. If $\varphi(x)$ is below $\alpha$ for $x \to -\infty$ and above $\alpha$ for $x \to \infty$, then the solution $u(x, t)$ converges uniformly to a travelling front solution $U(x - ct)$.
We have the following:
Consider \( f \) with only three zeroes, at \( x = 0, \alpha, 1, \)

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2. If \( \varphi(x) \) is below \( \alpha \) for \( x \to -\infty \) and above \( \alpha \) for \( x \to \infty \), then the solution \( u(x, t) \) converges uniformly to a travelling front solution \( U(x - ct) \).

3. If \( \varphi(x) \) is bigger than \( \alpha \) on a bounded interval \( |x| < L \), then \( u(x, t) \) converges to a pair of fronts, moving in opposite directions.
We have the following:
Consider $f$ with only three zeroes, at $x = 0, \alpha, 1$,

1. If $\varphi(x) < \alpha$ or $\varphi(x) > \alpha$ for all $x$, then $u(x, t)$ converges to 0 or 1, resp.

2. If $\varphi(x)$ is below $\alpha$ for $x \to -\infty$ and above $\alpha$ for $x \to \infty$, then the solution $u(x, t)$ converges uniformly to a travelling front solution $U(x - ct)$.

3. If $\varphi(x)$ is bigger than $\alpha$ on a bounded interval $|x| < L$, then $u(x, t)$ converges to a pair of fronts, moving in opposite directions.

From the last statement, we see that $u(x, t)$ takes in the end on a large, but finite, interval the value 0 or 1.
Other possibilities under slightly different conditions:

- **Fife, Paul C. and McLeod, J.B.**

- **Fife, Paul C.**

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Thank you for your attention!