Convergence of Solutions of Bistable Nonlinear Diffusion Equations to Travelling Front Solutions

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May 23, 2012

Outline

1. Motivation
2. Travelling Fronts
3. Uniform Convergence to a Front
4. Diverging Fronts
5. Conclusions

Progression

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Problem

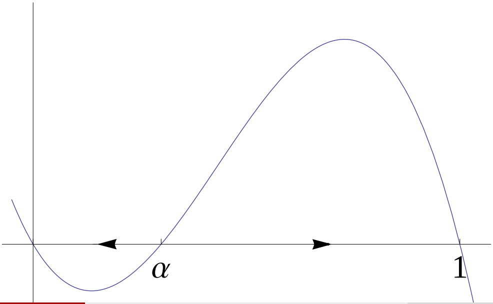
Asymptotic behavior as $t \to \infty$ of solutions $u(x,t)$ of the *bistable* nonlinear diffusion equation

$$\begin{aligned} u_t - u_{xx} - f(u) &= 0, \qquad x \in \mathbb{R}, t \in \mathbb{R}_+, \\ u(x,0) &= \varphi(x) \end{aligned} \tag{1}$$

where

$$f(0) = f(1) = 0, \qquad f'(0) < 0, f'(1) < 0. \tag{2}$$

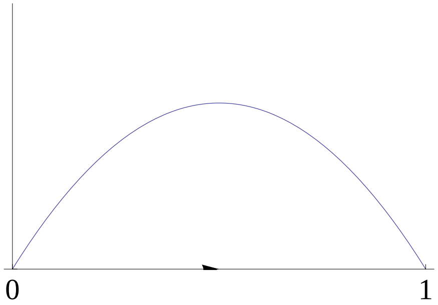
Moreover, $f \in C^1$ and has only one zero for $u = \alpha \in (0,1)$.



Simple Examples

Example

1. Fisher's equation: $f(u) = u(1-u)$
   to describe the spreading of biological populations. (not $f'(0) < 0$)
2. Newell-Whitehead-Segel equation: $f(u) = u(1-u^2)$
3. Zeldovich equation: $f(u) = u(1-u)(u-\alpha)$ and $0 < \alpha < 1$



Simple Examples

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1. Fisher's equation: $f(u) = u(1-u)$
2. Newell-Whitehead-Segel equation: $f(u) = u(1-u^2)$
   to describe Rayleigh-Benard convection. (not $f'(0) < 0$)
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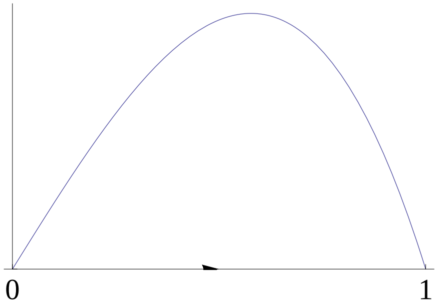

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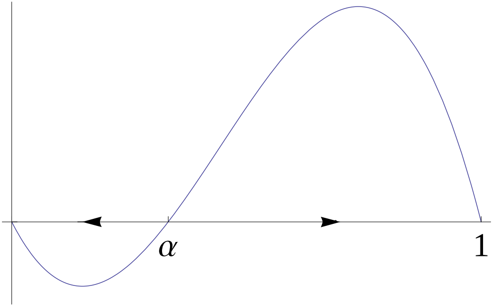
2. Newell-Whitehead-Segel equation: $f(u) = u(1-u^2)$
3. Zeldovich equation: $f(u) = u(1-u)(u-\alpha)$ and $0 < \alpha < 1$ that arises in combustion theory.



Result (Uniqueness of Solution)

If φ piecewise continuous and $0 \leq \varphi(x) \leq 1$, then there exists one and only one bounded classical solution $u(x,t)$ of

$$\begin{aligned} u_t - u_{xx} - f(u) &= 0, \qquad && x \in \mathbb{R}, t \in \mathbb{R}_+, \\ u(x,0) &= \varphi(x), && \end{aligned} \tag{3}$$

with $0 \leq u(x,t) \leq 1$ for all x, t.

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Fix these conditions on φ, f, so that we are concerned only with this unique bounded solution.

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Travelling Fronts

Definition (Travelling Front)

A *travelling front* is a solution U of

$$\begin{aligned} u_t - u_{xx} - f(u) = 0, \qquad & x \in \mathbb{R}, t \in \mathbb{R}_+, \\ u(x,0) = \varphi(x) \qquad & \end{aligned} \tag{4}$$

of the form

$$u(x,t) = U(x - ct) = U(\xi), \tag{5}$$

with $U(-\infty) = 0, U(\infty) = 1$.
c is *speed* with opposite sign as $\int_0^1 f(u)du$.

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c is *speed* with opposite sign as $\int_0^1 f(u)du$.

Limits of U when $x \to \infty$ should exist and be unequal.
Connects the homogeneous states.
These solutions move with constant speed without changing their shape.

Let U be a travelling front. Then $P := \frac{dU}{d\xi}$ satisfies

$$\begin{aligned} P' + \frac{f}{P} &= -c \\ P(0) = P(1) &= 0, \end{aligned} \tag{6}$$

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Lemma

Let α be small, and let $P_1(U), P_2(U)$ be solutions of (6) with corresponding speed c_1, c_2. Assume $P_1(U), P_2(U) > 0$ for $U \in (0, U_0]$ we have

$$P_1(U) \leq P_2(U) \qquad \text{if } c_1 \leq c_2. \tag{7}$$

Moreover, with our conditions on f, there exists at most one solution which is positive in $(0, 1)$.

Uniqueness

Theorem

Suppose for $\alpha \in (0,1)$ that one of the following holds:

(a) $f \leq 0$ *in* $(0,\alpha)$, $f > 0$ *in* $(\alpha,1)$, $\int_0^1 f(u)du > 0$;

(b) $f < 0$ *in* $(0,\alpha)$, $f \geq 0$ *in* $(\alpha,1)$, $\int_0^1 f(u)du < 0$;

(c) $f < 0$ *in* $(0,\alpha)$, $f > 0$ *in* $(\alpha,1)$.

Then there is a unique solution of (6) which is positive in $(0,1)$.

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Suppose for $\alpha \in (0,1)$ that one of the following holds:

$$\begin{array}{ll} (a) & f \leq 0 \textit{ in } (0,\alpha),\ f > 0 \textit{ in } (\alpha,1), \int_0^1 f(u)du > 0; \\ (b) & f < 0 \textit{ in } (0,\alpha),\ f \geq 0 \textit{ in } (\alpha,1), \int_0^1 f(u)du < 0; \\ (c) & f < 0 \textit{ in } (0,\alpha),\ f > 0 \textit{ in } (\alpha,1). \end{array}$$

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Note: we can reconstruct U from a solution P by integrating:
$U'(\xi) = P(U), U(0) = \frac{1}{2}$.

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Then there is a unique solution of (6) which is positive in $(0,1)$.

Note: we can reconstruct U from a solution P by integrating: $U'(\xi) = P(U), U(0) = \frac{1}{2}$.
Can have different types of convergence. Will concentrate only on two possibilities:

1. Convergence to travelling wave front with only one zero-point α
2. Starting with a function which has sufficiently large part above α. This yields two diverging travelling fronts.

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Uniform Convergence

Theorem

Let, as before, $f(u) < 0$ for $0 < u < \alpha$; $f(u) > 0$ for $\alpha < u < 1$.
Let U be a travelling front solution with speed c and suppose

$$\limsup_{x \to -\infty} \varphi(x) < \alpha, \qquad \liminf_{x \to \infty} \varphi(x) > \alpha, \tag{8}$$

then solution $u(x, t)$ of (1) satisfies

$$|u(x, t) - U(x - ct - \xi_0)| < Ke^{-\omega t}, \tag{9}$$

for some constants $K, \omega > 0$ and ξ_0.

Proof of Uniform Convergence

Lemma

There exists constants ξ_1, ξ_2 and $q_0, \mu \geq 0$ such that

$$U(\xi - \xi_1) - q_0 e^{-\mu t} \leq v(\xi, t) \leq U(\xi - \xi_2) + q_0 e^{-\mu t}. \tag{10}$$

Proof.

Define $v(\xi, t) = u(x, t), \xi = x - ct$; satisfies:

$$\begin{aligned} v_t - v_{\xi\xi} - cv_\xi - f(v) = 0, \qquad & \xi \in \mathbb{R}, t \in \mathbb{R}_+, \\ v(\xi, 0) = \varphi(\xi). & \end{aligned} \tag{11}$$

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Construct subsolution $\underline{v}(\xi, t) := \max(0, U(\xi - z(t)) - q(t))$, for suitably chosen $q(t) \geq 0$ and $z(t)$.

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If $\underline{v} > 0$ then for $q(t) = q_0 e^{-\mu t}$ and with a clever choice of $z(t), \xi_1, \xi_2$ this results in

$$\underline{v}_t - \underline{v}_{\xi\xi} - c\underline{v}_\xi - f(\underline{v}) \leq 0.$$

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Hence,

$$U(\xi - \xi_1) - q_0 e^{-\mu t} = U(\xi - \xi_1) - q(t) \leq \underline{v}(\xi, t) \leq v(\xi, t). \tag{12}$$

$\square$

Can take $q_0 = O(\varepsilon)$ such that $|v(\xi, t) - U(\xi - \xi_0)| < \omega(\varepsilon)$, for a constant ξ_0 and a function ω.

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Moreover, we can estimate for $\pm z < 0$:

$$|1 \pm v(\xi,t)|, \quad |v_\xi(\xi,t)|, \quad |v_{\xi\xi}(\xi,t)|, \quad |v_t(\xi,t)| < C\left(e^{\left(-\frac{c}{2}\pm\sigma\right)z} + e^{-\mu t}\right), \tag{13}$$

for positive constants $\sigma > \frac{|c|}{2}, C, \mu$.

Lemma

There exists a value ξ_0 such that

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Proof.

Let $\varepsilon > 0$ satisfy $|c|\varepsilon < 2\mu$. Define truncated function w by

$$w(\xi, t) = \begin{cases} 0 \text{ for } z \leq -\varepsilon t - 1, \\ v(\xi, t) \text{ for } |z| \leq \varepsilon t, \\ 1 \text{ for } z \geq \varepsilon t + 1, \end{cases} \tag{15}$$

with a smooth connection between the different parts.

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Then w can be used to find a limit function $\widetilde{v}(\xi) = \lim_n w(\cdot, t_n')$, which satisfies $\widetilde{v}_{\xi\xi} + c\widetilde{v}_\xi + f(\widetilde{v}) = 0$.

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Moreover, $\widetilde{v}(-\infty) = 0, \widetilde{v}(\infty) = 1$, so by uniqueness of travelling fronts, $\widetilde{v}(\xi) = U(\xi - \xi_0)$ for some ξ_0. □

Proof of Exponential Rate

Define

$$h(\xi, t) := w(\xi, t) - U(\xi - \xi_0 - \alpha(t)),$$

where $\alpha(t)$ is chosen so that h is orthogonal to $e^{c\xi}$ for large t. Existence of α follows from the *Implicit Function Theorem*.

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imply that w converges exponentially to $U(\xi - \xi_0)$.
From (13) and the definition of w it follows that

$$|v(\xi, t) - w(\xi, t)| < Ce^{-\widetilde{\nu} t}.$$

Hence, $v(\xi, t)$ converges exponentially to $U(\xi - \xi_0)$.

$\square$

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Let $\varphi(x) \leq \alpha - \delta < \alpha$, then u is bounded by the supersolution $\overline{u}(t)$ defined by

$$\begin{cases} \overline{u}(t) = f'(\overline{u}), \\ \overline{u}(0) = \alpha - \delta, \end{cases} \tag{17}$$

and the subsolution $\underline{u}(t)$ of the same equation, with $\underline{u}(0) = \inf \varphi(x)$. Obviously, $\underline{u}, \overline{u} \to 0$ as $t \to \infty$.

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Obviously, $\underline{u}, \overline{u} \to 0$ as $t \to \infty$.
Similarly, if $\alpha < \varphi(x) \leq 1$ for all x, then

$$\lim_{t\to\infty} u(x,t) = 1. \tag{18}$$

Statement

Theorem

Let f as before, with $\int_0^1 f(u)du > 0$. Let φ satisfy

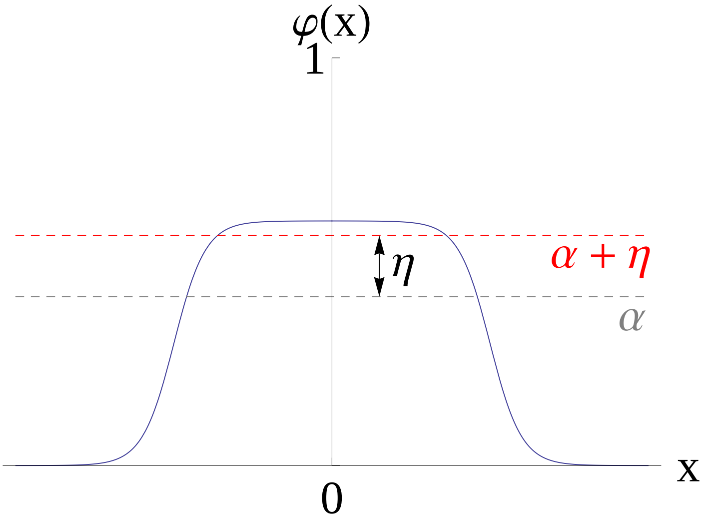
$$\limsup_{|x|\to\infty} \varphi(x) < \alpha \qquad \varphi(x) > \alpha + \eta \text{ for } |x| < L, \tag{19}$$

where $\eta, L > 0$.
If $L(\eta, f)$ large enough, then solution $u(x,t)$ of (1) satisfies

$$\begin{aligned} |u(x,t) - U(x - ct - \xi_0)| &< Ke^{-\omega t}, \qquad & x < 0, \\ |u(x,t) - U(-x - ct - \xi_1)| &< Ke^{-\omega t}, \qquad & x > 0, \end{aligned} \tag{20}$$

for some constants $K, \omega > 0$ and ξ_0, ξ_1.

Situation



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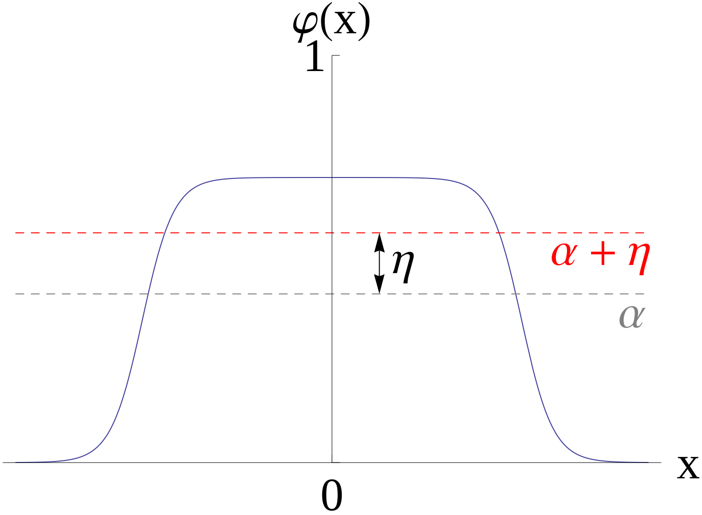

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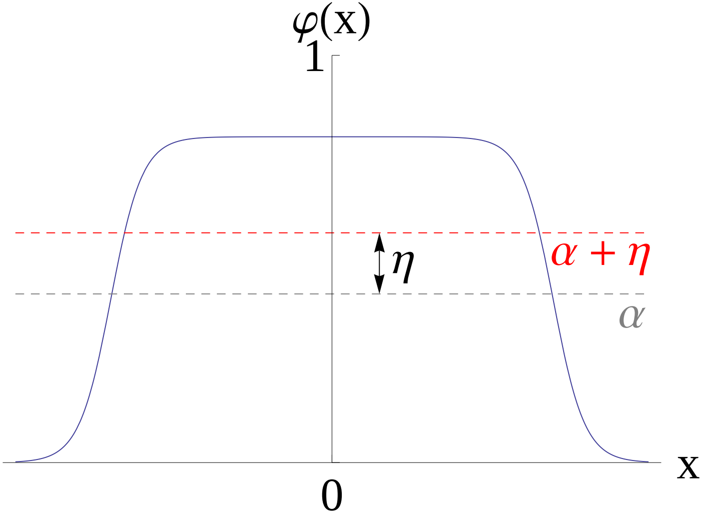

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Lemma

There exist constants $q_0, \mu > 0$ and ξ_1, ξ_2, such that

$$\begin{aligned} U(x-ct-\xi_1)+&U(-x-ct-\xi_1)-1-q_0 e^{-\mu t} \leq u(x,t) \\ &\leq U(x-ct-\xi_2)+U(-x-ct-\xi_2)-1+q_0 e^{-\mu t}. \end{aligned} \tag{21}$$

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Lemma

There exist functions $\omega(\varepsilon)$, $T(\varepsilon)$, defined for small $\varepsilon > 0$ and with $\lim_{\varepsilon\downarrow 0}\omega(\varepsilon) = 0$, such that if

$$|u(x,t_0) - U(x-ct_0-x_0)| < \varepsilon, \tag{22}$$

for some $t_0 > T(\varepsilon)$, some x_0 and all $x < 0$, then

$$|u(x,t) - U(x-ct-x_0)| < \omega(\varepsilon), \tag{23}$$

for all $t > t_0, x < 0$.

Proof of the Theorem (continued)

Define the *left truncated function* by

$$u_l(x,t) = \begin{cases} u(x,t) & x < 0, \\ 1 - \zeta(x)(1 - u(x,t)) & x \geq 0, \end{cases} \tag{24}$$

with $\zeta(x) \in C^\infty(\mathbb{R}), \zeta(x) \equiv 1$ for $x \leq 0$ and $\zeta(x) \equiv 0$ for $x \geq 1$.

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Moreover, $v_l(\xi, t) = u_l(x,t) = u_l(\xi + ct, t)$.

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Moreover, $v_l(\xi,t) = u_l(x,t) = u_l(\xi + ct, t)$.
The rest of the proof follows by slightly modifying the proofs of the lemma's in the uniform convergence case.

□

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2. If $\varphi(x)$ is below α for $x \to -\infty$ and above α for $x \to \infty$, then the solution $u(x,t)$ converges uniformly to a travelling front solution $U(x-ct)$.

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3. If $\varphi(x)$ is bigger than α on a bounded interval $|x| < L$, then $u(x,t)$ converges to a pair of fronts, moving in opposite directions.

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3. If $\varphi(x)$ is bigger than α on a bounded interval $|x| < L$, then $u(x,t)$ converges to a pair of fronts, moving in opposite directions.

From the last statement, we see that $u(x,t)$ takes in the end on a large, but finite, interval the value 0 or 1.

Further Reading

Other possibilities under slightly different conditions:

- Fife, Paul C. and McLeod, J.B.
  The Approach of Solutions of Nonlinear Diffusion Equations to Travelling Front Solutions Arch. Rational Mech. Anal. 65, 1977.
- Fife, Paul C.
  Long Time Behavior of Solutions of Bistable Nonlinear Diffusion Equations 1978.
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Thank you for your attention!