

Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields

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2 May 2012

Motivation

- Shun-ichi Amari, Dynamics of Pattern Formation in Lateral-Inhibition Type Neural Fields
- Biological Cybernetics 27, 1977
- Lots of citations
- Nice results

Table of contents

- 1 Introduction
- 2 Model
- 3 Equilibrium solutions
- 4 Stationary input
- 5 Summary

Content

- 1 Introduction
- 2 Model**
- 3 Equilibrium solutions
- 4 Stationary input
- 5 Summary

Neurofield equation

- Continuous model, 1D
- Average membrane potential: $u(x, t)$

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x, y) Z \left(y, t - \frac{|x - y|}{c} \right) dy + h + s(x, t)$$

Neurofield equation

- Continuous model, 1D
- Average membrane potential: $u(x, t)$
- Connectivity: $w(x, y)$
- Pulse emission rate: $Z(x, t)$

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x, y) Z \left(y, t - \frac{|x - y|}{c} \right) dy + h + s(x, t)$$

Neurofield equation

- Continuous model, 1D
- Average membrane potential: $u(x, t)$
- Connectivity: $w(x, y)$
- Pulse emission rate: $Z(x, t)$
- Stimulus: $\bar{s} + s(x, t)$
- Equilibrium state $h = \bar{s} - r$, where r resting potential

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x, y) Z \left(y, t - \frac{|x - y|}{c} \right) dy + h + s(x, t)$$

Assumptions

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x, y) Z \left(y, t - \frac{|x - y|}{c} \right) dy + h + s(x, t)$$

- No time delay

Assumptions

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x, y) Z(y, t) dy + h + s(x, t)$$

- No time delay
- Pulse emission:

$$Z(x, t) = f[u(x, t)] = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0 \end{cases}$$

Assumptions

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(\mathbf{x}, \mathbf{y}) f[u(\mathbf{y}, t)] d\mathbf{y} + h + s(\mathbf{x}, t)$$

- No time delay
- Pulse emission:

$$Z(x, t) = f[u(x, t)] = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0 \end{cases}$$

- Homogeneous field: $w(\mathbf{x}, \mathbf{y}) = w(\mathbf{x} - \mathbf{y})$

Assumptions

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x - y) f[u(y, t)] dy + h + s(x, t)$$

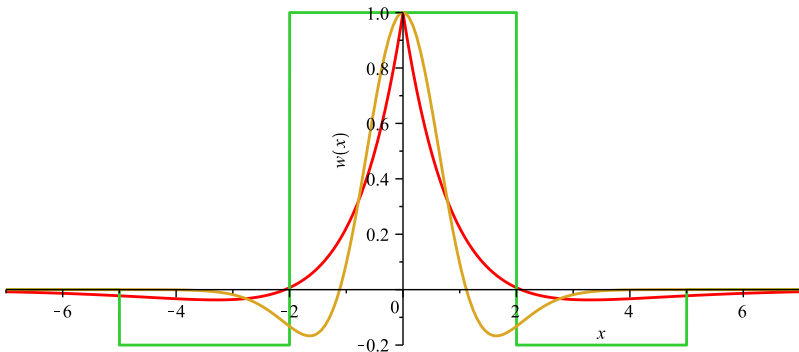
- No time delay
- Pulse emission:

$$Z(x, t) = f[u(x, t)] = \begin{cases} 0, & u \leq 0 \\ 1, & u > 0 \end{cases}$$

- Homogeneous field: $w(x, y) = w(x - y)$

About w

- w symmetric
- Excitatory connections nearby
- Inhibitory connections larger distance
- Integrable



Content

- 1 Introduction
- 2 Model
- 3 Equilibrium solutions**
- 4 Stationary input
- 5 Summary

Conditions

$$\tau \frac{\partial u(x, t)}{\partial t} = -u + \int_{-\infty}^{\infty} w(x-y)f[u(y, t)]dy + h + s(x, t)$$

- No inhomogeneous input: $s(x, t) = 0$
- $\partial u(x, t)/\partial t = 0$

$$u = \int_{-\infty}^{\infty} w(x-y)f[u(y, t)]dy + h$$

Excited region

Definition (Excited region)

The excited region $R[u]$ of an equilibrium solution u is defined as:

$$R[u] := \{x | u(x) > 0\} = \{x | f(x) = 1\}.$$

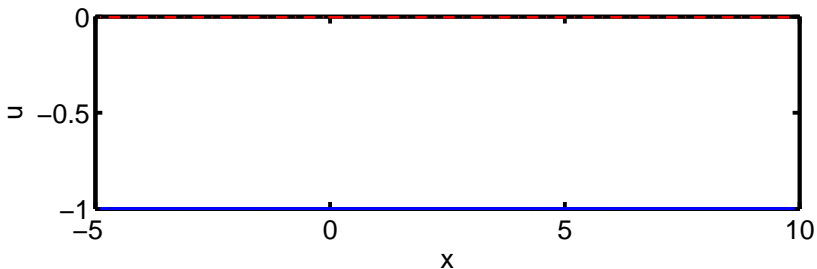
$$u = \int_{-\infty}^{\infty} w(x-y)f[u(y,t)]dy + h = \int_{R[u]} w(x-y)dy + h$$

ϕ -solutionDefinition (ϕ -solution)

A solution is called ϕ -solution if:

$$R[u] = \emptyset,$$

so $u(x) \leq 0$ for all x .



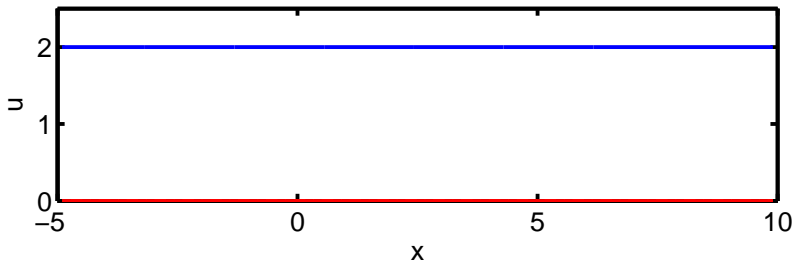
∞ -solution

Definition (∞ -solution)

A solution is called ∞ -solution if:

$$R[u] = (-\infty, \infty),$$

so $u(x) > 0$ for all x



a -solution

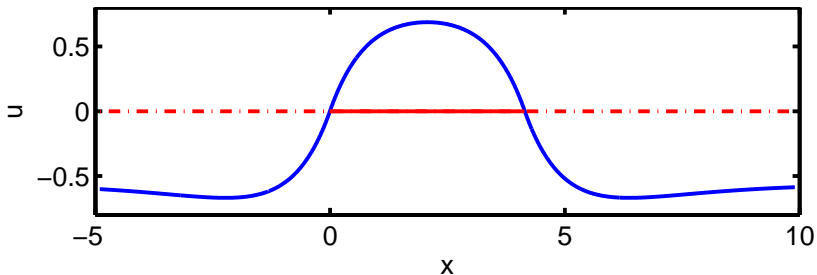
Definition (a -solution)

A solution is called an a -solution if $a_2 - a_1 = a$ and

$$R[u] = (a_1, a_2).$$

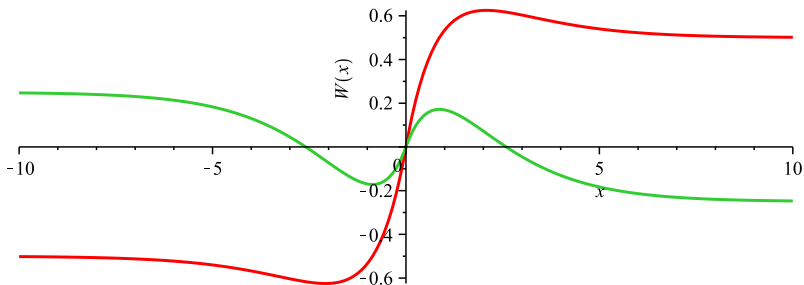
Remark

WLOG assume $R[u] = [0, a]$



W

- $W(x) := \int_0^x w(y)dy$
- $W(x) = -W(-x)$
- $W_m := \max_{x>0} W(x)$
- $W_\infty := \lim_{x \rightarrow \infty} W(x)$



Existence of equilibrium types:

Theorem

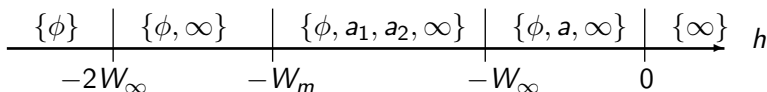
In absence of input:

- 1 There exist a ϕ -solution $\iff h < 0$
- 2 There exist a ∞ -solution $\iff 2W_\infty > -h$
- 3 There exist an a -solution $\iff h < 0$ and $a > 0$ satisfies:

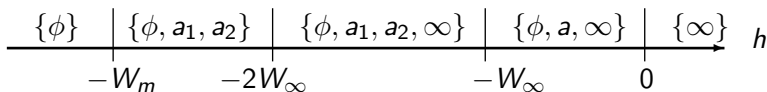
$$W(a) + h = 0$$

Overview equilibria

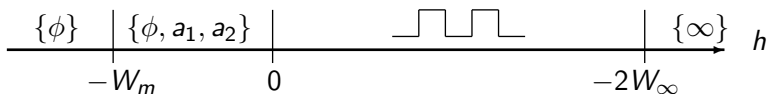
- Case 1a: $2W_\infty > W_m$



- Case 1b: $W_m > 2W_\infty > 0$



- Case 2: $W_\infty < 0$



Boundary movement

- Assume $R[u(x, t)] = (x_1(t), x_2(t))$
- $c_1 := \frac{\partial u(x_1, t)}{\partial x} > 0$, $c_2 := -\frac{\partial u(x_2, t)}{\partial x} > 0$
- $\frac{dx_1}{dt} = \frac{-1}{\tau c_1} [W(x_2 - x_1) + h]$, $\frac{dx_2}{dt} = \frac{1}{\tau c_2} [W(x_2 - x_1) + h]$
- Interval length $a(t) := x_2(t) - x_1(t)$
- $\frac{da}{dt} = \frac{1}{\tau} \left(\frac{1}{c_1} + \frac{1}{c_2} \right) [W(a) + h]$

Stability $a(t)$

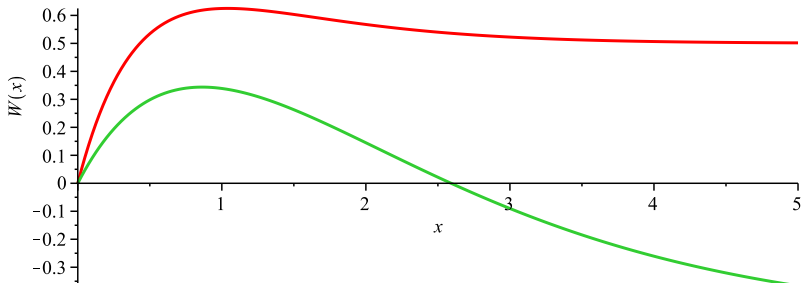
$$\frac{da}{dt} = \frac{1}{\tau} \left(\frac{1}{c_1} + \frac{1}{c_2} \right) [W(a) + h]$$

- Equilibrium length: $W(a_*) + h = 0$
- $\frac{d^2a}{dt^2} = \frac{dW(a_*)}{da} = w(a_*)$
- Stable if $\frac{dW(a_*)}{da} < 0$
- Unstable if $\frac{dW(a_*)}{da} > 0$

Check stability

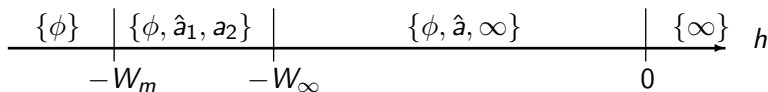
$$\frac{da}{dt} = \frac{1}{\tau} \left(\frac{1}{c_1} + \frac{1}{c_2} \right) [W(a) + h]$$

- Easy check: look at graph W
- Solutions $0 < a_1 < a_2$ then a_1 unstable, a_2 stable
- Finite region grows to ∞ -solution needs $W_\infty + h > 0$

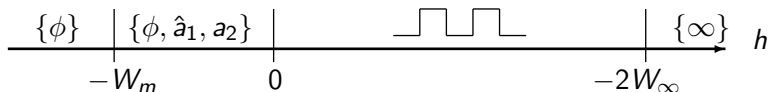


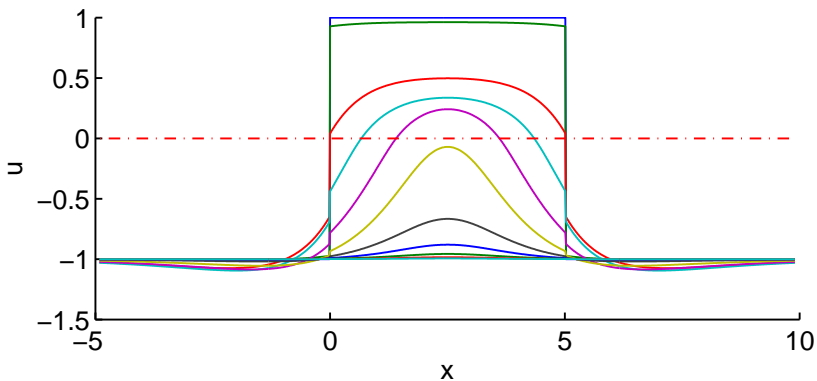
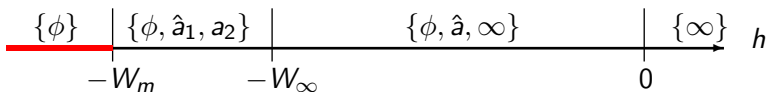
Overview stability equilibria

- Remove ∞ -solutions with $W_\infty + h < 0$
- Case 1: $W_\infty > 0$

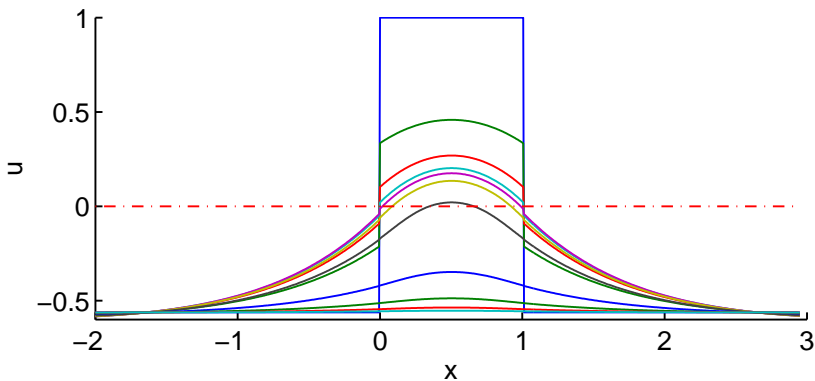
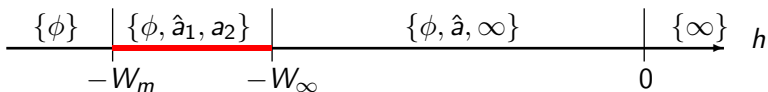


- Case 2: $W_\infty < 0$

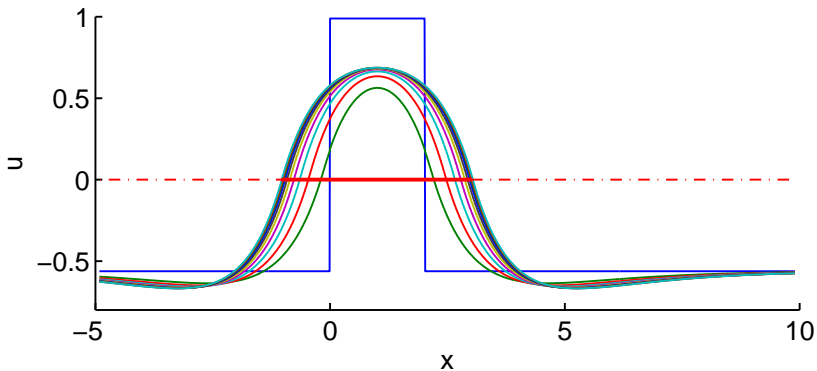
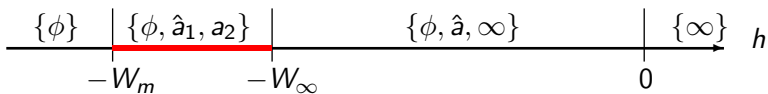


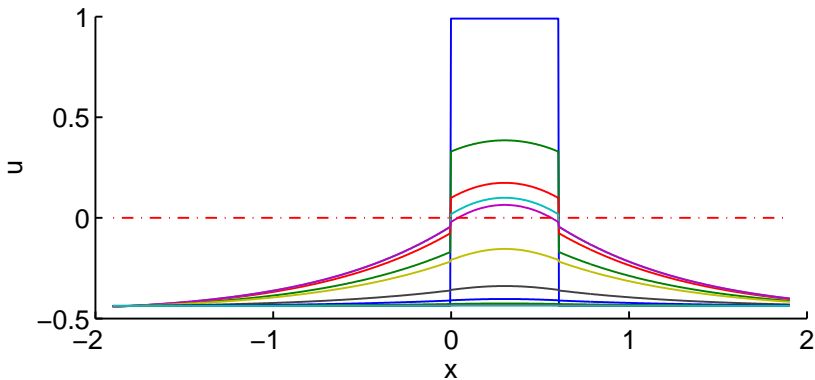
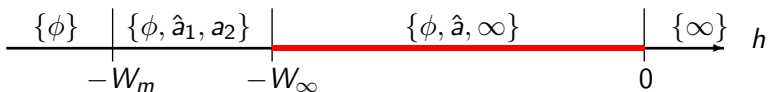
Case 1: $h < -W_m$ 

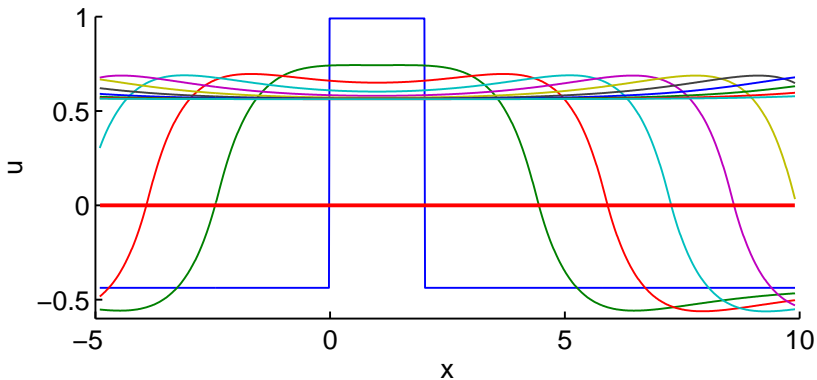
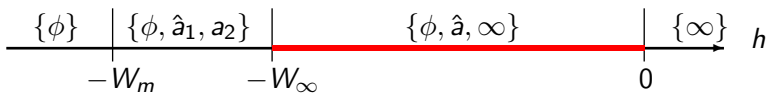
Case 1: $-W_m < h < -W_\infty$

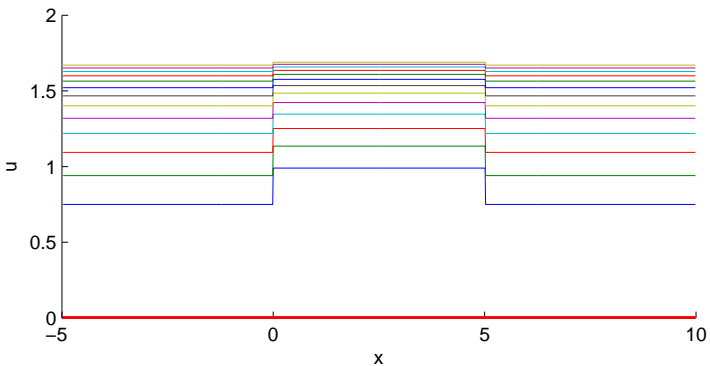
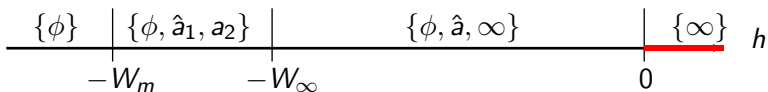


Case 1: $-W_m < h < -W_\infty$



Case 1: $-W_\infty < h < 0$ 

Case 1: $-W_\infty < h < 0$ 

Case 1: $h > 0$ 

Content

- 1 Introduction
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- 3 Equilibrium solutions
- 4 Stationary input**
- 5 Summary

Assumptions

- Space dependent input: $s(x) = s(x, t)$
- Small: $\epsilon s(x)$
- Assume a_0 -solution, stable equilibrium
- Solution of form: $u(x, t) = u_0(x) + \epsilon u_1(x, t) + O(\epsilon^2)$

Boundary movement

- $\frac{dx_1}{dt} = \frac{-1}{\tau c_1} [W(x_2 - x_1) + h + \epsilon s(x_1)],$
- $\frac{dx_2}{dt} = \frac{1}{\tau c_1} [W(x_2 - x_1) + h + \epsilon s(x_2)]$
- Symmetry: $c_1 = c + O(\epsilon), c_2 = c + O(\epsilon)$
- Assume $a(t) = a_0 + \epsilon a_1 + O(\epsilon^2)$
- Center speed $\frac{1}{2} \frac{dx_1 + x_2}{dt} = \frac{\epsilon}{2\tau c} [s(x_2) - s(x_1)]$
- Length change:

$$\frac{dx_2 - x_1}{dt} = \epsilon \frac{da_1}{dt} = \frac{\epsilon}{\tau c} [2w(a_0)a_1 + s(x_1) + s(x_2)]$$

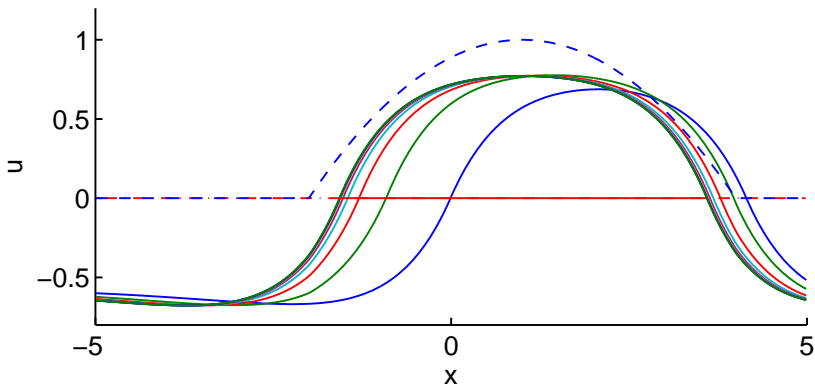
Interpretation

$$\frac{1}{2} \frac{dx_1 + x_2}{dt} = \frac{\epsilon}{2\tau c} [s(x_2) - s(x_1)]$$

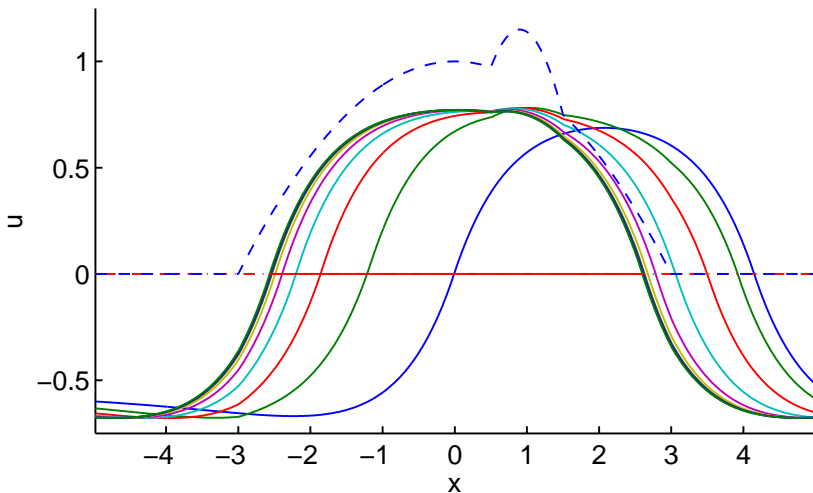
$$\frac{da_1}{dt} = \frac{\epsilon}{\tau c} [2w(a_0)a_1 + s(x_1) + s(x_2)]$$

- Move to higher stimulus level
- Equilibrium: $s(x_1) = s(x_2)$
- Equilibrium length: $a_0 - \epsilon \frac{s(x_1) + s(x_2)}{w(a_0)}$
- Stable equilibrium: $w(a_0) < 0$
- Stimulus peak width $> a_0$

Move to peak



Double peak



Content

- 1 Introduction
- 2 Model
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Results

- Classification equilibria
- Stability equilibria
- 5 types of behavior
- Behavior to small stimulus
- Numerical examples

Questions

Questions?