Bifurcations of Maps
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Overview

General introduction

Local Codimension 1 Bifurcations
  Period-Doubling Route to chaos

Invariant Manifolds and Homoclinic tangencies
  Generalized Hénon Map

Bifurcations of Invariant curves
  Phase-locking
  Bifurcations of Invariant Curves

Codim 2 Bifurcations
  1-Dimensional codim 2 bifurcations
  Strong Resonances 1:1 & 1:2
Overview

Part 1
▶ Setting and stability
▶ Local codimension 1 bifurcations
▶ Invariant manifolds and homoclinic bifurcations
▶ Chaos and Lyapunov exponents

Part 2
▶ Bifurcations on and of invariant curves
▶ Codim 2 bifurcations as organizing centers
Maps: Examples

- Models of discrete-time nature; Logistic map, Population biology with off-spring only once per year, Models from economy/game theory with policy adaptation every round.
- Derived from ODE’s: Periodically forced ODE’s, Poincaré maps or with an Euler-step, bipedal walkers
Setting

Consider a map with parameter $\alpha$

$$x \mapsto f(x, \alpha) \in \mathbb{R}^n, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m.$$

**Orbit:** Sequence of points defined by iterating initial point

$$x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \ldots, x_k := f^k(x_0), \ldots$$

**Fixed point:** $f(x_0, \alpha_0)^k - x_0 = 0$.

Minimal period $k$; fixed point if $k = 1$ or cycle if $k > 1$.

We study dynamics near a fixed point as the parameter $\alpha$ varies and set w.l.o.g. $k = 1$. We will mostly ignore phenomena induced by non-invertibility.
Stability of fixed point (cycle)

Consider evolution of a small perturbation $x_n = x_0 + u_n$:

$$u_{n+1} = f(x_n)^k - x_0 = f(x_0)^k - x_0 + Au_n + O(\|u_n\|^2),$$

where $A = f_x(x_0, \alpha_0)^k$. So: Near a fixed point the dynamics is given by the linearized mapping

$$u \mapsto Au.$$

The fixed point has multipliers (eigenvalues of $A$)

$$\{\mu_1, \mu_2, \ldots, \mu_n\} = \sigma(A),$$

The fixed point is stable if $\forall i : |\mu_i| < 1$. 

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Some Linear Phase Portraits

- $0 < \mu_{1,2} < 1$
- $-1 < \mu_2 < 0 < \mu_1 < 1$
- $0 < \mu_1 < 1 < \mu_2$
- $\mu_{1,2} = \rho e^{i\theta}, \rho < 1$
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Occurrence of codim 1 bifurcations

Follow a family of fixed points with the defining system

\[ F := f(x, \alpha)^k - x. \]

The stability of a fixed point may change as a multiplier crosses the unit circle when a parameter is varied:

<table>
<thead>
<tr>
<th>Limit Point</th>
<th>Period-Doubling</th>
<th>Neimark-Sacker</th>
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</thead>
<tbody>
<tr>
<td>( \mu_1 = 1 )</td>
<td>( \mu_1 = -1 )</td>
<td>( \mu_{1,2} = e^{\pm i\theta} )</td>
</tr>
</tbody>
</table>

\[ \Re(\mu) \quad \Im(\mu) \]

\[ \Re(\mu) \quad \Im(\mu) \]

\[ \Re(\mu) \quad \Im(\mu) \]
Limit Point Bifurcation

The fixed point has a simple multiplier $\mu_1 = 1$ and no other multipliers on the unit circle. The simplest example is

$$\xi \mapsto \alpha + \xi + a\xi^2,$$

where $a \neq 0$.

Other names: saddle-node or fold bifurcation.

The Implicit Function Theorem guarantees the existence of a branch of fixed points $x(\alpha)$ of $f(x, \alpha)^k - x = 0$ as long as $1 \notin \sigma(A)$. 
Limit Point Bifurcation

As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.

\[ \alpha < \alpha_0 \]
Limit Point Bifurcation

As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.

\[ \alpha = \alpha_0 \]
Limit Point Bifurcation

As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.

\[ \alpha > \alpha_0 \]
Limit Point Bifurcation

As the parameter crosses the critical value, two fixed points, one stable, one unstable, coalesce and disappear.

\[ \alpha < \alpha_0 \]

\[ \alpha = \alpha_0 \]

\[ \alpha > \alpha_0 \]
Period-Doubling Bifurcation

The fixed point has a simple multiplier $\lambda_1 = -1$ and no other multipliers on the unit circle. The simplest example is

$$\xi \mapsto -\xi (1 - \alpha) + b\xi^3,$$

where $b \neq 0$. Other names: flip bifurcation.

Branches of cycles; $\xi^* = 0$ and $\xi^* = \pm \sqrt{\frac{\alpha}{b}}$
Period-Doubling Bifurcation

When the parameter crosses the critical value, a cycle of period 2 bifurcates from the fixed point. 2-cycle stable if \( b > 0 \).

\[
\begin{align*}
\alpha &< \alpha_0 \\
\alpha &= \alpha_0 \\
\alpha &> \alpha_0
\end{align*}
\]
Neimark-Sacker bifurcation

Suppose for a critical value of the parameter $\alpha = \alpha_0$

- the fixed point has critical multipliers $\mu_{1,2} = e^{\pm i\theta_0}$ and no other eigenvalues on the unit circle.
- $e^{iq\theta_0} \neq 1$, for $q = 1, 2, 3, 4$, i.e. no strong resonances.

The simplest example is given by

$$z \mapsto ze^{i\theta_0} \left(1 + \alpha + d|z|^2\right),$$

where $z = x + iy = \rho e^{i\phi}$ is a complex variable and $d$ a complex constant.
Neimark-Sacker bifurcation

If the first Lyapunov coefficient $c := Re(d) \neq 0$, then a unique closed invariant curve appears around the fixed point when the parameter crosses the critical value. Supercritical case $c < 0$: the invariant curve is stable.
Neimark-Sacker bifurcation

Remarks:

▶ Subcritical case $c > 0$: an unstable invariant curve disappears as the fixed point becomes unstable when $\alpha$ increases.

▶ The dynamics on the invariant curve may be a rigid rotation $\phi \mapsto \phi + \theta$ if the rotation number $\theta/(2\pi)$ is (sufficiently) irrational.

▶ If $\theta/(2\pi)$ is close to rational, the dynamics may be more complicated, see tomorrow.

▶ Other names used in literature: Hopf (for maps), secondary Hopf, Torus bifurcation
Period-doubling bifurcation revisited

Cobweb: Plot graph and iterate by plotting 
\( (x, x) \rightarrow (x, f(x)) \rightarrow (f(x), f(x)) \).

\[
\begin{align*}
\xi & \quad \xi \\
\beta < 0 & \quad \beta = 0 & \quad \beta > 0 \\
g_\beta(\xi) & \quad \lambda_1 = -1 & \quad g_\beta(\xi)
\end{align*}
\]
Logistic Map

Mapping the unit-interval $[0, 1]$ to itself with parameter $0 < r < 4$

$$x \mapsto f(x, r) := rx(1 - x)$$

$r=2.5$

![Graph of the logistic map with $r=2.5$.](image)
Logistic Map: Period-Doublings

Second and Fourth iterate shown in yellow. As $r$ increases, see Movie.
Lyapunov exponents $\lambda$

Measure of rate of separation of orbits near $\{x_k\}$
For every iterate consider $f(x_k + \delta \vec{v}) = x_{k+1} + \delta Df(x_k)\vec{v} + ...$
Define growth rate $r_k = \|Df(x_k)\vec{v}\|$

$$\lambda(\vec{v}) := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} r_k$$

- Choose large $N$ so that average converges.
- There are $n$ exponents for $n$-dimensional systems.
- $\lambda < 0$ indicates stability, $\lambda = 0$ corresponds to a neutral direction (higher-dimensions), $\lambda > 0$ indicates chaos.
Logistic Map: Positive exponents suggest chaos.

Coordinates of attractor

Lyapunov exponent

\[
\begin{align*}
 r_2 &\quad 3 & r_{16} &\quad 3.5644072 \\
 r_4 &\quad 3.44948974 & r_{32} &\quad 3.5687594 \\
 r_8 &\quad 3.54409035 & r_{64} &\quad 3.5696916 \\
 & & r_\infty &\approx 3.57
\end{align*}
\]
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(Un)Stable Manifolds

Consider a saddle fixed point $x_0$ with
\[ |\mu_1| < \cdots < |\mu_i| < 1 < |\mu_{i+1}| < \cdots < |\mu_n|. \]

A point $x$ in the stable manifold $W^s$ satisfies
\[ \lim_{j \to \infty} f(x)^j = x_0. \]

A point $x$ in the unstable manifold $W^u$ (if defined) satisfies
\[ \lim_{j \to \infty} f(x)^{-j} = x_0. \]

The (un)stable manifold near the fixed point can be approximated by the (un)stable eigenspace of the linearization.
Connecting Orbits

Consider stable and unstable manifolds of saddles $x^\pm$.

A heteroclinic orbit $\{x_i\}$ satisfies $\lim_{j \to -\infty} x_j = x^-$ and $\lim_{j \to \infty} x_j = x^+$. For a homoclinic orbit we have $x^- = x^+$. 
Homoclinic orbits come in pairs

A transversal intersection of manifolds persists for small parameter variations.
A transversal homoclinic orbit allows the construction of Smale’s Horseshoe $\implies$ Chaotic dynamics
Global bifurcation: Homoclinic tangency

A transversal intersection (dis)appears through a primary tangency of the stable and unstable manifolds.

Let’s turn to the dynamics near a homoclinic tangency bifurcation curve.
Generalized Hénon Map (GHM): Setting

(A) Map $f_0$ has a saddle fixed point $O$ with eigenvalues $\gamma, \lambda$, such that $0 < |\lambda| < 1 < |\gamma|;$

(B) the saddle quantity $\sigma \equiv |\lambda \gamma| = 1;$

(C) the invariant manifolds $W^u(O)$ and $W^s(O)$ have a quadratic tangency at points of a homoclinic orbit $\Gamma.$
Consider a \((n_0 + k)\)-round orbit:
Start at \(\sigma_0^k\) in \(\Pi^+\) ending at \(\sigma_1^k\) in \(\Pi^-\) after \(k\) iterations of \(f_0\). Next iterate \(n_0\)-times along the homoclinic orbit to come back at \(\Pi^+\).
GHM and bifurcations of fixed points

Approximate return map near $\sigma_k^0$ defined by

\[
\begin{pmatrix}
  x \\
  y 
\end{pmatrix} \mapsto \begin{pmatrix}
  y \\
  \alpha_1 - \alpha_2 x - y^2 + Rxy 
\end{pmatrix},
\]

where $\alpha$ is a parameter close to the homoclinic tangency bifurcation curve $\mu_1 = 0$ in the original map $f$.

- LP bifurcation for $\alpha_1 = \frac{(\alpha_2 + 1)^2}{4(R - 1)}$
- PD bifurcation for $\alpha_1 = \frac{1}{4}(\alpha_2 + 1)^2(3 - R)$
- NS bifurcation for $\alpha_1 = \frac{(\alpha_2 - 1)(\alpha_2 - 1 + 2R)}{R^2}$
- Open regions with stable fixed points.
GHM and Newhouse regions

Away from the homoclinic tangency bifurcation curve $\mu_1 = 0$, and for saddle quantity $\sigma < 1$, there is a parameter set with infinitely many stable fixed points with high period.
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Neimark-Sacker in Delayed Logistic Map

\[ F := \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} rx(1 - y) \\ x \end{pmatrix} \]

Fixed Point at \( x^* = y^* = \frac{r-1}{r} \).

\[ DF = \begin{pmatrix} r(1 - y) & -rx \\ 1 & 0 \end{pmatrix} \implies \det(DF(x^*, y^*)) = rx^* = r - 1 \]

So there is a Neimark-Sacker bifurcation at \( r = 2 \). See Movie.
Model accounting for Phase-Locking

Suppose rotation number $\rho \approx 2\pi p/q$. Include higher order terms in model to describe period $q$ cycles

$$z \mapsto z(1 + \beta_1) e^{i\theta(\beta)} + \left( \sum_{m=1}^{\lfloor \frac{(q-1)}{2} \rfloor} A_m(\beta) z|z|^{2m} \right) + B(\beta) z^{q-1} + ...$$

Only really higher order if $q \geq 5$.

Model has the following bifurcations

- **Neimark-Sacker bifurcation** for $\beta_1 = 0$
- **Saddle-Node of period q**: $\beta_2 = C_1 \beta_1 \pm C_2 \beta_1^{(q-2)/2}$ for some constants $C_{1,2}$ depending on $A_m, B$. 
Resonance tongue from NS-bifurcation

As parameters vary through the tongue, a saddle and a node cycle of period \( q \) appear on the invariant curve.
Resonance Tongues in 3D Map

Adaptive Control Map (Frouzakis et al. IJBC 1991)

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}
\mapsto
\begin{pmatrix}
    y \\
    bx + k + yz \\
    z - \frac{ky}{c+y^2}(bx + k + yz - 1)
\end{pmatrix}
\]

Resonance tongues emerge from the Neimark-Sacker bifurcation at \( b = -\frac{c+1}{c+2} \). Fix \( c = 0.1 \).
Bifurcations of Invariant Curves 1

Quasi-periodic Saddle-Node bifurcation

Quasi-periodic Torus bifurcation
Bifurcations of Invariant Curves 2

Quasi-periodic Doubling bifurcation (two options)
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Why bother about codim 2 bifurcations?

- Numerical continuation yields bifurcation curves depending on two parameters.
- Bifurcation curves divide the parameter plane into several regions with qualitatively different dynamics. Such local and global bifurcations curves come together at codim 2 bifurcation points acting as organizing centers.
- For codim 2 bifurcations the normal forms are known, but unfoldings differ depending on critical coefficients.
- The idea is to come to a consistent picture of phase portraits when going around the codim 2 point in the parameter plane.
Recap codim 1 bifurcations

The simplest model systems are so-called normal forms

LP: \[ \xi \mapsto \alpha + x + a\xi^2, \]
PD: \[ \xi \mapsto -\xi(1 - \alpha) + b\xi^3, \]
NS: \[ z \mapsto ze^{i\theta_0} \left(1 + \alpha + d|z|^2\right), \]

with \(a, b, \text{Re}(d) \neq 0\) and \(q\theta_0 \neq 2\pi\) for \(q = 1, 2, 3, 4\).

Codim 2 bifurcations appear through additional multipliers or degeneracies.
List of local codim 2 bifurcations

<table>
<thead>
<tr>
<th>Case</th>
<th>degeneracy or additional multipliers</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$\mu_1 = 1, a = 0$</td>
<td>cusp</td>
</tr>
<tr>
<td>(2)</td>
<td>$\mu_1 = -1, b = 0$</td>
<td>generalized flip</td>
</tr>
<tr>
<td>(3)</td>
<td>$\mu_{1,2} = e^{\pm i\theta_0}, c = Re(d) = 0$</td>
<td>Chenciner</td>
</tr>
<tr>
<td>(4)</td>
<td>$\mu_1 = \mu_2 = 1$</td>
<td>1:1 resonance</td>
</tr>
<tr>
<td>(5)</td>
<td>$\mu_1 = \mu_2 = -1$</td>
<td>1:2 resonance</td>
</tr>
<tr>
<td>(6)</td>
<td>$\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{2\pi}{3}$</td>
<td>1:3 resonance</td>
</tr>
<tr>
<td>(7)</td>
<td>$\mu_{1,2} = e^{\pm i\theta_0}, \theta_0 = \frac{\pi}{2}$</td>
<td>1:4 resonance</td>
</tr>
<tr>
<td>(8)</td>
<td>$\mu_1 = 1, \mu_2 = -1$</td>
<td>fold-flip</td>
</tr>
<tr>
<td>(9)</td>
<td>$\mu_1 = 1, \mu_{2,3} = e^{\pm i\theta_0}$</td>
<td>fold-NS</td>
</tr>
<tr>
<td>(10)</td>
<td>$\mu_1 = -1, \mu_{2,3} = e^{\pm i\theta_0}$</td>
<td>flip-NS</td>
</tr>
<tr>
<td>(11)</td>
<td>$\mu_{1,2} = e^{\pm i\theta_1}, \mu_{3,4} = e^{\pm i\theta_2}$</td>
<td>double NS</td>
</tr>
</tbody>
</table>
Cusp: normal form and unfolding

When the quadratic coefficient vanishes along an LP-bifurcation curve, the following normal form characterizes nearby dynamics

\[ \xi \mapsto \beta_1 + \xi + \beta_2 \xi^2 + d\xi^3 + \cdots, \quad d \neq 0 \]
Degenerate Period-Doubling

The normal form is given by

\[ w \mapsto G(w) = -w(1 + \beta_1) + \beta_2 w^3 + ew^5 + \cdots, \quad e \neq 0 \]
Strong resonances

If $q\theta_0 = 2\pi$, $q = 1, 2, 3, 4$ then a Neimark-Sacker bifurcation becomes a strong resonance.

- The critical normal form for the 1:1 resonance is:

  \[
  f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ y + a_1 x^2 + b_1 xy \end{pmatrix} + \cdots 
  \]

- The critical normal form for the 1:2 resonance is:

  \[
  f := \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -x + y \\ -y + c_1 x^3 + d_1 x^2 y \end{pmatrix} + \cdots 
  \]

Observation: composition $Df^{-1}(0) \circ f$ is close to the identity.
Vector field Approximation

Theorem (Takens, Neimark): Suppose \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism and \( D\Phi(0) \) has all eigenvalues on the unit circle. Denote by \( S \) the semi-simple part of \( D\Phi(0) \). Then there exists a diffeomorphism \( \Psi \) and a vector field \( X \) such that

\[
\Psi \circ \Phi \circ \Psi^{-1} = \phi_X(t = 1) \circ S
\]

in the sense of Taylor series.


Remark:
- \( \Phi \) is the time-1 map of the flow of the vector field \( X \).
- parameters can be included.
1:1 Resonance Approximately Unfolded

The unfolding of the approximating vector field involves Saddle-node, Hopf and a global homoclinic bifurcation.
1:1 Resonance; Map intricacies
In practice, you may observe the following diagrams:

Bifurcation curves

Lyapunov exponents

Note phase-locking (green/yellow), chaos (red), invariant curve (magenta)
1:2 Resonance: normal form

The normal form $G$ (including parameters) is:

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix} \mapsto \begin{pmatrix}
  -x + y \\
  \beta_1 + (-1 + \beta_2)y + c_1 x^3 + d_1 x^2 y
\end{pmatrix} + \cdots
$$

Non-degenerate if $c_1 \neq 0$ and $d_1 + c_1 \neq 0$. If $c_1 < 0$ a codim 1 branch of Neimark-Sacker bifurcation of double period emanates.

Asymptotic expression of the new branch

$$
H^{(2)} : \left( x^2, y, \beta_1, \beta_2 \right) = \left( -\frac{1}{c_1}, 0, 1, \left( 2 + \frac{d_1}{c_2} \right) \right) \varepsilon
$$
Unfolding $c_1 > 0$

Only NS and PD branches.
Unfolding $c_1 < 0$:

New codim 1 branch $H^{(2)}$ (local bifurcation)
Perturbed 1:2 Resonance Normal Form

In practice, you may observe the following diagrams:

Bifurcation curves

Lyapunov exponents

Note periods 1,2 (yellow/orange), phase-locking (green/dark blue), chaos (red), invariant curve (magenta)