USER GUIDE TO SLIDECONT 2.0

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ABSTRACT

SLIDECONT, an AUTO97 driver for sliding bifurcation analysis of discontinuous piecewise-smooth autonomous systems, known as Filippov systems, is described in detail. Sliding bifurcations are those in which some sliding on the discontinuity boundary is critically involved. The software allows for detection and continuation of codimension-1 sliding bifurcations as well as detection of some codimension-2 singularities, with special attention to planar systems ($n = 2$). Some bifurcations are also supported for n-dimensional systems.

This document gives a brief introduction to Filippov systems, describes the structure of SLIDECONT and all computations supported by SLIDECONT 2.0, provides a user guide, and describes several tutorial examples, which are distributed together with the source code of SLIDECONT 2.0.

Key words & Phrases: Sliding bifurcations, discontinuous piecewise smooth systems, Filippov systems, continuation techniques, AUTO97

1 INTRODUCTION

SLIDECONT (version 2.0) is a suite of routines accompanying AUTO97 (Doedel & Kernévez, 1986; Doedel *et al.*, 1997) which allow one to perform bifurcation analysis of generic discontinuous piecewise smooth autonomous systems (Filippov, 1964, 1988), here called *Filippov systems*, with special attention to planar systems.

Bifurcation analysis of Filippov systems is important in many applications in various fields of science and engineering. Unfortunately, the complete catalogue of sliding bifurcations in n-dimensional systems is not yet available. There is a growing number of interesting results on bifurcations of periodic solutions in specific 3-dimensional and in general n-dimensional Filippov systems (see, for example Feigin (1994); Bernardo di *et al.* (1998a,b, 1999, 2001), and, in particular, Bernardo di *et al.* (2002)).

For the case of planar systems ($n = 2$), codimension-1 sliding bifurcations have been recently completely analyzed (Kuznetsov *et al.*, 2003) and suitable *defining systems* have been proposed for the numerical computation of bifurcation curves with standard continuation techniques. We provide their implementation in AUTO97, indicating explicitly when they are also applicable to general n-dimensional systems. Moreover, defining systems for continuing periodic orbits with a sliding segment in n-dimensional systems are implemented.

The document is organized as follows. In the next section we recall the definition of Filippov systems and some of their properties (for details and references, see Kuznetsov *et al.* (2003)). Then, we focus on SLIDECONT, assuming that the reader is acquainted with AUTO97. In particular, we give an overview of the capabilities and limitations of SLIDECONT (Section 3) and we describe its structure (Section 4), as well as the problems it can solve (Section 5). Then, we present a brief programmer's guide (Section 6), including the information on availability of the software and its installation. Finally, we consider several tutorial examples from mechanics and ecological modelling (Section 7).

2 Preliminaries

We consider a generic Filippov system (Filippov, 1964)

$$\dot{x} = \begin{cases} f^{(1)}(x), & x \in S_1, \\ f^{(2)}(x), & x \in S_2, \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$,

$$S_1 = \{x \in \mathbf{R}^n : H(x) < 0\}, \;\; S_2 = \{x \in \mathbf{R}^n : H(x) > 0\},$$

H is a smooth scalar function with non-vanishing gradient $H_x(x)$ on the discontinuity boundary

$$\Sigma = \{x \in \mathbf{R}^n : H(x) = 0\}\,,$$

and $f^{(i)} : \mathbf{R}^n \to \mathbf{R}^n, \; i = 1, 2$, are smooth functions. For $n = 2$ we denote

$$H_x^{\perp} = \begin{pmatrix} H_{x_2} \\ -H_{x_1} \end{pmatrix}.$$

Solutions of (1) can be constructed by concatenating *standard solutions* in $S_{1,2}$ and *sliding solutions* on Σ obtained with the Filippov convex method (Filippov, 1964, 1988; Kuznetsov *et al.*, 2003) described below. Let

$$\sigma(x) = \langle H_x(x), f^{(1)}(x)\rangle\langle H_x(x), f^{(2)}(x)\rangle, \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbf{R}^n$. The *crossing set* $\Sigma_c \subset \Sigma$ is defined by

$$\Sigma_c = \{x \in \Sigma : \sigma(x) > 0\}.$$

By definition, at points in Σ_c the orbit of (1) crosses Σ. The *sliding set* Σ_s is the complement to Σ_c in Σ, i.e.

$$\Sigma_s = \{x \in \Sigma : \sigma(x) \leq 0\}.$$

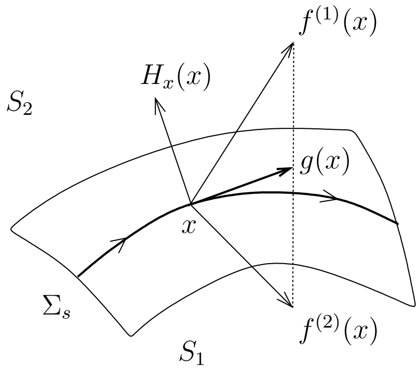

Figure 1: Filippov construction.

Points $x \in \Sigma_s$, where

$$\langle H_x(x), f^{(2)}(x) - f^{(1)}(x) \rangle = 0$$

are called *singular sliding points*. At such points, either both vectors $f^{(1)}(x)$ and $f^{(2)}(x)$ are tangent to Σ, or one of them vanishes while the other is tangent to Σ, or they both vanish. The Filippov method (see Fig. 1) associates the following convex combination $g(x)$ of the two vectors $f^{(i)}(x)$ to each nonsingular sliding point $x \in \Sigma_s$:

$$g(x) = \lambda f^{(1)}(x) + (1-\lambda) f^{(2)}(x), \;\; \lambda = \frac{\langle H_x(x), f^{(2)}(x) \rangle}{\langle H_x(x), f^{(2)}(x) - f^{(1)}(x) \rangle}. \tag{3}$$

Thus,

$$\dot{x} = g(x), \;\; x \in \Sigma_s, \tag{4}$$

is a smooth system of differential equations in codimension-1 domains of Σ_s which are composed of nonsingular sliding points. Solutions of this system are called *sliding solutions*.

Equilibria of (4), where the vectors $f^{(i)}(x)$ are transversal to Σ_s and anti-collinear, are called *pseudo-equilibria* of (1). An equilibrium X of (4), where one of the vectors $f^{(i)}(X)$ vanishes, is called a *boundary equilibrium*. In this setting, all isolated singular sliding points are equilibria of (4). The boundary of a sliding domain is composed of *tangent points*, T, where both vectors $f^{(i)}(T)$ are nonzero but one of them is tangent to Σ, i.e.

$$\langle H_x(T), f^{(i)}(T) \rangle = 0,$$

boundary equilibria, and singular sliding points. Tangent points are called *visible* (*invisible*) if the orbits of $\dot{x} = f^{(i)}(x)$ starting from them at time $t = 0$ belong to S_i (S_j, $j \neq i$) for all sufficiently small $|t| \neq 0$.

Orbits of (1) can overlap when sliding. Three types of periodic orbits can occur in (1): *standard*, *crossing* (i.e. passing through both domains S_i but with no points in Σ_s), and *sliding* (i.e. with at least one point in Σ_s).

Two Filippov systems are called *topologically equivalent*, if there is a homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ that maps the discontinuity boundary of one system onto the discontinuity boundary of the other, and that maps orbits of one system onto the corresponding orbits of the other, preserving the time direction and mapping standard and sliding segments of any orbit onto the corresponding segments of its image. Now, consider a Filippov system depending on parameters

$$\dot{x} = \begin{cases} f^{(1)}(x,\alpha), & x \in S_1(\alpha), \\ f^{(2)}(x,\alpha), & x \in S_2(\alpha), \end{cases} \tag{5}$$

where $x \in \mathbf{R}^n, \alpha \in \mathbf{R}^m$, and $f^{(i)}, i = 1, 2,$ are smooth functions of (x, α), while

$$S_1(\alpha) = \{x \in \mathbf{R}^n : H(x,\alpha) < 0\}\,,\; S_2(\alpha) = \{x \in \mathbf{R}^n : H(x,\alpha) > 0\}\,,$$

for some smooth function $H(x, \alpha)$ with $H_x(x, \alpha) \neq 0$ for all (x, α) such that $H(x, \alpha) = 0$. System (5) exhibits a *bifurcation* at $\alpha = \alpha_0$ if by an arbitrarily small parameter perturbation we get a topologically nonequivalent system. All bifurcations of (5) are classified as *local* or *global*. A local bifurcation can be detected by looking at an arbitrarily small neighborhood of a point in the state space. All other bifurcations are called global.

3 OVERVIEW

SLIDECONT can be used to perform a partial bifurcation analysis of n-dimensional Filippov systems (5) and a much more complete bifurcation analysis in the planar case ($n = 2$). No more than two *control parameters* are allowed ($m \leq 2$). Specifically, using the terminology introduced in Kuznetsov *et al.* (2003), SLIDECONT can:

- compute the boundary Σ in planar systems ($n = 2$) for fixed parameter values;

- compute a curve of tangent points in three-dimensional systems ($n = 3$) for fixed parameter values;
- continue a tangent point in planar systems ($n = 2$) in one control parameter;
- continue a standard equilibrium in one control parameter;
- continue a pseudo-equilibrium in one control parameter;
- continue a standard periodic solution in one control parameter;
- continue a standard orbit, possibly crossing the boundary Σ, connecting a tangent point with the boundary Σ in planar systems ($n = 2$) (e.g. the standard part of a sliding cycle) in one control parameter;
- continue a standard orbit, possibly crossing the boundary Σ, connecting a pseudo-equilibrium with the boundary Σ in one control parameter;
- continue a standard orbit, possibly crossing the boundary Σ, connecting a standard saddle with the boundary Σ in planar systems ($n = 2$) in one control parameter;
- continue a crossing periodic solution in one control parameter;
- continue a sliding periodic solution, possibly crossing the boundary Σ, in one control parameter;
- continue a boundary equilibrium in two control parameters;
- continue a pseudo-saddle-node bifurcation in two control parameters;
- continue a double tangency bifurcation in planar systems ($n = 2$) in two control parameters;
- continue coinciding tangent points in planar systems ($n = 2$) in two control parameters;
- continue a touching (grazing) bifurcation of periodic solutions in two control parameters;
- continue a standard orbit, possibly crossing the boundary Σ, connecting two tangent points of the same vector field in planar systems ($n = 2$) (e.g. a crossing-crossing bifurcation of sliding periodic solutions) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting two tangent points of different vector fields in planar systems ($n = 2$) (e.g. a buckling (switching) or sliding-crossing bifurcation of sliding periodic solutions) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a tangent point with a pseudo-equilibrium (e.g. the standard part of a sliding homoclinic orbit to a pseudo-saddle in planar systems ($n = 2$)) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a tangent point with a standard saddle in planar systems ($n = 2$) (e.g. the standard part of a sliding homoclinic orbit to a saddle) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a pseudo-equilibrium with a tangent point in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting two pseudo-equilibria in planar systems ($n = 2$) in two control parameters;

- continue a standard orbit, possibly crossing the boundary Σ, connecting a pseudo-equilibrium with a standard saddle with one-dimensional unstable manifold in two control parameters;

Accurate detection of additional local degeneracies along these computations is supported together with switching possibilities between different types of problems (see Section 5 for details).

4 STRUCTURE OF SLIDECONT

In this section the structure of SLIDECONT is presented, together with some comments on its implementation (which is further described in the next two sections). SLIDECONT solves, through numerical continuation techniques, several problems, a list of which has been summarized in the previous section. The general idea is that SLIDECONT sets up the proper defining equations of the user-selected problem in AUTO97 format, so that the computation can be performed by means of standard AUTO97 routines. This is why SLIDECONT is an AUTO97 driver. The driving process and the overall structure of SLIDECONT are illustrated in Figure 2.

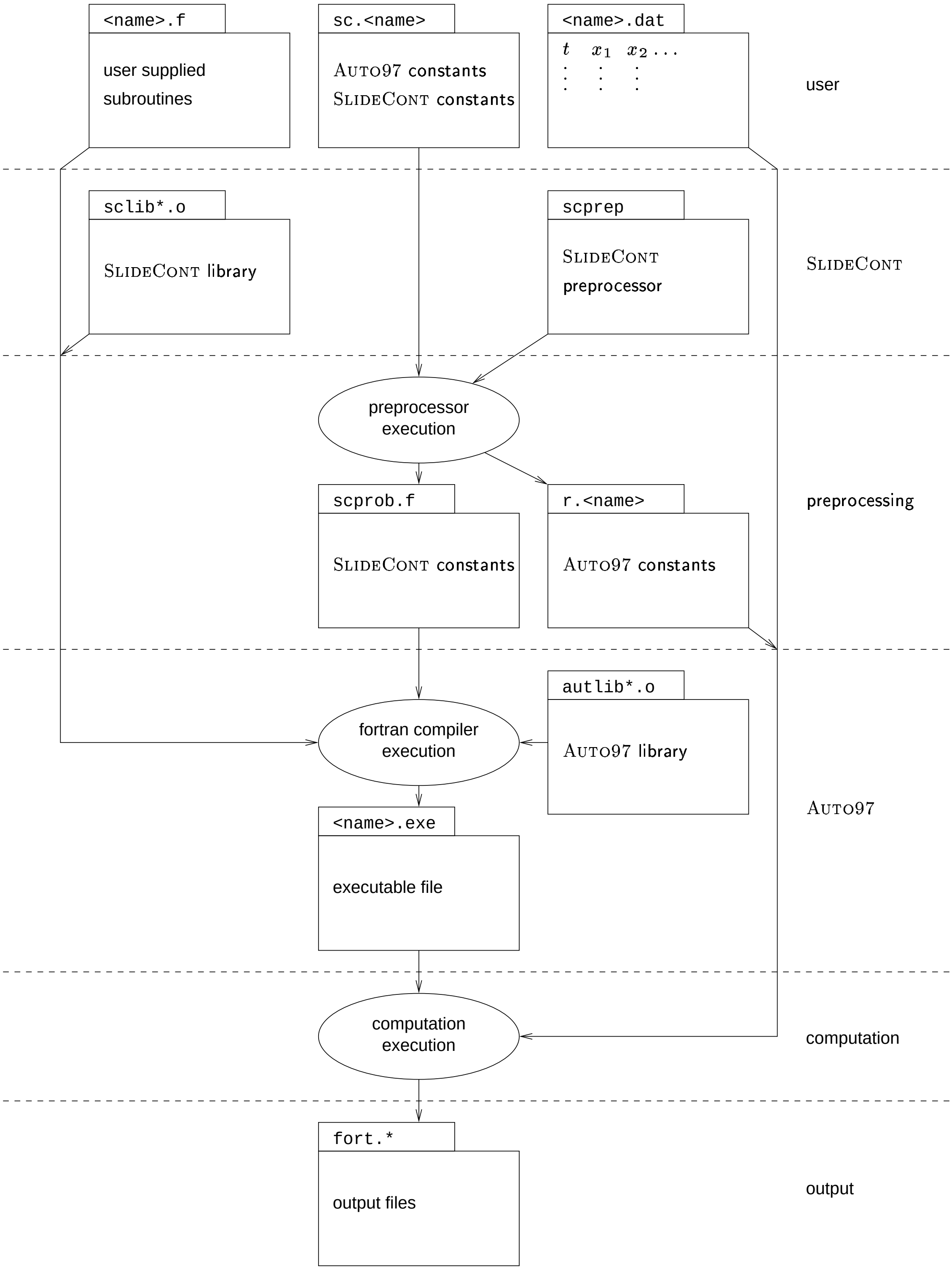

Figure 2: SLIDECONT implementation structure.

As in AUTO97, the user must provide three files. An equations file (`<name>.f`, where `<name>` is a user-selected name), a constants file (`sc.<name>`), and, possibly, a data file (`<name>.dat`) (see Fig. 2). The equations file contains a set of Fortran subroutines specifying model (5), namely the two vector fields, $f^{(1)}$ and $f^{(2)}$, and the scalar function H, the starting solution, either analytically or numerically, and possible state and parameter user functions to be monitored during continuation. Analytical derivatives of $f^{(1)}$, $f^{(2)}$, and H are required by some problems (see Section 5). The constants file specifies all parameters qualifying the AUTO97 continuation algorithms plus some SLIDECONT specific constants, including, of course, the problem type, namely a constant indicating the problem to be solved. As in AUTO97, problem types are coded by means of integer numbers. This coding is done in such a way that SLIDECONT problem types do not overlap with those of AUTO97, so that the SLIDECONT user can access all AUTO97 facilities. Finally, the data file is required to numerically specify the starting solution of boundary-value problems.

As shown in Figure 2, SLIDECONT is composed of two parts: the SLIDECONT preprocessor (`scprep`) and the SLIDECONT library (`sclib*.o`). Preprocessing takes the user constants file and produces the corresponding AUTO97 constants file (`r.<name>`) and a problem specific Fortran file (`scprob.f`) containing the definition of all SLIDECONT constants, as global variables, and an initialization subroutine which sets these variables at the values specified by the user in the constants file. The initialization subroutine is called by the subroutine of the SLIDECONT library first called by AUTO97, so that during the computation SLIDECONT constants are well defined. The library contains the standard AUTO97 user subroutines for each problem and additional support routines.

The compilation of the user and problem specific Fortran files and the linking with the SLIDECONT and AUTO97 libraries produce the executable file (`<name>.exe`), whose execution finally gives the standard AUTO97 output files (`fort.*`).

5 PROBLEM DESCRIPTION

Among other things, AUTO97 computes curves of solutions to *algebraic problems*:

$$F(U,\mu) = 0, \quad U, F \in \mathbf{R}^{n_d},\ \mu \in \mathbf{R}^1,$$

which we rewrite as

$$F(x,\alpha,\beta)=0, \qquad x\in\mathbf{R}^n,\ F\in\mathbf{R}^{n_d},\ \alpha\in\mathbf{R}^m,\ \beta\in\mathbf{R}^{m_d}, \tag{6}$$

as well as paths of solutions to *boundary-value problems* with non-separated boundary conditions:

$$\dot U(\tau)-F(U(\tau),\alpha,\beta)=0, \qquad U,F\in\mathbf{R}^{n_d},\ \alpha\in\mathbf{R}^m,\ \beta\in\mathbf{R}^{m_d},\ \tau\in[0,1] \tag{7}$$

$$b(U(0),U(1),\alpha,\beta)=0, \qquad b\in\mathbf{R}^{n_b}. \tag{8}$$

In both cases, control parameters $\alpha_i,\ i=1,2,\ldots,m$, are allowed to vary and the following conditions on dimensions are imposed: $n+m+m_d=n_d+1$ for equation (6) and $m+m_d=n_b-n_d+1$ for equations (7)–(8). Moreover, AUTO97 can accurately locate zeros along the solution branch of several test functions (see AUTO97 documentation).

For each SLIDECONT problem, we present in a separated subsection the corresponding defining system (6) or (7)–(8) and some details on its implementation. As for the defining system, we report its analytical formulation and specify the following informations: the state space dimension n for which the defining system is valid; the number m of control parameters; the list of other active parameters β (i.e., different from $\alpha_1,\ldots,\alpha_m$) and their total number m_d; for algebraic problems, the dimension n_d of the defining system; for boundary-value problems, the number n_d of differential conditions and the number n_b of boundary conditions.

As for the implementation, we specify, in accordance with AUTO97 notation, the following informations: the AUTO97 problem type `IPS` used to perform the computation; the order `SCIDIFF` up to which analytical derivatives of $f^{(1)}$, $f^{(2)}$, and H are required; the problem dimension `NDIM` (`NDIM`=n_d, except for Subsections 5.1 and 5.2 where an extra state variable is used); the composition of the state vector `U(1)`, …, `U(NDIM)`; the right-hand side vector `F(1)`, …, `F(NDIM)`; the total number of active parameters `NICP` and the list of active parameter indexes `ICP(1)`, …, `ICP(NICP)`, denoting by I_i the index of the user parameter α_i ($i=1,\ldots,m$) and reporting other active parameter symbols in parenthesis after the corresponding indexes; the list of test functions for detecting additional local degeneracies with switching possibilities (a switch to a problem is denoted by a reference to the problem subsection, indicating the vector field(s) at which the problem is applied, and adding a star ($*$) if the switch is not automatic, namely the user must set up the starting solution manually). Notice that boundary conditions FB(1), ..., FB(NBC) (NBC=n_b) are not reported, since their definition is always clear. By ξ_k we denote computed points on a standard cycle at all (principle and intermediate) mesh points; ν_q are the eigenvalues of a (standard or boundary) equilibrium.

It is worth to remark that for boundary-value problems there can be more switching possibilities than those listed, which, however, are not possible with certainty when the corresponding test function vanishes. For example, continuing, for $n = 2$, an orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with the boundary Σ (see Subsection 5.7), a zero of test function 3 detects a tangent point of $f^{(i)}$ at the right boundary-value. If this tangent point is the same tangent point present at the left boundary-value, a condition that does not imply a codimension-2 bifurcation, then left and right boundary-values coincide and one can switch to the continuation of a standard cycle (Subsection 5.6) or of a touching bifurcation (Subsection 5.20). Thus, when a test function vanishes during the continuation of a boundary-value problem, one should check if the critical solution satisfies the defining system of some problems not listed among the switching possibilities.

A common practice in AUTO97 is 'parameter overspecification', namely the number of active parameters NICP is allowed to be greater than the number required by the specified problem. In such cases, the extra activated parameters located at the end of the ICP list are not true continuation parameters, but their values appear in the output. Overspecified parameters are denoted in the parameter index list ICP by indicating in parenthesis the value at which they are set (see Subsections 5.16 and 5.19). Parameter overspecification is also used by SLIDECONT for the implementation of test functions and by the user for defining user test functions (see Section 6), so that the actual number of active parameters NICP can be greater than the value specified here.

Notice that the defining systems for standard equilibrium and cycle continuation (see Subsections 5.4 and 5.6) are not reported, since they correspond to AUTO97 built-in problems. Similarly, the defining systems for the continuation of pseudo-saddle-node and double tangency bifurcations (see Subsections 5.17 and 5.18) are not reported, since they are implemented indirectly by using the defining systems for pseudo-equilibrium and tangent point continuation (Subsections 5.5 and 5.3) and enabling AUTO97 limit point continuation (ISW=2, see AUTO97 documentation).

All the informations described above are organized in a two-columns tabular format where the left column reports symbols or names of the informations, while the right column reports their values or expressions. Recall that SLIDECONT does not support the automatic switches marked by a star ($*$) in the right column. Finally, refer to Figures 3-15 for a graphical representation of boundary-value problems in the case of planar systems ($n = 2$).

5.1 The discontinuity boundary

$$H(x, \alpha) = 0. \tag{9}$$

n, m	2, 0
m_d	0
n_d	1
`IPS`, `SCIDIFF`, `NDIM`	0, 0, 2
`U(1)`, $\ldots$, `U(NDIM)`	x_1, x_2
`F(1)`	$H(x, \alpha)$
`F(2)`	$x_1 - \texttt{PAR}(63)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	1, (63)
t. f. 1: tangent point of $f^{(1)}$ switch to	$\langle H_x(x, \alpha), f^{(1)}(x, \alpha)\rangle$ problem 5.3 for $f^{(1)}$
t. f. 2: tangent point of $f^{(2)}$ switch to	$\langle H_x(x, \alpha), f^{(2)}(x, \alpha)\rangle$ problem 5.3 for $f^{(2)}$
t. f. 3: pseudo-equilibrium switch to	$\langle H_x^{\perp}(x, \alpha), g(x, \alpha)\rangle$ problem 5.5

5.2 A curve of tangent points of vector field $f^{(i)}$ in three-dimensional systems

$$\left\{ \begin{array}{rcl} H(x, \alpha) & = & 0, \\ \langle H_x(x, \alpha), f^{(i)}(x, \alpha)\rangle & = & 0. \end{array} \right. \tag{10}$$

n, m	3, 0
m_d	0
n_d	2
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 3
`U(1)`, $\ldots$, `U(NDIM)`	x_1, x_2, x_3
`F(1)`	$H(x, \alpha)$
`F(2)`	$\langle H_x(x, \alpha), f^{(i)}(x, \alpha)\rangle$
`F(3)`	$x_1 - \texttt{PAR}(63)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	1, (63)
t. f. 1: tangent point of $f^{(j)}$, $j \neq i$ switch to	$\langle H_x(x, \alpha), f^{(j)}(x, \alpha)\rangle$ problem 5.2 for $f^{(j)}$

5.3 *A tangent point of vector field* $f^{(i)}$

$$\begin{cases} H(x,\alpha) = 0, \\ \langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle = 0. \end{cases} \quad (11)$$

n, m	2, 1
m_d	0
n_d	2
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 2
`U(1)`, $\ldots$, `U(NDIM)`	x_1, x_2
`F(1)`	$H(x,\alpha)$
`F(2)`	$\langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	1, (I_1)
t. f. 1: boundary equilibrium of $f^{(i)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(i)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$ switch to	$\langle H_x(x,\alpha), f^{(j)}(x,\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.19
t. f.: double tangency switch to	AUTO97 limit point problem 5.18 for $f^{(i)}$

5.4 *A standard equilibrium of vector field* $f^{(i)}$

n, m	n, 1
`IPS`, `SCIDIFF`, `NDIM`	1, 0, n
`U(1)`, $\ldots$, `U(NDIM)`	$x_1, \ldots, x_n$
`F(`k`)`, $k = 1, \ldots, n$	$f_k^{(i)}(x,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	1, (I_1)
t. f. 1: boundary equilibrium of $f^{(i)}$ switch to	$H(x,\alpha)$ - problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f.: branch switch to	AUTO97 branch problem 5.4 for $f^{(i)}$ (branch switching)
t. f.: limit point	AUTO97 limit point

switch to	continuation of a limit point bifurcation
t. f.: Hopf switch to	AUTO97 Hopf continuation of a Hopf bifurcation

5.5 *A pseudo-equilibrium*

$$\left\{ \begin{array}{rcl} H(x,\alpha) & = & 0, \\ \lambda_1 f^{(1)}(x,\alpha) + \lambda_2 f^{(2)}(x,\alpha) & = & 0, \\ \lambda_1 + \lambda_2 - 1 & = & 0. \end{array} \right. \tag{12}$$

n, m	n, 1
m_d, $(\beta_1, \dots, \beta_{m_d})$	2, (λ_1, λ_2)
n_d	$n+2$
`IPS`, `SCIDIFF`, `NDIM`	0, 0, $n+2$
`U(1)`, ..., `U(NDIM)`	$x_1, \dots, x_n, \lambda_1, \lambda_2$
`F(`k`)`, $k = 1, \dots, n$	$\lambda_1 f_k^{(1)}(x,\alpha) + \lambda_2 f_k^{(2)}(x,\alpha)$
`F(`$n+1$`)`	$H(x,\alpha)$
`F(`$n+2$`)`	$\lambda_1 + \lambda_2 - 1$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	1, (I_1)
t. f. 1: boundary equilibrium of $f^{(1)}$ switch to	λ_2 - problem 5.2 for $f^{(1)}$ ($n=3$) - problem 5.3 for $f^{(1)}$ ($n=2$) - problem 5.4 for $f^{(1)}$ - problem 5.16 for $f^{(1)}$
t. f. 2: boundary equilibrium of $f^{(2)}$ switch to	λ_1 - problem 5.2 for $f^{(2)}$ ($n=3$) - problem 5.3 for $f^{(2)}$ ($n=2$) - problem 5.4 for $f^{(2)}$ - problem 5.16 for $f^{(2)}$
t. f. 3: singular sliding point switch to	$\langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$)
t. f.: pseudo-saddle-node bifurcation switch to	AUTO97 limit point problem 5.17

5.6 *A standard cycle of vector field $f^{(i)}$*

n, m	n, 1
`IPS`, `SCIDIFF`, `NDIM`	2, 0, n
`U(1)`, $\ldots$, `U(NDIM)`	$x_1, \ldots, x_n$
`F(`k`)`, $k = 1, \ldots, n$	$f_k^{(i)}(x, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1,11)
t. f. 1: touching bifurcation switch to	$\min_{\{k=1,\ldots,NTST*NCOL\}}\{(-1)^i H(\xi_k, \alpha)\}$ - problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) * - problem 5.14 for $f^{(i)}$ * - problem 5.20 for $f^{(i)}$ *
t. f.: branch switch to	AUTO97 branch problem 5.6 for $f^{(i)}$ (branch switching)
t. f.: limit point switch to	AUTO97 limit point continuation of a limit point bifurcation
t. f.: period doubling switch to	AUTO97 period doubling continuation of a period doubling bifurcation
t. f.: torus switch to	AUTO97 torus continuation of a torus bifurcation

5.7 *An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with the boundary Σ*

$$\begin{cases} \dot{u} - Tf^{(i)}(u, \alpha) &= 0, \\ H(u(0), \alpha) &= 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle &= 0, \\ H(u(1), \alpha) &= 0. \end{cases} \tag{13}$$

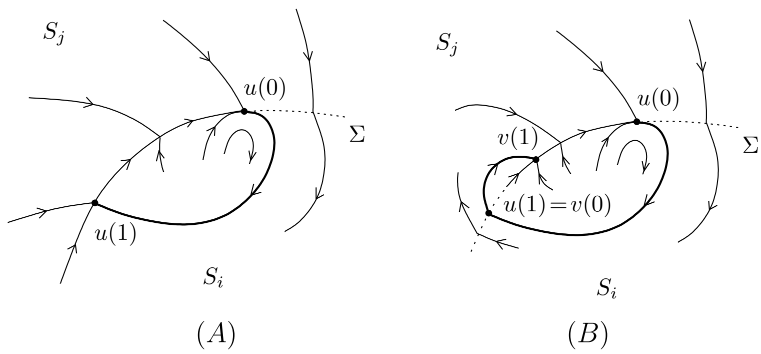

Figure 3: Boundary-value problems corresponding to (A): Subsection 5.7; (B): Subsection 5.8.

n, m	2, 1
m_d, $(\beta_1, \ldots, \beta_{m_d})$	1, (T)
n_d, n_b	2, 3
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2
`U(1)`, $\ldots$, `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1, 61(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.19
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$
t. f. 4: tangent point of $f^{(j)}$, $j \neq i$, at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.8 for $f^{(i)}$, $f^{(j)}$ * - problem 5.23 for $f^{(i)}$, $f^{(j)}$
t. f. 5: pseudo-equilibrium at $u(1)$ switch to	$\langle H_x^{\perp}(u(1), \alpha), g(u(1), \alpha)\rangle$ - problem 5.5 - problem 5.25 for $f^{(i)}$

5.8 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with the boundary Σ

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u, \alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v, \alpha) & = & 0, \\ H(u(0), \alpha) & = & 0, \\ \langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle & = & 0, \\ H(u(1), \alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1), \alpha) & = & 0. \end{array}\right. \tag{14}$$

n, m	2, 1
m_d, $(\beta_1, \ldots, \beta_{m_d})$	2, (T_i, T_j)
n_d, n_b	4, 6
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`, $\ldots$, `U(NDIM)`	u_1, u_2, v_1, v_2

`F(`k`)`, $k=1,2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=3,4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, …, `ICP(NICP)`)	3, (I_1, 61(T_i), 62(T_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.19
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.23 for $f^{(i)}$, $f^{(j)}$ *
t. f. 5: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.22 for $f^{(i)}$, $f^{(j)}$
t. f. 6: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.24 for $f^{(i)}$, $f^{(j)}$
t. f. 7: pseudo-equilibrium at $v(1)$ switch to	$\langle H_x^{\perp}(v(1),\alpha), g(v(1),\alpha)\rangle$ - problem 5.5 - problem 5.26 for $f^{(i)}$, $f^{(j)}$

5.9 An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with the boundary Σ

$$\left\{\begin{array}{rcll} \dot{u} - T f^{(i)}(u,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ H(u(1),\alpha) & = & 0. & \end{array}\right. \tag{15}$$

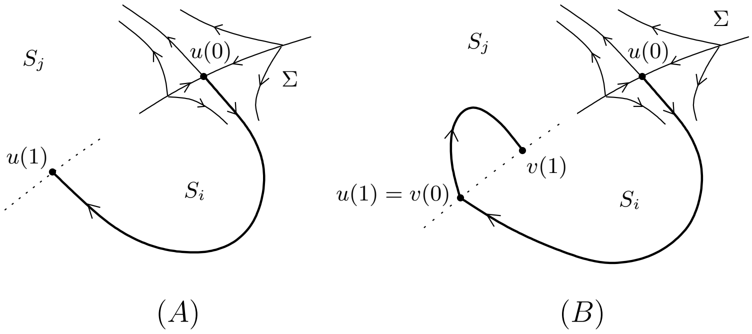

Figure 4: Boundary-value problems corresponding to (A): Subsection 5.9; (B): Subsection 5.10.

n, m	n, 1
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+3$
`IPS`, `SCIDIFF`, `NDIM`	4, 0, n
`U(1)`, $\ldots$, `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	4, (I_1, 61(T), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) - problem 5.19 ($n = 2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.29 for $f^{(i)}$
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.31 for $f^{(i)}$, $f^{(j)}$
t. f. 6: pseudo-equilibrium at $u(1)$ ($n = 2$)	$\langle H_x^{\perp}(u(1), \alpha), g(u(1), \alpha)\rangle$

switch to	- problem 5.5 - problem 5.33 for $f^{(i)}$
note	the initial value of $u(0)$ must be specified in PAR(67),..., PAR(67+n-1) in the user subroutine SCSTPNT

5.10 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with the boundary Σ

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0. \end{array}\right. \tag{16}$$

n, m	n, 1
m_d, $(\beta_1,\ldots,\beta_{m_d})$	4, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+4$
IPS, SCIDIFF, NDIM	4, 0, $2n$
U(1),..., U(NDIM)	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
F(k), $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
F(k), $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
NICP, (ICP(1),..., ICP(NICP))	5, (I_1, 61(T_i), 62(T_j), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(i)}$ ($n=2$) * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$)

t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.29 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(j)}$ ($n=2$) * - problem 5.31 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.30 for $f^{(i)}$, $f^{(j)}$
t. f. 7: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.32 for $f^{(i)}$, $f^{(j)}$
t. f. 8: pseudo-equilibrium at $v(1)$ ($n=2$) switch to	$\langle H_x^{\perp}(v(1),\alpha), g(v(1),\alpha)\rangle$ - problem 5.5 - problem 5.34 for $f^{(i)}$, $f^{(j)}$
note	the initial value of $u(0)$ must be specified in `PAR(67)`, $\ldots$, `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.11 An orbit of vector field $f^{(i)}$ connecting a saddle of $f^{(i)}$ with the boundary Σ

$$\left\{\begin{array}{rcll} \dot{u} - T f^{(i)}(u,\alpha) & = & 0, & \\ f^{(i)}(y,\alpha) & = & 0, & \\ \left[f_x^{(i)}(y,\alpha)\right]^T w - \nu w & = & 0, & \nu < 0, \\ \langle w, w\rangle - 1 & = & 0, & \\ \langle w, y - u(0)\rangle & = & 0, & \\ H(u(1),\alpha) & = & 0. & \end{array}\right. \tag{17}$$

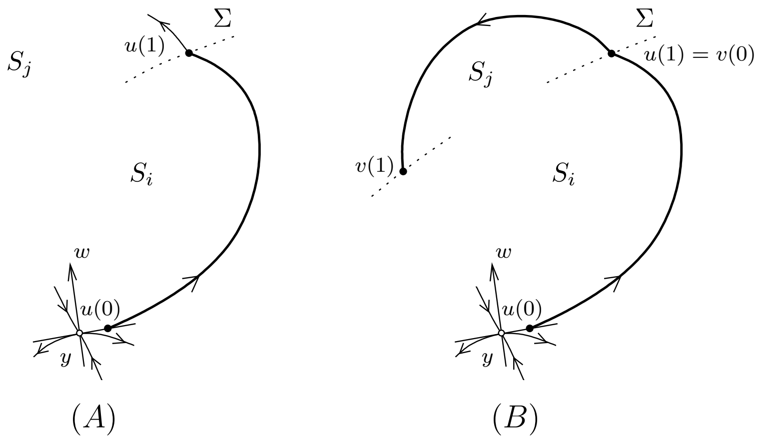

Figure 5: Boundary-value problems corresponding to (A): Subsection 5.11; (B): Subsection 5.12.

n, m	2, 1
m_d, $(\beta_1, \ldots, \beta_{m_d})$	5, $(y_1, y_2, w_1, w_2, \nu)$
n_d, n_b	2, 7
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2
`U(1)`$, \ldots,$ `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	6, (I_1, 67(y_1), 68(y_2), 69(w_1), 70(w_2), 71(ν))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$H(u(0), \alpha)$ - problem 5.1 - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.7 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: branch/limit point switch to	$\mathrm{Re}\nu_p \ : \ p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.27 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$, $j \neq i$, at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$
t. f. 5: pseudo-equilibrium at $u(1)$ switch to	$\langle H_x^{\perp}(u(1), \alpha), g(u(1), \alpha)\rangle$ - problem 5.5 - problem 5.35 for $f^{(i)}$ *
note	the initial value of y must be specified in `PAR(67)`$, \ldots,$ `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.12 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a saddle of $f^{(i)}$ with the boundary Σ

$$\begin{cases} \dot{u} - T_i f^{(i)}(u,\alpha) &= 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) &= 0, \\ f^{(j)}(y,\alpha) &= 0, \\ \left[f_x^{(j)}(y,\alpha)\right]^T w - \nu w &= 0, \quad \nu < 0, \\ \langle w, w\rangle - 1 &= 0, \\ \langle w, y - u(0)\rangle &= 0, \\ H(u(1),\alpha) &= 0, \\ u(1) - v(0) &= 0, \\ H(v(1),\alpha) &= 0. \end{cases} \tag{18}$$

n, m	2, 1
m_d, $(\beta_1,\ldots,\beta_{m_d})$	6, $(T_j, y_1, y_2, w_1, w_2, \nu)$
n_d, n_b	4, 10
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`$,\ldots,$`U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k=1,2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=3,4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$ `ICP(NICP)`)	7, (I_1, 62(T_j), 67(y_1), 68(y_2), 69(w_1), 70(w_2), 71(ν))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$H(u(0),\alpha)$ - problem 5.1 - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.7 for $f^{(i)}$ * - problem 5.8 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: branch/limit point switch to	$\mathrm{Re}\nu_p \;:\; p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.27 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ *
t. f. 5: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$
t. f. 6: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.28 for $f^{(j)}$, $f^{(i)}$ *

t. f. 7: pseudo-equilibrium at $v(1)$ switch to	$\langle H_x^{\perp}(v(1),\alpha), g(v(1),\alpha)\rangle$ - problem 5.5 - problem 5.36 for $f^{(j)}$, $f^{(i)}$ *
note	the initial value of y must be specified in `PAR(67)`, ..., `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.13 A crossing cycle

$$\left\{\begin{array}{rcl} \dot{u} - T_1 f^{(1)}(u,\alpha) & = & 0, \\ \dot{v} - T_2 f^{(2)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ v(1) - u(0) & = & 0. \end{array}\right. \tag{19}$$

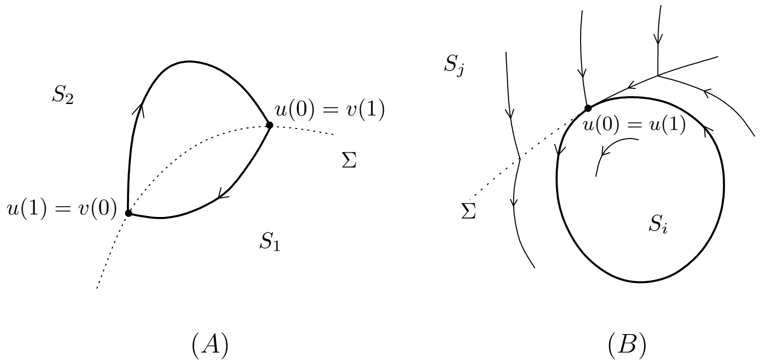

Figure 6: Boundary-value problems corresponding to (A): Subsection 5.13; (B): Subsection 5.20.

n, m	n, 1
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, (T_1,T_2)
n_d, n_b	$2n$, $2n+2$
`IPS`, `SCIDIFF`, `NDIM`	4, 0, $2n$
`U(1)`, ..., `U(NDIM)`	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
`F(`k`)`, $k=1,\ldots,n$	$T_1 f_k^{(1)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_2 f_k^{(2)}(v,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	3, (I_1, 61(T_1), 62(T_2))
t. f. 1: tangent point of $f^{(1)}$ at $u(0)$	$\langle H_x(u(0),\alpha), f^{(1)}(u(0),\alpha)\rangle$

switch to	- problem 5.2 for $f^{(1)}$ ($n = 3$) - problem 5.3 for $f^{(1)}$ ($n = 2$) - problem 5.7 for $f^{(1)}$ ($n = 2$) * - problem 5.8 for $f^{(1)}$, $f^{(2)}$ ($n = 2$) - problem 5.15 for $f^{(1)}$, $f^{(2)}$ * - problem 5.22 for $f^{(1)}$, $f^{(2)}$ ($n = 2$)
t. f. 2: tangent point of $f^{(2)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(2)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(2)}$ ($n = 3$) - problem 5.3 for $f^{(2)}$ ($n = 2$)
t. f. 3: tangent point of $f^{(1)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(1)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(1)}$ ($n = 3$) - problem 5.3 for $f^{(1)}$ ($n = 2$)
t. f. 4: tangent point of $f^{(2)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(2)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(2)}$ ($n = 3$) - problem 5.3 for $f^{(2)}$ ($n = 2$) - problem 5.7 for $f^{(2)}$ ($n = 2$) * - problem 5.8 for $f^{(2)}$, $f^{(1)}$ ($n = 2$) * - problem 5.15 for $f^{(2)}$, $f^{(1)}$ * - problem 5.22 for $f^{(2)}$, $f^{(1)}$ ($n = 2$) *

5.14 A sliding cycle with a standard orbit of vector field $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{s} - T_0 g(s,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ s(0) - u(1) & = & 0, \\ s(1) - u(0) & = & 0, \end{array}\right. \tag{20}$$

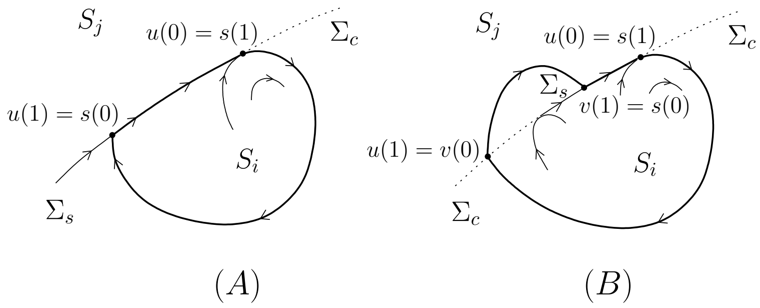

Figure 7: Boundary-value problems corresponding to (A): Subsection 5.14; (B): Subsection 5.15.

n, m	n, 1
m_d	2, (T_i, T_0)
n_d, n_b	$2n, 2n+2$
`IPS`, `SCIDIFF`, `NDIM`, `NBC`	4, 1, $2n$, $2n+2$
`U(1)`$, \dots,$ `U(NDIM)`	$u_1, \dots, u_n$, $s_1, \dots, s_n$
`F(`k`)`, $k = 1, \dots, n$	$T_i f_k^{(i)}(u, \alpha)$
`F(`k`)`, $k = n+1, \dots, 2n$	$T_0 g(s, \alpha)$
`NICP`, (`ICP(1)`$, \dots,$ `ICP(NICP)`)	3, (1(α), 60(T_s), 61(T_i))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.9 for $f^{(i)}$ ($n = 2$) * - problem 5.19 ($n = 2$)
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.21 for $f^{(i)}$ ($n = 2$) *
t. f. 4: tangent point of $f^{(j)}$, $j \neq i$, at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.15 for $f^{(i)}$, $f^{(j)}$ *
t. f. 5: pseudo-equilibrium at $u(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(1), \alpha), g(u(1), \alpha)\rangle$ - problem 5.5 - problem 5.25 for $f^{(i)}$ *

5.15 A sliding cycle with standard orbits of vector fields $f^{(i)}$ and $f^{(j)}$ ($j \neq i$)

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ \dot{s} - T_0 g(s,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ H(u(1),\alpha) & = & 0, \\ v(0) - u(1) & = & 0, \\ s(0) - v(1) & = & 0, \\ s(1) - u(0) & = & 0, \end{array}\right. \tag{21}$$

n, m	n, 1
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, (T_i, T_j, T_0)
n_d, n_b	$3n$, $3n+3$
`IPS`, `SCIDIFF`, `NDIM`, `NBC`	4, 1, $3n$, $3n+3$
`U(1)`, $\ldots$, `U(NDIM)`	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$, $s_1,\ldots,s_n$
`F(`k`)`, $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(u,\alpha)$
`F(`k`)`, $k=2n+1,\ldots,3n$	$T_0 g(s,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	4, (1(α), 60(T_s), 61(T_i), 62(T_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n=2$) switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.9 for $f^{(i)}$ ($n=2$) * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.19 ($n=2$)
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.21 for $f^{(i)}$ ($n=2$) *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(j)}$ ($n=2$) * - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n=2$) *
t. f. 5: tangent point of $f^{(i)}$ at $v(1)$	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$

switch to	- problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.22 for $f^{(i)}$, $f^{(j)}$ ($n=2$) *
t. f. 6: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.24 for $f^{(i)}$, $f^{(j)}$ ($n=2$) *
t. f. 7: pseudo-equilibrium at $v(1)$ ($n=2$) switch to	$\langle H_x^{\perp}(v(1),\alpha), g(v(1),\alpha)\rangle$ - problem 5.5 - problem 5.26 for $f^{(i)}$, $f^{(j)}$ *

5.16 A boundary equilibrium of vector field $f^{(i)}$

$$\begin{cases} f^{(i)}(x,\alpha) &=& 0, \\ H(x,\alpha) &=& 0. \end{cases} \tag{22}$$

n, m	n, 2
m_d	0
n_d	$n+1$
`IPS`, `SCIDIFF`, `NDIM`	0, 0, $n+1$
`U(1)`, $\ldots$, `U(NDIM)`	$x_1,\ldots,x_n$, α_2
`F(`k`)`, $k=1,\ldots,n$	$f_k^{(i)}(x,\alpha)$
`F(`$n+1$`)`	$H(x,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1, I_2(`PAR(`I_2`)`=`U(`$n+1$`)`))
t. f. 1: Hopf switch to	$\mathrm{Re}\nu_p\ :\ p=\arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\ :\ \mathrm{Im}\nu_q\neq 0\}$ continuation of a Hopf bifurcation *
t. f. 2: branch/limit point switch to	$\mathrm{Re}\nu_p\ :\ p=\arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\ :\ \mathrm{Im}\nu_q=0\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *
t. f. 3: tangent point of $f^{(j)}$, $j\neq i$ switch to	$\langle H_x(x,\alpha), f^{(j)}(x,\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$)

5.17 A pseudo-saddle-node bifurcation

n, m	n, 2
`IPS`, `SCIDIFF`, `NDIM`	0, 0, $n+2$
`U(1)`, $\ldots$, `U(NDIM)`	$x_1,\ldots,x_n,\lambda_1,\lambda_2$

`F(`k`)`, $k = 1, \ldots, n$	$\lambda_1 f_k^{(1)}(x,\alpha) + \lambda_2 f_k^{(2)}(x,\alpha)$
`F(`$n+1$`)`	$H(x,\alpha)$
`F(`$n+2$`)`	$\lambda_1 + \lambda_2 - 1$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	2, (I_1, I_2)
t. f. 1: boundary equilibrium of $f^{(1)}$ switch to	λ_2 - problem 5.2 for $f^{(1)}$ ($n = 3$) - problem 5.3 for $f^{(1)}$ ($n = 2$) - problem 5.4 for $f^{(1)}$ - problem 5.16 for $f^{(1)}$
t. f. 2: boundary equilibrium of $f^{(2)}$ switch to	λ_1 - problem 5.2 for $f^{(2)}$ ($n = 3$) - problem 5.3 for $f^{(2)}$ ($n = 2$) - problem 5.4 for $f^{(2)}$ - problem 5.16 for $f^{(2)}$
t. f. 3: singular sliding point switch to	$\langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.19 ($n = 2$)
note	`ISW`$= 2$

5.18 A double tangency bifurcation of vector field $f^{(i)}$

n, m	2, 2
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 2
`U(1)`, ..., `U(NDIM)`	x_1, x_2
`F(1)`	$H(x,\alpha)$
`F(2)`	$\langle H_x(x,\alpha), f^{(i)}(x,\alpha)\rangle$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	2, (I_1, I_2)
t. f. 1: boundary equilibrium of $f^{(i)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(i)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$ switch to	$\langle H_x(x,\alpha), f^{(j)}(x,\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.19
note	`ISW`$= 2$

5.19 Coinciding tangent points

$$\left\{\begin{array}{rcl} H(x,\alpha) & = & 0, \\ \langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle & = & 0, \\ \langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle & = & 0. \end{array}\right. \tag{23}$$

n, m	2, 2
m_d	0
n_d	3
`IPS`, `SCIDIFF`, `NDIM`	0, 1, 3
`U(1)`, $\ldots$, `U(NDIM)`	x_1, x_2, α_2
`F(1)`	$H(x,\alpha)$
`F(2)`	$\langle H_x(x,\alpha), f^{(1)}(x,\alpha)\rangle$
`F(3)`	$\langle H_x(x,\alpha), f^{(2)}(x,\alpha)\rangle$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	2, (I_1, I_2(`PAR(`I_2`)=U(3)`))
t. f. 1: boundary equilibrium of $f^{(1)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(1)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(1)}$ - problem 5.16 for $f^{(1)}$
t. f. 2: boundary equilibrium of $f^{(2)}$ switch to	$\langle H_x^{\perp}(x,\alpha), f^{(2)}(x,\alpha)\rangle$ - problem 5.4 for $f^{(2)}$ - problem 5.16 for $f^{(2)}$
t. f.: double tangency switch to	AUTO97 limit point problem 5.18 for $f^{(1)}$ or for $f^{(2)}$ *

5.20 A touching (grazing) cycle of vector field $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ u(0) - u(1) & = & 0. \end{array}\right. \tag{24}$$

n, m	n, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	1, (T)
n_d, n_b	n, $n+2$
`IPS`, `SCIDIFF`, `NDIM`	7, 1, n
`U(1)`, $\ldots$, `U(NDIM)`	$u_1,\ldots,u_n$
`F(`k`)`, $k=1,\ldots,n$	$Tf_k^{(i)}(u,\alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	3, (I_1, I_2, 61(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n=2$)	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$

switch to	- problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.15 for $f^{(i)}$, $f^{(j)}$ * - problem 5.9 for $f^{(i)}$ ($n=2$) - problem 5.10 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.19 ($n=2$) - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.25 for $f^{(i)}$ ($n=2$) - problem 5.29 for $f^{(i)}$ ($n=2$) - problem 5.31 for $f^{(i)}$, $f^{(j)}$ ($n=2$)
t. f.: branch switch to	AUTO97 branch problem 5.20 for $f^{(i)}$ (branch switching)
t. f.: limit point switch to	AUTO97 limit point continuation of a limit point bifurcation
t. f.: period doubling switch to	AUTO97 period doubling continuation of a period doubling bifurcation
t. f.: torus switch to	AUTO97 torus continuation of a torus bifurcation

5.21 An orbit of vector field $f^{(i)}$ connecting two tangent points of $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) &=& 0,\\ H(u(0),\alpha) &=& 0,\\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &=& 0,\\ H(u(1),\alpha) &=& 0,\\ \langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle &=& 0. \end{array}\right. \tag{25}$$

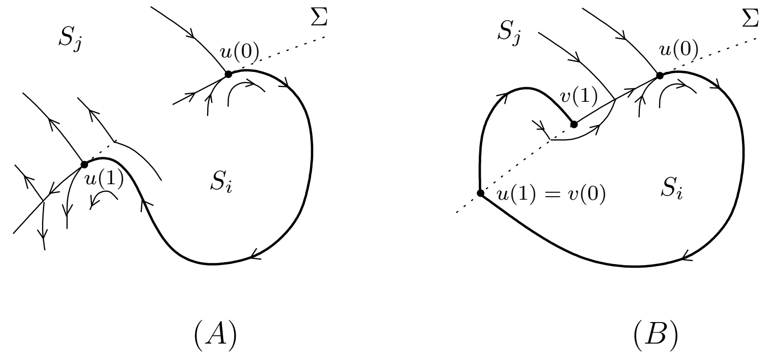

Figure 8: Boundary-value problems corresponding to (A): Subsection 5.21; (B): Subsection 5.22.

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	1, (T)
n_d, n_b	2, 4
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2
`U(1)`$, \ldots ,$ `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots ,$ `ICP(NICP)`)	3, (I_1, I_2, 61(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.19 - problem 5.29 for $f^{(i)}$
t. f. 3: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x^{\perp}(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$ - problem 5.25 for $f^{(i)}$
t. f. 4: tangent point of $f^{(j)}$, $j \neq i$, at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.8 for $f^{(i)}$, $f^{(j)}$ * - problem 5.19 - problem 5.23 for $f^{(i)}$, $f^{(j)}$ - problem 5.25 for $f^{(i)}$

5.22 *A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two tangent points of $f^{(i)}$*

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1)-v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle & = & 0. \end{array} \right. \tag{26}$$

n, m	2, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, (T_i, T_j)
n_d, n_b	4, 7
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`$,\ldots,$`U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k=1,2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=3,4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$`ICP(NICP)`)	4, (I_1, I_2, 61(T_i), 62(T_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.19 - problem 5.30 for $f^{(i)}$, $f^{(j)}$
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.23 for $f^{(i)}$, $f^{(j)}$ * - problem 5.23 for $f^{(j)}$, $f^{(i)}$ *
t. f. 5: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x^{\perp}(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 6: tangent point of $f^{(j)}$ at $v(1)$	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$

switch to	- problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.19 - problem 5.24 for $f^{(i)}$, $f^{(j)}$ - problem 5.26 for $f^{(i)}$, $f^{(j)}$

5.23 An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$ ($j \neq i$)

$$\left\{ \begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ H(u(1),\alpha) & = & 0, \\ \langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle & = & 0. \end{array} \right. \quad (27)$$

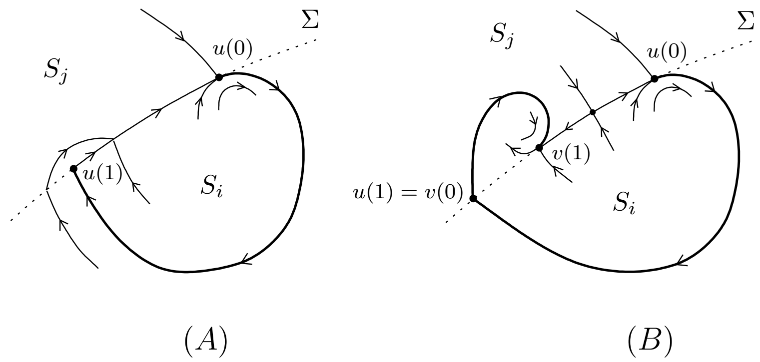

Figure 9: Boundary-value problems corresponding to (A): Subsection 5.23; (B): Subsection 5.24.

n, m	2, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	1, (T)
n_d, n_b	2, 4
IPS, SCIDIFF, NDIM	4, 1, 2
U(1), $\ldots$, U(NDIM)	u_1, u_2
F(k), $k = 1,2$	$Tf_k^{(i)}(u,\alpha)$
NICP, (ICP(1), $\ldots$, ICP(NICP))	3, (I_1, I_2, 61(T))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$

switch to	- problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.19 - problem 5.31 for $f^{(i)}$, $f^{(j)}$
t. f. 3: boundary equilibrium of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x^{\perp}(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.16 for $f^{(j)}$ - problem 5.25 for $f^{(i)}$
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.5 - problem 5.19 - problem 5.21 for $f^{(i)}$ - problem 5.25 for $f^{(i)}$

5.24 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) &=& 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) &=& 0, \\ H(u(0),\alpha) &=& 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle &=& 0, \\ H(u(1),\alpha) &=& 0, \\ u(1) - v(0) &=& 0, \\ H(v(1),\alpha) &=& 0, \\ \langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle &=& 0. \end{array} \right. \tag{28}$$

n, m	2, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	2, (T_i, T_j)
n_d, n_b	4, 7
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`$,\ldots,$`U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k=1,2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=3,4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`$,\ldots,$`ICP(NICP)`)	4, (I_1, I_2, 61(T_i), 62(T_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$

t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.19 - problem 5.32 for $f^{(i)}$, $f^{(j)}$
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$ *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.21 for $f^{(j)}$ * - problem 5.23 for $f^{(i)}$, $f^{(j)}$ *
t. f. 5: boundary equilibrium of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x^{\perp}(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.16 for $f^{(j)}$
t. f. 6: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.5 - problem 5.19 - problem 5.22 for $f^{(i)}$, $f^{(j)}$

5.25 An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium

$$\left\{\begin{array}{rcll} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, & \\ H(u(1),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(1),\alpha) + \lambda_j f^{(j)}(u(1),\alpha) & = & 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 & = & 0. & \end{array}\right. \qquad (29)$$

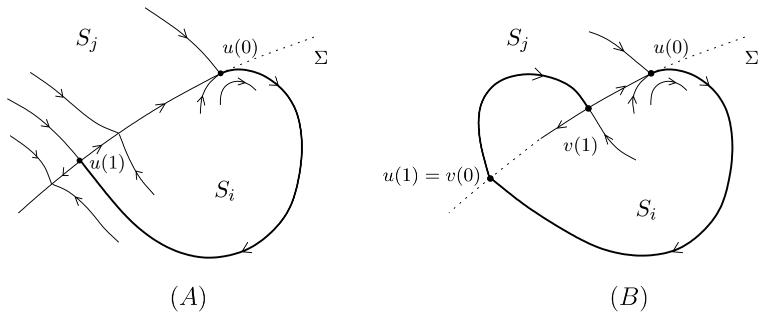

Figure 10: Boundary-value problems corresponding to (A): Subsection 5.25; (B): Subsection 5.26.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`, $\ldots$, `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`, $\ldots$, `ICP(NICP)`)	5, (I_1, I_2, 61(T), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.9 for $f^{(i)}$ ($n = 2$) - problem 5.19 ($n = 2$)
t. f. 3: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 4: boundary equilibrium of $f^{(j)}$ at $u(1)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.16 for $f^{(j)}$ - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 5: singular sliding point at $u(1)$	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$

switch to	- problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.19 ($n=2$) - problem 5.21 for $f^{(i)}$ ($n=2$) - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n=2$)
note	the initial value of $u(1)$ must be specified in `PAR(67+`n`)`, ..., `PAR(67+2`n`-1)` in the user subroutine `SCSTPNT`

5.26 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \lambda_i f^{(i)}(v(1),\alpha) + \lambda_j f^{(j)}(v(1),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0. \end{array} \right. \tag{30}$$

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	4, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+5$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, $2n$
`U(1)`, ..., `U(NDIM)`	$u_1, \ldots, u_n, v_1, \ldots, v_n$
`F(`k`)`, $k = 1, \ldots, n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k = n+1, \ldots, 2n$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	6, (I_1, I_2, 61(T_i), 62(T_j) 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ ($n=2$) switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.9 for $f^{(i)}$ ($n=2$) * - problem 5.10 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$)

t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.21 for $f^{(i)}$ ($n=2$) *
t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(j)}$ ($n=2$) * - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n=2$) * - problem 5.25 for $f^{(j)}$ *
t. f. 5: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$ - problem 5.22 for $f^{(i)}$, $f^{(j)}$ ($n=2$)
t. f. 6: boundary equilibrium of $f^{(j)}$ at $v(1)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 7: singular sliding point at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$) - problem 5.22 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.24 for $f^{(i)}$, $f^{(j)}$ ($n=2$)
note	the initial value of $v(1)$ must be specified in `PAR(67+`n`)`, …, `PAR(67+2`n`-1)` in the user subroutine `SCSTPNT`

5.27 An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ to a saddle

$$\left\{\begin{array}{rcl}
\dot{u} - Tf^{(i)}(u,\alpha) & = & 0,\\
H(u(0),\alpha) & = & 0,\\
\langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0,\\
f^{(i)}(y,\alpha) & = & 0,\\
\left[f_x^{(i)}(y,\alpha)\right]^T w - \nu w & = & 0, \quad \nu > 0,\\
\langle w,w\rangle - 1 & = & 0,\\
\langle w, y-u(1)\rangle & = & 0.
\end{array}\right. \tag{31}$$

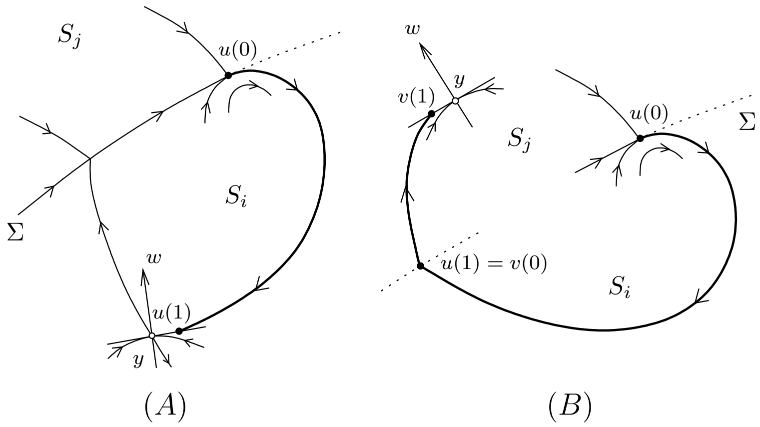

Figure 11: Boundary-value problems corresponding to (A): Subsection 5.27; (B): Subsection 5.28.

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	5, $(y_1, y_2, w_1, w_2, \nu)$
n_d, n_b	2, 8
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 2
`U(1)`$, \ldots,$ `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	7, (I_1, I_2, 67(y_1), 68(y_2), 69(w_1), 70(w_2), 71(ν))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 2: tangent point of $f^{(j)}$, $j \neq i$, at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(j)}(u(0), \alpha) \rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.19 - problem 5.35 for $f^{(i)}$
t. f. 3: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	$H(u(1), \alpha)$ - problem 5.1 - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.7 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$ - problem 5.25 for $f^{(i)}$
t. f. 4: branch/limit point switch to	$\text{Re}\nu_p \, : \, p = \arg\min_q \{ \lvert \text{Re}\nu_q \rvert \}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *

note	the initial value of y must be specified in `PAR(67+`n`)`, ..., `PAR(67+`$2n$`-1)` in the user subroutine `SCSTPNT`

5.28 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ to a saddle of $f^{(j)}$

$$\left\{ \begin{array}{rcll} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, & \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle & = & 0, & \\ H(u(1),\alpha) & = & 0, & \\ u(1) - v(0) & = & 0, & \\ f^{(j)}(y,\alpha) & = & 0, & \\ \left[f_x^{(j)}(y,\alpha)\right]^T w - \nu w & = & 0, & \nu > 0, \\ \langle w,w\rangle - 1 & = & 0, & \\ \langle w, y - v(1)\rangle & = & 0. & \end{array} \right. \tag{32}$$

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	6, $(T_i, y_1, y_2, w_1, w_2, \nu)$
n_d, n_b	4, 11
`IPS`, `SCIDIFF`, `NDIM`	4, 1, 4
`U(1)`, ..., `U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k = 1, 2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k = 3, 4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	8, (I_1, I_2, 61(T_i), 67(y_1), 68(y_2), 69(w_1), 70(w_2), 71(ν))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	$\langle H_x^{\perp}(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$ - problem 5.35 for $f^{(j)}$ *
t. f. 2: tangent point of $f^{(j)}$ at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(j)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ * - problem 5.19 - problem 5.36 for $f^{(i)}$, $f^{(j)}$
t. f. 3: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.21 for $f^{(i)}$ *

t. f. 4: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.23 for $f^{(i)}$, $f^{(j)}$ * - problem 5.27 for $f^{(j)}$ *
t. f. 5: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	$H(v(1),\alpha)$ - problem 5.1 - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.8 for $f^{(i)}$, $f^{(j)}$ - problem 5.16 for $f^{(j)}$ - problem 5.24 for $f^{(i)}$, $f^{(j)}$ - problem 5.26 for $f^{(i)}$, $f^{(j)}$
t. f. 6: branch/limit point switch to	$\mathrm{Re}\nu_p \, : \, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert\}$ problem 5.4 for $f^{(j)}$ (branch switching) */ continuation of a limit point bifurcation *
note	the initial value of y must be specified in `PAR(67+`n`)`, ..., `PAR(67+`$2n$`-1)` in the user subroutine `SCSTPNT`

5.29 An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$

$$\begin{cases} \dot{u} - Tf^{(i)}(u,\alpha) &= 0, \\ H(u(0),\alpha) &= 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) &= 0, \quad j \neq i, \\ \lambda_i + \lambda_j - 1 &= 0, \\ H(u(1),\alpha) &= 0, \\ \langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle &= 0. \end{cases} \tag{33}$$

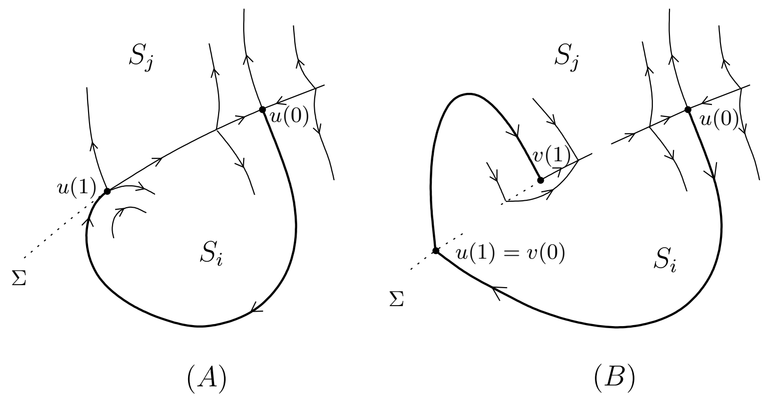

Figure 12: Boundary-value problems corresponding to (A): Subsection 5.29; (B): Subsection 5.30.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`$, \ldots,$ `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	5, (I_1, I_2, 61(T), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) - problem 5.19 ($n = 2$) - problem 5.21 for $f^{(i)}$ ($n = 2$)
t. f. 4: boundary equilibrium of $f^{(i)}$ at $u(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.19 ($n = 2$) - problem 5.31 for $f^{(i)}$, $f^{(j)}$
note	the initial value of $u(0)$ must be specified in `PAR(67)`$, \ldots,$ `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.30 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$

$$\left\{\begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) &=& 0,\\ \dot{v} - T_j f^{(j)}(v,\alpha) &=& 0,\\ H(u(0),\alpha) &=& 0,\\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) &=& 0,\\ \lambda_i + \lambda_j - 1 &=& 0,\\ H(u(1),\alpha) &=& 0,\\ u(1) - v(0) &=& 0,\\ H(v(1),\alpha) &=& 0,\\ \langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle &=& 0. \end{array}\right. \tag{34}$$

n, m	n, 4
m_d, $(\beta_1,\ldots,\beta_{m_d})$	4, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+5$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, $2n$
`U(1)`, ..., `U(NDIM)`	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
`F(`k`)`, $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	6, (I_1, I_2, 61(T_i), 62(T_j), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(i)}$ ($n=2$) * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$) - problem 5.22 for $f^{(i)}$, $f^{(j)}$ ($n=2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.29 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$

switch to	- problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(j)}$ ($n=2$) * - problem 5.23 for $f^{(j)}$, $f^{(i)}$ ($n=2$) * - problem 5.31 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(i)}$ at $v(1)$ ($n=2$) switch to	$\langle H_x^{\perp}(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.16 for $f^{(i)}$ - problem 5.34 for $f^{(i)}$, $f^{(j)}$
t. f. 7: tangent point of $f^{(j)}$ at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.5 ($n=2$) - problem 5.19 ($n=2$) - problem 5.32 for $f^{(i)}$, $f^{(j)}$
note	the initial value of $u(0)$ must be specified in `PAR(67)`, ..., `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.31 An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$ ($j \neq i$)

$$\left\{\begin{array}{rcl} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ \langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle & = & 0. \end{array}\right. \tag{35}$$

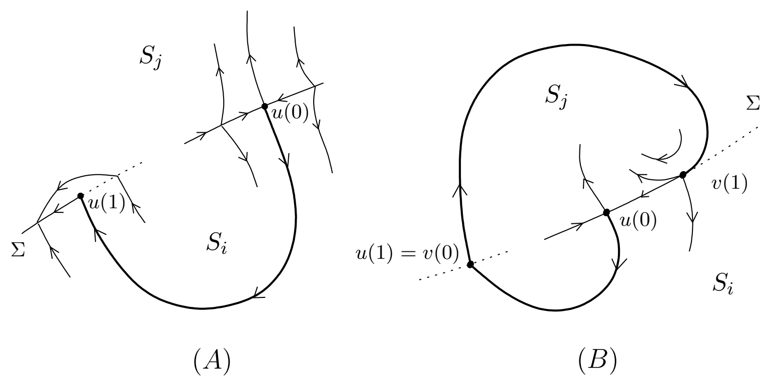

Figure 13: Boundary-value problems corresponding to (A): Subsection 5.31; (B): Subsection 5.32.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	3, $(T, \lambda_i, \lambda_j)$
n_d, n_b	n, $n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`$, \ldots,$ `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	5, (I_1, I_2, 61(T), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n = 2$) * - problem 5.19 ($n = 2$) - problem 5.23 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 4: boundary equilibrium of $f^{(j)}$ at $u(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.16 for $f^{(j)}$ - problem 5.33 for $f^{(i)}$
t. f. 5: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.19 ($n = 2$) - problem 5.29 for $f^{(i)}$
note	the initial value of $u(0)$ must be specified in `PAR(67)`$, \ldots,$ `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.32 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \langle H_x(v(1),\alpha), f^{(j)}(v(1),\alpha)\rangle & = & 0. \end{array} \right. \tag{36}$$

n, m	n, 4
m_d, $(\beta_1,\ldots,\beta_{m_d})$	4, $(T_i, T_j, \lambda_i, \lambda_j)$
n_d, n_b	$2n$, $2n+5$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, $2n$
`U(1)`, ..., `U(NDIM)`	$u_1,\ldots,u_n$, $v_1,\ldots,v_n$
`F(`k`)`, $k=1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k=n+1,\ldots,2n$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, (`ICP(1)`, ..., `ICP(NICP)`)	6, (I_1, I_2, 61(T_i), 62(T_j), 63(λ_i), 64(λ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.7 for $f^{(i)}$ ($n=2$) * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$) - problem 5.24 for $f^{(i)}$, $f^{(j)}$ ($n=2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.29 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$

switch to	- problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(j)}$ ($n = 2$) * - problem 5.21 for $f^{(j)}$ ($n = 2$) * - problem 5.31 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(j)}$ at $v(1)$ ($n = 2$) switch to	$\langle H_x^{\perp}(v(1), \alpha), f^{(j)}(v(1), \alpha)\rangle$ - problem 5.4 for $f^{(j)}$ - problem 5.5 - problem 5.16 for $f^{(j)}$
t. f. 7: tangent point of $f^{(i)}$ at $v(1)$ switch to	$\langle H_x(v(1), \alpha), f^{(i)}(v(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.5 ($n = 2$) - problem 5.19 ($n = 2$) - problem 5.30 for $f^{(i)}$, $f^{(j)}$
note	the initial value of $u(0)$ must be specified in `PAR(67)`, ..., `PAR(67+`n`-1)` in the user subroutine `SCSTPNT`

5.33 An orbit of vector field $f^{(i)}$ connecting two pseudo-equilibria

$$\left\{\begin{array}{rcll} \dot{u} - Tf^{(i)}(u, \alpha) & = & 0, & \\ H(u(0), \alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0), \alpha) + \lambda_j f^{(j)}(u(0), \alpha) & = & 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ H(u(1), \alpha) & = & 0, & \\ \mu_i f^{(i)}(u(1), \alpha) + \mu_j f^{(j)}(u(1), \alpha) & = & 0, & \\ \mu_i + \mu_j - 1 & = & 0. & \end{array}\right. \quad (37)$$

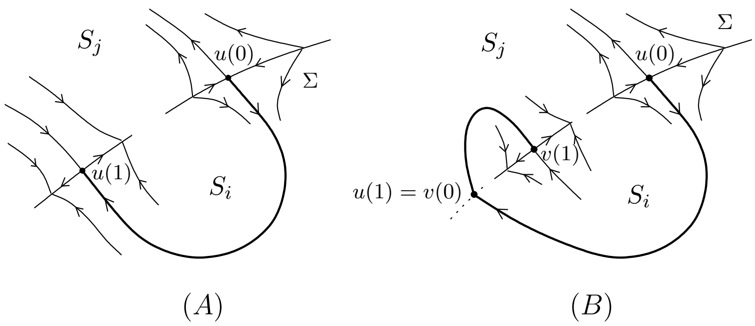

Figure 14: Boundary-value problems corresponding to (A): Subsection 5.33; (B): Subsection 5.34.

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	5, $(T, \lambda_i, \lambda_j, \mu_i, \mu_j)$
n_d, n_b	2, 8
`IPS`, `SCIDIFF`, `NDIM`	4, 0, 2
`U(1)`$, \ldots,$ `U(NDIM)`	u_1, u_2
`F(`k`)`, $k = 1, 2$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	7, (I_1, I_2, 61(T), 63(λ_i), 64(λ_j), 65(μ_i), 66(μ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.7 for $f^{(i)}$ - problem 5.19 - problem 5.25 for $f^{(i)}$
t. f. 4: boundary equilibrium of $f^{(i)}$ at $u(1)$ switch to	μ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 5: boundary equilibrium of $f^{(j)}$ at $u(1)$ switch to	μ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.16 for $f^{(j)}$ - problem 5.31 for $f^{(i)}$, $f^{(j)}$
t. f. 6: singular sliding point at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha) \rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.10 for $f^{(i)}$, $f^{(j)}$ * - problem 5.19 - problem 5.29 for $f^{(i)}$ - problem 5.31 for $f^{(i)}$, $f^{(j)}$
note	the initial values of $u(0)$ and $u(1)$ must be specified in `PAR(67)`$, \ldots,$ `PAR(67+2`n`-1)` in the user subroutine `SCSTPNT`

5.34 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two pseudo-equilibria

$$\left\{ \begin{array}{rcl} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, \\ H(u(0),\alpha) & = & 0, \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, \\ \lambda_i + \lambda_j - 1 & = & 0, \\ H(u(1),\alpha) & = & 0, \\ u(1) - v(0) & = & 0, \\ H(v(1),\alpha) & = & 0, \\ \mu_i f^{(i)}(v(1),\alpha) + \mu_j f^{(j)}(v(1),\alpha) & = & 0, \\ \mu_i + \mu_j - 1 & = & 0. \end{array} \right. \tag{38}$$

n, m	2, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	6, $(T_i, T_j, \lambda_i, \lambda_j, \mu_i, \mu_j)$
n_d, n_b	4, 11
`IPS`, `SCIDIFF`, `NDIM`	4, 0, 4
`U(1)`, ..., `U(NDIM)`	u_1, u_2, v_1, v_2
`F(`k`)`, $k = 1, 2$	$T_i f_k^{(i)}(u,\alpha)$
`F(`k`)`, $k = 3, 4$	$T_j f_k^{(j)}(v,\alpha)$
`NICP`, `(ICP(1)`, ..., `ICP(NICP))`	8, (I_1, I_2, 61(T_i), 62(T_j), 63(λ_i), 64(λ_j), 65(μ_i), 66(μ_j))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0),\alpha), f^{(i)}(u(0),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.7 for $f^{(i)}$ * - problem 5.8 for $f^{(i)}$, $f^{(j)}$ - problem 5.19 - problem 5.26 for $f^{(i)}$, $f^{(j)}$
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(i)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ - problem 5.29 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1),\alpha), f^{(j)}(u(1),\alpha)\rangle$ - problem 5.3 for $f^{(j)}$ - problem 5.7 for $f^{(j)}$ * - problem 5.25 for $f^{(j)}$ * - problem 5.31 for $f^{(i)}$, $f^{(j)}$ *

t. f. 6: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	μ_j - problem 5.3 for $f^{(i)}$ - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$ - problem 5.30 for $f^{(i)}$, $f^{(j)}$
t. f. 7: boundary equilibrium of $f^{(j)}$ at $v(1)$ switch to	μ_i - problem 5.3 for $f^{(j)}$ - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 8: singular sliding point at $v(1)$ switch to	$\langle H_x(v(1),\alpha), f^{(i)}(v(1),\alpha)\rangle$ - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ - problem 5.19 - problem 5.30 for $f^{(i)}$, $f^{(j)}$ - problem 5.32 for $f^{(i)}$, $f^{(j)}$
note	the initial values of $u(0)$ and $u(1)$ must be specified in `PAR(67)`, ..., `PAR(67+`$2n$`-1)` in the user subroutine `SCSTPNT`

5.35 An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium to a saddle with a one-dimensional unstable manifold

$$\left\{ \begin{array}{rcll} \dot{u} - Tf^{(i)}(u,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, & j \neq i, \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ f^{(i)}(y,\alpha) & = & 0, & \\ \left[f_x^{(i)}(y,\alpha)\right]^T w - \nu w & = & 0, & \nu > 0, \\ \langle w, w\rangle - 1 & = & 0, & \\ \langle w, y - u(1)\rangle & = & 0. & \end{array} \right. \quad (39)$$

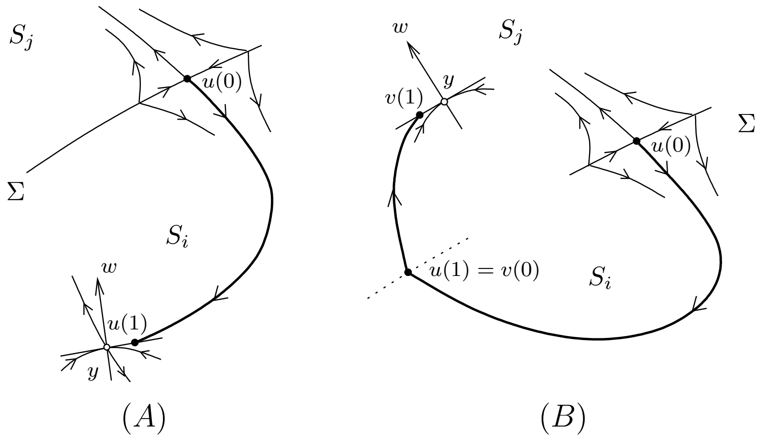

Figure 15: Boundary-value problems corresponding to (A): Subsection 5.35; (B): Subsection 5.36.

n, m	n, 2
m_d, $(\beta_1, \ldots, \beta_{m_d})$	$2n+3$, $(\lambda_i, \lambda_j, y_1, \ldots, y_n, w_1, \ldots, w_n, \nu)$
n_d, n_b	n, $3n+4$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, n
`U(1)`$, \ldots,$ `U(NDIM)`	$u_1, \ldots, u_n$
`F(`k`)`, $k = 1, \ldots, n$	$Tf_k^{(i)}(u, \alpha)$
`NICP`, (`ICP(1)`$, \ldots,$ `ICP(NICP)`)	$2n+5$, (I_1, I_2, 63(λ_i), 64(λ_j), 67(y_1), $\ldots$, ($67+n-1$)(y_n), ($67+n$)(w_1), $\ldots$, ($67+2n-1$)(w_n), ($67+2n$)(ν))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ ($n=2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(j)}$ ($n=2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha) \rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n=3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n=2$) - problem 5.19 ($n=2$) - problem 5.27 for $f^{(i)}$ ($n=2$)
t. f. 4: boundary equilibrium of $f^{(i)}$ at $u(1)$	$H(u(1), \alpha)$

switch to	- problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.9 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$ - problem 5.29 for $f^{(i)}$ - problem 5.33 for $f^{(i)}$ ($n = 2$)
t. f. 5: Hopf switch to	$\mathrm{Re}\nu_p \, : \, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \, : \, \mathrm{Im}\nu_q \neq 0\}$ continuation of a Hopf bifurcation *
t. f. 6: branch/limit point switch to	$\mathrm{Re}\nu_p \, : \, p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \, : \, \mathrm{Im}\nu_q = 0\}$ problem 5.4 for $f^{(i)}$ (branch switching) */ continuation of a limit point bifurcation *
note	the saddle y is assumed to have a one-dimensional unstable manifold
note	the initial values of $u(0)$ and y must be specified in `PAR(67)`, ..., `PAR(67+2n-1)` in the user subroutine `SCSTPNT`

5.36 A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium to a saddle with a one-dimensional unstable manifold

$$\left\{\begin{array}{rcll} \dot{u} - T_i f^{(i)}(u,\alpha) & = & 0, & \\ \dot{v} - T_j f^{(j)}(v,\alpha) & = & 0, & \\ H(u(0),\alpha) & = & 0, & \\ \lambda_i f^{(i)}(u(0),\alpha) + \lambda_j f^{(j)}(u(0),\alpha) & = & 0, & \\ \lambda_i + \lambda_j - 1 & = & 0, & \\ H(u(1),\alpha) & = & 0, & \\ u(1) - v(0) & = & 0, & \\ f^{(i)}(y,\alpha) & = & 0, & \\ \left[f_x^{(i)}(y,\alpha)\right]^T w - \nu w & = & 0, & \nu > 0, \\ \langle w,w\rangle - 1 & = & 0, & \\ \langle w, y - v(1)\rangle & = & 0. & \end{array}\right. \qquad (40)$$

n, m	n, 2
m_d, $(\beta_1,\ldots,\beta_{m_d})$	$2n+4$, $(T_i,\lambda_i,\lambda_j,y_1,\ldots,y_n,w_1,\ldots,w_n,\nu)$
n_d, n_b	$2n$, $4n+5$
`IPS`, `SCIDIFF`, `NDIM`	4, 1, $2n$
`U(1)`, ..., `U(NDIM)`	$u_1,\ldots,u_n, v_1,\ldots,v_n$
`F(k)`, $k = 1,\ldots,n$	$T_i f_k^{(i)}(u,\alpha)$

`F(`k`)`, $k = n+1, \dots, 2n$	$T_j f_k^{(j)}(v, \alpha)$
`NICP`, (`ICP(1)`, $\dots$, `ICP(NICP)`)	$2n+6$, (I_1, I_2, 61(T_i), 63(λ_i), 64(λ_j), 67(y_1), $\dots$, $(67+n-1)$(y_n), $(67+n)$(w_1), $\dots$, $(67+2n-1)$(w_n), $(67+2n)$(ν))
t. f. 1: boundary equilibrium of $f^{(i)}$ at $u(0)$ switch to	λ_j - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.16 for $f^{(i)}$ - problem 5.35 for $f^{(j)}$ *
t. f. 2: boundary equilibrium of $f^{(j)}$ at $u(0)$ switch to	λ_i - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.4 for $f^{(j)}$ - problem 5.16 for $f^{(j)}$
t. f. 3: singular sliding point at $u(0)$ switch to	$\langle H_x(u(0), \alpha), f^{(i)}(u(0), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ and for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ and for $f^{(j)}$ ($n = 2$) - problem 5.7 for $f^{(i)}$ ($n = 2$) * - problem 5.19 ($n = 2$) - problem 5.28 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 4: tangent point of $f^{(i)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(i)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.29 for $f^{(i)}$ *
t. f. 5: tangent point of $f^{(j)}$ at $u(1)$ switch to	$\langle H_x(u(1), \alpha), f^{(j)}(u(1), \alpha)\rangle$ - problem 5.2 for $f^{(j)}$ ($n = 3$) - problem 5.3 for $f^{(j)}$ ($n = 2$) - problem 5.27 for $f^{(j)}$ ($n = 2$) * - problem 5.31 for $f^{(i)}$, $f^{(j)}$ *
t. f. 6: boundary equilibrium of $f^{(i)}$ at $v(1)$ switch to	$H(v(1), \alpha)$ - problem 5.1 ($n = 2$) - problem 5.2 for $f^{(i)}$ ($n = 3$) - problem 5.3 for $f^{(i)}$ ($n = 2$) - problem 5.4 for $f^{(i)}$ - problem 5.5 - problem 5.10 for $f^{(i)}$, $f^{(j)}$ - problem 5.16 for $f^{(i)}$ - problem 5.32 for $f^{(i)}$, $f^{(j)}$ - problem 5.34 for $f^{(i)}$, $f^{(j)}$ ($n = 2$)
t. f. 7: Hopf switch to	$\mathrm{Re}\nu_p \ : \ p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \ : \ \mathrm{Im}\nu_q \neq 0\}$ continuation of a Hopf bifurcation *
t. f. 8: branch/limit point switch to	$\mathrm{Re}\nu_p \ : \ p = \arg\min_q\{\lvert\mathrm{Re}\nu_q\rvert \ : \ \mathrm{Im}\nu_q = 0\}$ problem 5.4 for $f^{(j)}$ (branch switching) */ continuation of a limit point bifurcation *

note	the saddle y is assumed to have a one-dimensional unstable manifold
note	the initial values of $u(0)$ and y must be specified in PAR(67), $\ldots$, PAR(67+$2n$-1) in the user subroutine SCSTPNT

6 PROGRAMMER'S GUIDE

A programmer's guide for the software SLIDECONT (version 2.0) is presented in this section. The reader should refer to the previous sections, as well as to Kuznetsov *et al.* (2003) and references therein for the nomenclature and the theory behind the methods.

6.1 Disclaimer

SLIDECONT is freely available for non-commercial use on an "as is" basis. In no circumstances can the authors be held liable for any deficiency, fault or other mishappening with regard to the use or performance of SLIDECONT.

6.2 System requirements

SLIDECONT requires that AUTO97 is installed under UNIX. Please, follow AUTO97 documentation to install and test the package. A pre-defined maximum value (`SCNDIMX`) of the user problem dimension (`SCNDIM`) is set in the header file `slidecont.h`. This maximum affects the run-time memory requirements and should not be set to unnecessarily large values. The restriction can be changed by editing `slidecont.h` followed by recompilation. Notice that the effective dimension of the defining systems and the number of active parameters depend upon the user problem dimension and they may exceed AUTO97 restrictions on problem size, requiring AUTO97 recompilation. In particullar, AUTO97 should be compiled with `NPARX`≥ 100 in both `auto.h` and `fcon.h` (see AUTO97 documentation). If the SLIDECONT maximum user problem dimension or the AUTO97 restrictions on problem size are exceeded in a SLIDECONT-run, then the computation halts with an error message.

6.3 Installation

SLIDECONT is freely available for download at

`http://www.math.uu.nl/people/kuznet/cm/slidecont.tar.gz`

The software can be extracted by running `tar -xzvf slidecont.tar.gz` in the directory in which SLIDECONT should be installed, where the directories `bin`, `cmds`, `examples`, `include`, `lib`, and `src` are created. The SLIDECONT header and source files are contained in `include` and `src`, respectively, `cmds` contains SLIDECONT commands (see Subsection 6.4) and the environment file `slc.env`, while `examples` contains a few tutorial examples, three of which are described in Section 7. The directories `bin` and `lib` are empty initially.

The environment variables AUTO_DIR and SC_DIR must be set to the absolute paths of the AUTO97 and SLIDECONT directories and the directories `$AUTO_DIR/cmds`, `$AUTO_DIR/bin`, `$SC_DIR/cmds`, and `$SC_DIR/bin` must be added to the system search path list. For this, the environment file `slc.env` should be appropriately edited and sourced before running SLIDECONT. For example, the following lines

```
#  tc shell
setenv AUTO_DIR /usr/local/auto/97
setenv SC_DIR /local/scratch/SlideCont/2.0
set    path=($AUTO_DIR/cmds $AUTO_DIR/bin $path)
set    path=($SC_DIR/cmds $SC_DIR/bin $path)
```

set up the necessary paths to both AUTO97 and SLIDECONT under the Unix shell `csh`, assuming that AUTO97 is installed in `/usr/local/auto/97` and SLIDECONT 2.0 in `/local/scratch/SlideCont/2.0`.

Then, SLIDECONT should be compiled by typing `make` in the `$SC_DIR/src` directory (`f77` is assumed to be the Fortran compiler command name). This produces SLIDECONT preprocessor and libraries and places them into `bin` and `lib`, repectively.

The user must provide three files. The equations file `<name>.f` containing the Fortran subroutines `SCFUNC`, `SCBOUND`, `SCSTPNT`, `SCPVLS`, `SCBCND`, `SCICND`, and `SCFOPT`, the constants file `sc.<name>`, and, possibly, the data file `<name>.dat` which specifies numerically the starting solution for boundary-value problems (`<name>` is a user-selected name).

The user-supplied subroutines may be regarded as higher-level input routines that are called by the standard AUTO97 routines contained in the SLIDECONT library. The purpose of the user-supplied subroutines is the following:

`SCFUNC` Fortran prototype

```
      SUBROUTINE SCFUNC(SCNDIM,X,PAR,SCIDIFF,FI,
     +  DFIDX,DFIDP,DFIDXDX,DFIDXDP,I)
```

defines the vector fields $f^{(1)}(x,\alpha)$ and $f^{(2)}(x,\alpha)$; on input, `SCNDIM`, `X(`k`)`, `PAR(`k`)`, `SCIDIFF`, and `I` contain the vector fields dimension n, the actual state and parameter values x and α, the order up to which derivatives must be analytically specified, and the index i of the vector field of interest, respectively; as in AUTO97, only parameters `PAR(1)`$,\dots,$ `PAR(9)` can be used by the user; on output, `FI(`k`)` contains the k-th component of $f^{(i)}$, while `DFIDX(`k,p`)`, `DFIDP(`k,p`)`, `DFIDXDX(`k,p,q`)`, and `DFIDXDP(`k,p,q`)` contain the derivatives of $f_k^{(i)}$, if provided, with respect to x_p, α_p, (x_p,x_q), and (x_p,α_q), respectively;

`SCBOUND` Fortran prototype

```
      SUBROUTINE SCBOUND(SCNDIM,X,PAR,SCIDIFF,
     +  H,DHDX,DHDP,DHDXDX,DHDXDP)
```

defines the discontinuity boundary function $H(x,\alpha)$. Input and output arguments are analogous to those of `SCFUNC`;

`SCSTPNT` Fortran prototype

```
      SUBROUTINE SCSTPNT(SCNDIM,X,PAR,T,I)
```

defines the starting solution (x, α); on input, SCNDIM and I contain the vector fields dimension n and the index i of the vector field of interest, respectively; on output, X(k) and PAR(k) contain state and parameters starting values; for boundary-value problems, only an analytically known solution X(k)=x_k(T) can be specified, where T denotes the independent time variable which, on input, takes values in the interval $[0, 1]$; in such cases the time length of the solution must be specified in PAR(11) or PAR(60)–PAR(62), depending upon the problem (see Section 5, and AUTO97 documentation for more details);

SCPVLS Fortran prototype

```
SUBROUTINE SCPVLS(SCNDIM,X,PAR,I)
```

defines user functions and solution measures (see AUTO97 documentation);

SCBCND Fortran prototype

```
SCBCND(NDIM,PAR,ICP,NBC,U0,U1,FB,IJAC,DBC)
```

is meaningful only for AUTO97 problem types;

SCICND Fortran prototype

```
    SCICND(NDIM,PAR,ICP,NINT,U,UOLD,UDOT,
+       UPOLD,FI,IJAC,DINT)
```

is meaningful only for AUTO97 problem types;

SCFOPT Fortran prototype

```
SCFOPT(NDIM,U,ICP,PAR,IJAC,FS,DFDU,DFDP)
```

is meaningful only for AUTO97 problem types.

For a fully documented equations file see, for example, $SC_DIR/examples/hppc/hppc.f.

The format of the constants file is the same for all problems. An example is listed below:

```
2 102 0 1                      NDIM,IPS,IRS,ILP
1 2                            NICP,(ICP(I),I=1,NICP)
0 0 0 1 1 0 0 0                NTST,NCOL,IAD,ISP,ISW,IPLT,NBC,NINT
50 0.0 1.0 0.0 100.0           NMX,RL0,RL1,A0,A1
10 0 2 8 7 5 0                 NPR,MXBF,IID,ITMX,ITNW,NWTN,JAC
1.e-8 1.e-8 1.e-7              EPSL,EPSU,EPSS
```

```
-1.e-3 1.e-10 0.1 1          DS,DSMIN,DSMAX,IADS
0                            NTHL,((ITHL(I),THL(I)),I=1,NTHL)
0                            NTHU,((ITHU(I),THU(I)),I=1,NTHU)
1                            NUZR,((IUZR(I),UZR(I)),I=1,NUZR)
2 0.33
0 2                          SCISTART,SCIDIFF
1 1                          SCNPSI,(SCIPSI(I),I=1,SCNPSI)
0                            SCNFIXED,(SCIFIXED(I),I=1,SCNFIXED)
```

The first part (ending with `NUZR`) is an AUTO97 compliant constants file, thus we refer to the AUTO97 documentation for the meaning of single constants. In particular, `IPS` is the problem type and `DS` is the starting step size of the continuation algorithm. If `DS` is positive (negative) then the computation is performed forward (backward) with respect to the first active parameter of the `ICP` list. A list of SLIDECONT problem types (with alphabetical labels useful for source code inspection) is reported in Table 1.

Label	Problem type	Problem description
`T3D`i	i`00`	A curve of tangent points of vector field $f^{(i)}$ in three-dimensional systems
`TAN`i	i`01`	A tangent point of vector field $f^{(i)}$ A double tangency bifurcation of vector field $f^{(i)}$
`EQL`i	i`02`	A standard equilibrium of vector field $f^{(i)}$
`BEQ`i	i`03`	A boundary equilibrium of vector field $f^{(i)}$
`CYC`i	i`10`	A standard cycle of vector field $f^{(i)}$
`TGB`i	i`11`	An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with the boundary Σ
`TCB`i	i`12`	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with the boundary Σ
`PEB`i	i`13`	An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with the boundary Σ
`PCB`i	i`14`	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with the boundary Σ
`TCH`i	i`15`	A touching (grazing) cycle of vector field $f^{(i)}$
`TGT`i	i`16`	An orbit of vector field $f^{(i)}$ connecting two tangent points of $f^{(i)}$
`TCT`i	i`17`	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two tangent points of $f^{(i)}$
`TTD`i	i`18`	An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$ ($j \neq i$)
`TCD`i	i`19`	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a tangent point of $f^{(j)}$

Table 1: (continue)

Label	Problem type	Problem description
TGPi	i20	An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium
TCPi	i21	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ with a pseudo-equilibrium
TGSi	i22	An orbit of vector field $f^{(i)}$ connecting a tangent point of $f^{(i)}$ to a saddle
TCSi	i23	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a tangent point of $f^{(i)}$ to a saddle of $f^{(j)}$
PETi	i24	An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$
PCTi	i25	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(i)}$
PTDi	i26	An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$ ($j \neq i$)
PCDi	i27	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium with a tangent point of $f^{(j)}$
PEPi	i28	An orbit of vector field $f^{(i)}$ connecting two pseudo-equilibria
PCPi	i29	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting two pseudo-equilibria
PESi	i30	An orbit of vector field $f^{(i)}$ connecting a pseudo-equilibrium to a saddle with a one-dimensional unstable manifold
PCSi	i31	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a pseudo-equilibrium to a saddle with a one-dimensional unstable manifold
SCYi	i32	A sliding cycle with a standard orbit of vector field $f^{(i)}$
SCCi	i33	A sliding cycle with standard orbits of vector fields $f^{(i)}$ and $f^{(j)}$ ($j \neq i$)
SBNi	i34	An orbit of vector field $f^{(i)}$ connecting a saddle of $f^{(i)}$ with the boundary Σ
SCBi	i35	A crossing orbit of vector fields $f^{(i)}, f^{(j)}$ ($j \neq i$) connecting a saddle of $f^{(i)}$ with the boundary Σ
BND	300	The discontinuity boundary
PEQ	301	A pseudo-equilibrium A pseudo-saddle-node bifurcation
CTG	302	Coinciding tangent points
CCY	310	A crossing cycle

Table 1: SLIDECONT problem labels, types, and descriptions.

SLIDECONT problem types are composed of three digits with the following meaning: the first digit is either 1 or 2 if the problem refers to vector field $f^{(i)}$ with $i = 1$ or $i = 2$ in the defining system (see Section 5), while it is equal to 3 if the problem does not refer to a particular vector field. The remaining two digits form a progressive number. Thus, SLIDECONT problem types do not overlap with AUTO97 ones, so that, all AUTO97 facilities are accessible through SLIDECONT. In fact, if `IPS` is an AUTO97 problem type, then the rest of the constants file (starting with `SCISTART`) is ignored, the vector field $f^{(1)}$ is assumed to be the right-hand side of the AUTO97 problem, and the user-supplied subroutines `SCBCND`, `SCICND`, and `SCFOPT` assume the role of the analogous AUTO97 subroutines.

By contrast, for SLIDECONT problems, the constants `NBC`, `NINT`, `JAC` are ignored while the significance of the remaining SLIDECONT constants is described below:

`SCISTART`	type of initial solution when switching from a boundary-value to an algebraic problem (see Section 5): -1, start from $u(0)$; -2, start from $u(1)$; -3, start from $v(1)$;
`SCIDIFF`	order up to which derivatives are provided;
`SCNPSI`, `SCIPSI`	number and numerical labels of monitored test functions (see Section 5 for the list of the test functions for each problem); if the test function k is monitored then its value is assigned to `PAR(40+`k`)`, which is an overspecified parameter that appears in the output; thus, zeros of the test function k are detected as zeros of `PAR(40+`k`)` and classified by AUTO97 as user-specified solution points (alphabetical code `UZ`, numerical code -4);
`SCNFIXED`, `SCIFIXED`	dummy constants.

SLIDECONT can be run only in command mode through the commands described below. However, for applications involving several computations, a more flexible approach is to use a program (e.g. *make*) for directing recompilation. The commands `@scprep` and `@scexe` are specially useful for this purpose.

`@sc`	Type `@sc <name>` to run SLIDECONT. Starting data, if needed, must be in `q.<name>` and SLIDECONT constants in `sc.<name>`. This is the simplest way to run SLIDECONT.

`@scprep` Type `@scprep <name>` to run the SLIDECONT preprocessor. SLIDECONT constants must be in `sc.<name>`. The corresponding AUTO97 constants file `r.<name>` and the problem specific Fortran file `scprob.f` (see Section 4) are created.

`@scexe` Type `@scexe <name>` to produce the executable file `<name>.exe` (see Section 4). SLIDECONT constants must be in `sc.<name>`. The corresponding AUTO97 constants file `r.<name>` is also created.

`@scdat` Type `@scdat <name>` to convert the user-supplied data file `<name>.dat` into the AUTO97 formatted file `q.<name>`. SLIDECONT constants must be in `sc.<name>`.

The command `@scdat` is necessary for each problem whose defining system consists of a boundary-value problem to start the computation from a numerically known solution not obtained by previous computations or as a solution of a different problem from which automatic switch is not supported. In such cases the user must provide a data (text) file containing numerical data representing the starting solution of the boundary-value problem. Each row in the data file contains NDIM$+1$ numbers, namely the time variable $t \in [0,1]$ and the AUTO97 state components `U(1)`$, \ldots,$ `U(NDIM)` at t. As detailed in Section 5, when only one vector field is involved in the differential equations of the boundary-value problem (i.e. when NDIM = SCNDIM) the time length of the starting solution must be specified in `PAR(61)` (`PAR(11)` for the continuation of a standard cycle, see Subsection 5.6). By contrast, when both vector fields are involved (i.e. when NDIM = 2*SCNDIM), the time lengths of the two connected parts of the starting solution must be specified in `PAR(61)` and `PAR(62)`. Finally, the time length of a sliding segment has to be given in `PAR(60)`, if solutions with such a segment are computed.

7 EXAMPLES

Each example directory in `$SC_DIR/examples` contains constant files `sc.*`$.k,\ k = 1,2,\ldots,$ for all successive computations, several equations files `*.f`$.k$, for computations requiring parameter and state initialization in subroutine `SCSTPNT`, and several data files `*.dat.`$.k$, for manually set up the starting solution of boundary-value problems when automatic switch from previous computations is not supported (`*.f.0` is used when no initialization is required). Browse `dryf.f.`k to see how the equations have been specified (in subroutine `SCFUNC` and `SCBOUND`) and how the starting parameter values have been set (in subroutine `SCSTPNT`).

To execute each of the prepared computations the user can simply enter `make` in the corresponding example directory. It is also possible to execute all computations separately by means of SLIDECONT commands (see Tables 2–4). Some differences in the resulting screen outputs are to be expected on different machines.

7.1 A simple dry-friction oscillator

Consider a linear dumped oscillator with dry friction proposed by Tondl (1970):

$$\ddot{x}_1 + x_1 = \alpha_1\alpha_3 \text{sgn}(\alpha_5 - \dot{x}_1) - \alpha_2\alpha_4(\alpha_5 - \dot{x}_1). \tag{41}$$

This defines a piecewise-linear Filippov system (5) with $x_2 = \dot{x}_1, H(x,\alpha) = x_2 - \alpha_5$,

$$S_1 = \{x \in \mathbf{R}^2 : x_2 - \alpha_5 < 0\},\ S_2 = \{x \in \mathbf{R}^2 : x_2 - \alpha_5 > 0\}$$

and

$$f^{(1)}(x) = \begin{pmatrix} x_2 \\ -x_1 + \alpha_1\alpha_3 - \alpha_2\alpha_4(\alpha_5 - x_2) \end{pmatrix}, f^{(2)}(x) = \begin{pmatrix} x_2 \\ -x_1 - \alpha_1\alpha_3 - \alpha_2\alpha_4(\alpha_5 - x_2) \end{pmatrix}.$$

Set

$$\alpha_1 = 0.1, \alpha_2 = 0.03, \alpha_3 = \alpha_4 = 4, \alpha_5 = 0.5$$

At these parameter values the system has a stable sliding cycle starting at the visible tangent point T_1 of $f^{(1)}$ and composed of a standard segment in S_1 and a horizontal sliding segment (see Fig. 16).

To execute the prepared computations simply enter `make` in the directory `$SC_DIR/examples/dryf`. The list of all commands to be typed is reported in Table 2.

k	Command	Action
$1, 3, 4$	`cp sc.dryf.`k `sc.dryf`	get the constants file
	`cp dryf.f.`k `dryf.f`	get the equations file
	`cp dryf.dat.`k `dryf.dat`	get the data file
	`@scdat dryf`	convert the data file into the starting solution file
	`cp q.dryf q.dat.`k	save the starting solution file
	`@sc dryf`	run SLIDECONT
	`@sv dryf.`k	save output to `p.dryf.`k, `q.dryf.`k, `d.dryf.`k
2,5	`cp sc.dryf.`k `sc.dryf`	get the constants file
	`cp dryf.f.0 dryf.f`	get the equations file
	`cp q.dat.`$(k-1)$ `q.dryf`	get the starting solution file
	`@sc dryf`	run SLIDECONT
	`@sv dryf.`k	save output to `p.dryf.`k, `q.dryf.`k, `d.dryf.`k

Table 2: Command list; `@sv` is an AUTO97 command.

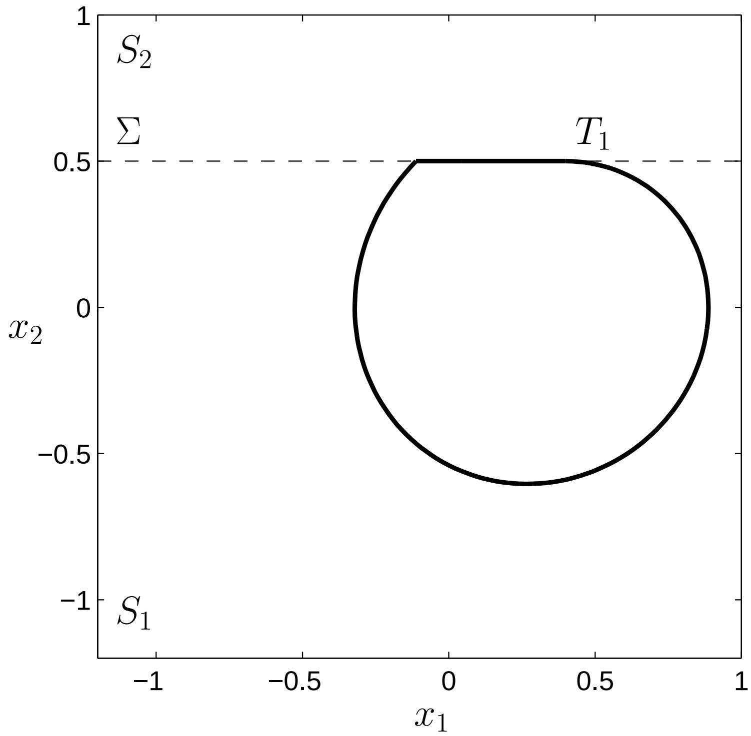

Figure 16: A sliding cycle of system (41).

First we continue the sliding cycle of Fig. 16 with respect to α_2 (see Fig. 17). Data files with the initial solutions to the boundary-value problem (20) can be easily obtained by numerical integration of the vector field $f^{(1)}$ and of the Filippov vector field g (3). Computation 1 (for decreasing α_2) gives:

PT	TY	LAB	PAR(2)	...	PAR(60)	PAR(61)	PAR(43)
10		2	2.72551E-02	...	9.62352E-01	5.49895E+00	4.81176E-01
20		3	4.16217E-03	...	3.31482E-01	5.96249E+00	1.65741E-01
24	LP	4	1.46358E-15	...	-7.99840E-09	6.28319E+00	-3.99920E-09
30		5	4.56498E-02	...	-1.57261E+00	7.25355E+00	-7.86307E-01
40		6	1.36890E-01	...	-8.03478E+00	7.75782E+00	-4.01739E+00
50	EP	7	1.80686E-01	...	-1.74344E+01	7.97053E+00	-8.71721E+00

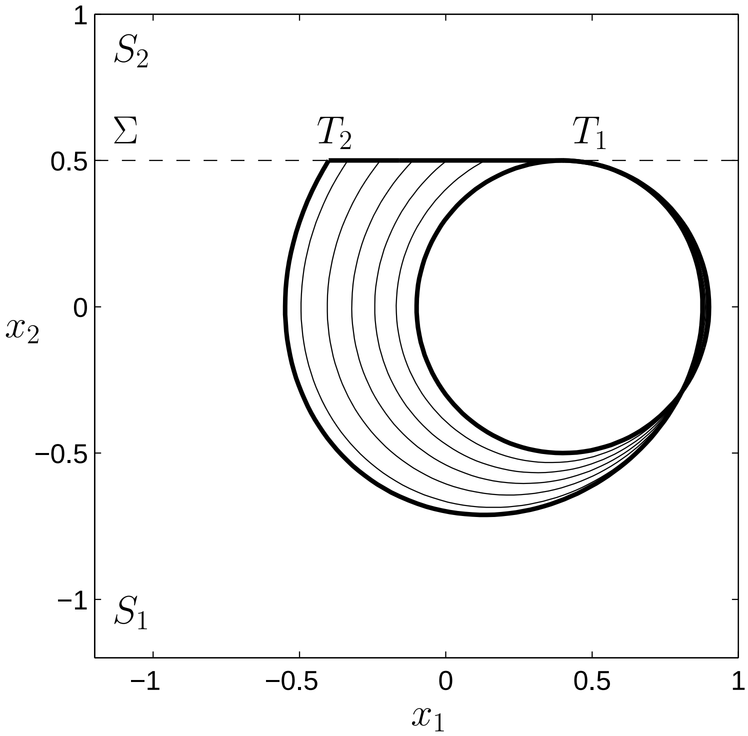

Figure 17: A family of sliding cycles of system (41): The right thick solution corresponds to a grazing bifurcation at $\alpha_2 = 0$, the left thick solution corresponds to a switching bifurcation at $\alpha_2 = 0.0557\ldots$.

The solution family ends at a touching (grazing) bifurcation at $\alpha_2 = 0$ (see label 4 where test function 3 vanishes). This bifurcation is degenerate, since at $\alpha_2 = 0$ the systems in both S_1 and S_2 become linear oscillators with closed orbits around $(\alpha_1\alpha_3, 0) = (0.4, 0)$ and $(-\alpha_1\alpha_3, 0) = (-0.4, 0)$, respectively (portrait O in Figure 18).

Computation 2 (for increasing α_2) gives:

PT	TY	LAB	PAR(2)	...	PAR(60)	PAR(61)	PAR(44)
10		2	3.28003E-02	...	1.08725E+00	5.43205E+00	2.56373E-01
19	UZ	3	5.57355E-02	...	1.60000E+00	5.22189E+00	-4.33287E-08
20		4	6.22664E-02	...	1.75086E+00	5.17563E+00	-7.54277E-02
30		5	1.49842E-01	...	4.47741E+00	4.85572E+00	-1.43870E+00
40		6	2.44375E-01	...	1.14592E+01	4.90712E+00	-4.92959E+00
50	EP	7	2.94622E-01	...	2.07868E+01	5.10207E+00	-9.59339E+00

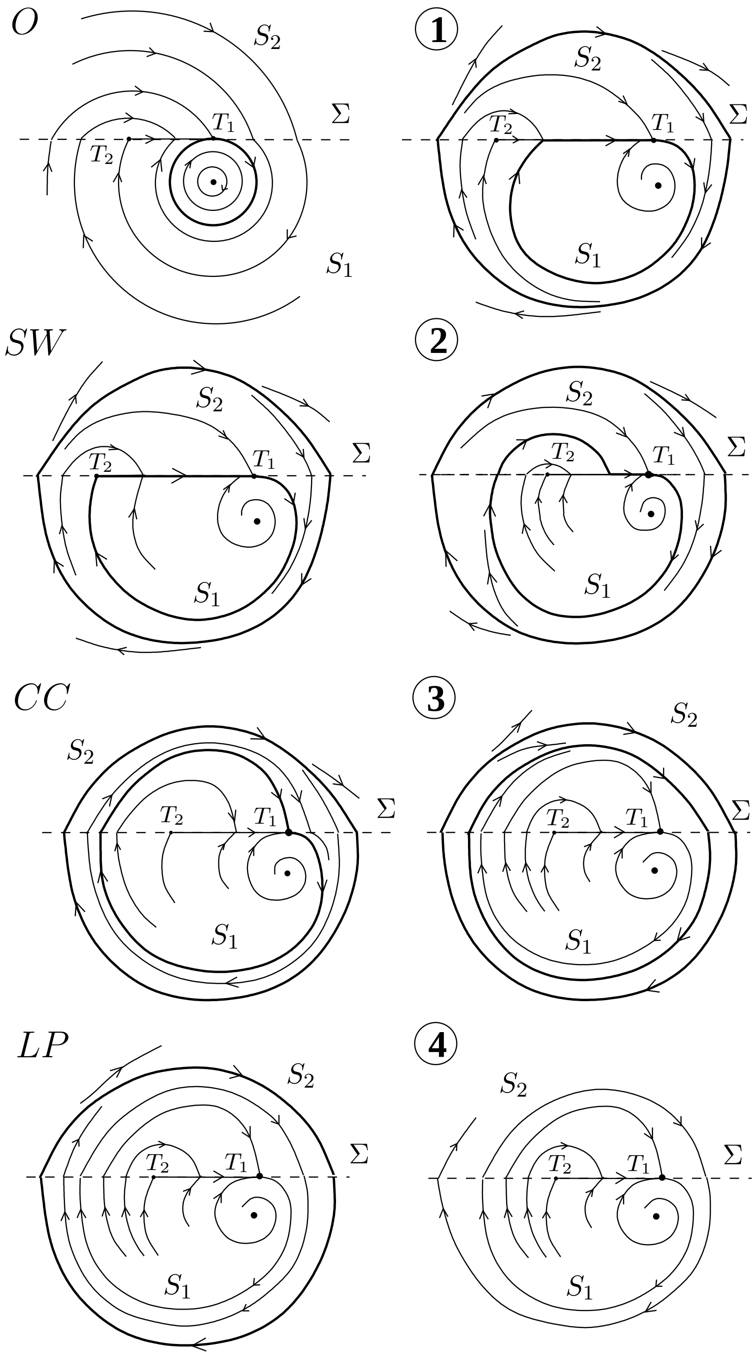

Figure 18: Bifurcation diagram of (41): O - centers ($\alpha_2 = 0$); **1** - a sliding cycle with the standard arc in S_1 and a crossing cycle ($0 < \alpha_2 < \alpha_2^{sw} = 0.0557\ldots$); SW - buckling (switching) bifurcation at α_2^{sw}; **2** - a sliding cycle with the standard arcs in both S_1 and S_2 and a crossing cycle ($\alpha_2^{sw} < \alpha_2 < \alpha_2^{cc}$); CC - crossing bifurcation at $\alpha_2^{cc} = 0.1023\ldots$; **3** - two crossing cycles ($\alpha_2^{cc} < \alpha_2 < \alpha_2^{lp}$); LP - limit point of the crossing cycles at $\alpha_2^{lp} = 0.1044\ldots$; **4** - no attractors ($\alpha_2 > \alpha_2^{lp}$).

Label 3 indicates a buckling (switching) bifurcation (zero of test function 4): At $\alpha_2^{sw} = 0.0557\ldots$ an invisible tangent point T_2 of $f^{(2)}$ is detected at the end point $u(1)$ (see Figure 18(SW)). Just below this parameter value the sliding cycle is located entirely in the domain S_1 (portrait **1** in Figure 18), while just above this parameter value the sliding cycles have also a standard segment in S_2 (portrait **2** in Figure 18) and, therefore, can be continued by means of the boundary-value problem (21).

Starting from the data file `dryf.dat.3` corresponding to label 3, computation 3 produces (see Fig. 19):

```
 PT   TY LAB    PAR(2)      ...    PAR(60)        PAR(61)        PAR(62)        PAR(45)
 10        2   5.71491E-02 ...  1.56748E+00   5.21147E+00   6.48471E-02   7.83742E-01
 20        3   7.67732E-02 ...  1.04430E+00   5.08803E+00   9.72593E-01   5.22152E-01
 27   UZ   4   1.02312E-01 ...  6.98102E-10   4.97317E+00   1.90398E+00   3.49050E-10
 30        5   1.28304E-01 ... -1.83052E+00   4.89559E+00   2.46968E+00  -9.15258E-01
 40        6   1.80925E-01 ... -1.05903E+01   4.83254E+00   3.04894E+00  -5.29517E+00
 50   EP   7   2.05960E-01 ... -1.97537E+01   4.84202E+00   3.22755E+00  -9.87687E+00
```

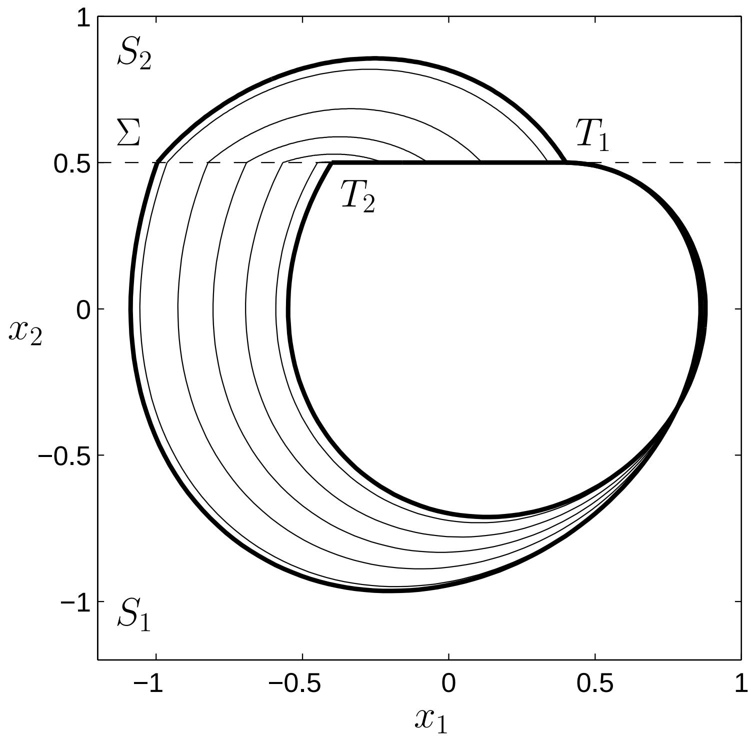

Figure 19: A family of sliding cycles of system (41). The thick solutions correspond to a switching bifurcation at $\alpha_2 = 0.0557\ldots$ and a crossing bifurcation at $\alpha_2 = 0.1023\ldots$

Label 4 indicates a crossing bifurcation (zero of test function 5): At $\alpha_2^{cc} = 0.1023\ldots$ a tangent point of $f^{(1)}$ is detected at the end point $v(1)$ that coincides with the starting tangent point T_1 (see portrait CC

in Figure 18). According to Kuznetsov *et al.* (2003), this implies that for α_2 slightly bigger than α_2^{cc} a *stable crossing cycle* replaces the sliding cycle.

Numerical integration at $\alpha_2 \approx 0.0716$, however, reveals an *unstable crossing cycle*. Starting from the corresponding data file `dryf.dat.4`, we can continue the crossing cycle (see Section 5.13) forward (computation 4):

```
PT  TY LAB    PAR(2)     ...   PAR(61)       PAR(62)       PAR(41)
10       2   7.10239E-02 ...  3.81133E+00   2.56183E+00   1.09438E+00
20       3   9.09305E-02 ...  4.13420E+00   2.34613E+00   5.71175E-01
26  LP   4   1.04455E-01 ...  4.73036E+00   2.02413E+00   1.20594E-01
27  UZ   5   1.02312E-01 ...  4.97317E+00   1.90398E+00  -1.34521E-08
30       6   1.87623E-02 ...  6.13818E+00   4.96953E-01  -6.72345E-01
31  EP   7  -4.36612E-03 ...  6.31774E+00  -1.22892E-01  -8.30773E-01
```

and backward (computation 5):

```
PT  TY LAB    PAR(2)        PAR(61)       PAR(62)       PAR(41)
10       8   6.92846E-02   3.78902E+00   2.57819E+00   1.14855E+00
20       9   5.38503E-02   3.61272E+00   2.71469E+00   1.74921E+00
30      10   3.30625E-02   3.41422E+00   2.88372E+00   3.32320E+00
40      11   1.63250E-02   3.27216E+00   3.01443E+00   7.33706E+00
50  EP  12   8.60323E-03   3.20972E+00   3.07440E+00   1.43668E+01
```

The result is shown graphically in Figure 20. Label 4 in the forward continuation corresponds to a *fold bifurcation of crossing cycles*, when two such cycles collide and disappear at $\alpha_2^{lp} = 0.1044\ldots$. Label 5 corresponds to the already detected crossing bifurcation (zero of test function 1). Therefore, the unstable crossing cycle exists for all $0 < \alpha_2 < \alpha_2^{lp}$ (see portraits **1**, SW, **2**, CC, and **3** in Figure 18). Its amplitude grows to infinity as $\alpha_2 \to 0$. This crossing cycle collides with the stable inner crossing cycle (born at the CC bifrcation at α_2^{cc}) and disappears via the fold bifurcation, so that no periodic motion is present above the critical parameter value α_2^{lp} (portrait **4** in Figure 18). This completes construction of the one-parameter bifurcation diagram of (41).

7.2 Forced dry friction oscillations

This example illustrates the two-parameter continuation of a touching (grazing) cycle in a 4-dimensional Filippov system.

Following Yoshitake & Sueoka (2002) and Bernardo di *et al.* (2003), consider a dry friction oscillator

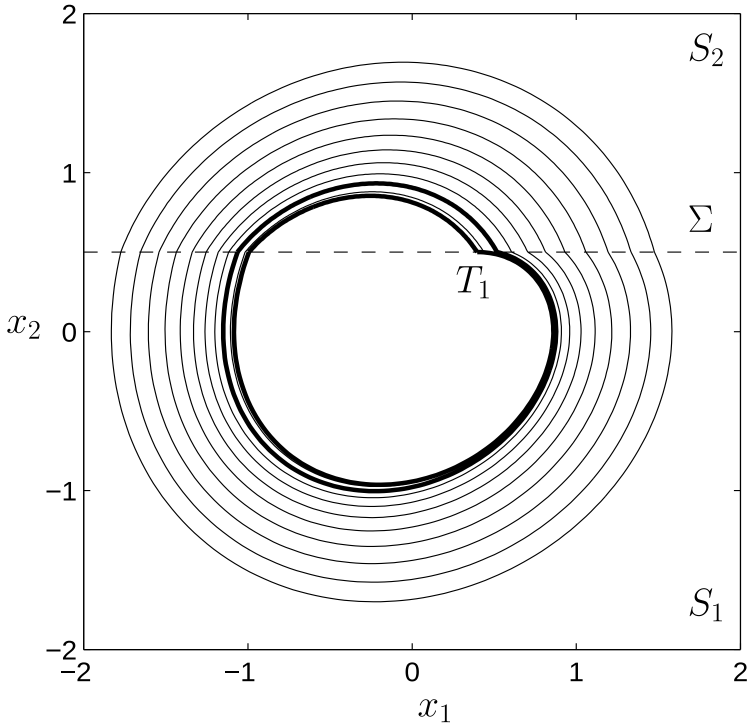

Figure 20: A family of crossing cycles of (41). The thick solutions correspond to the crossing bifurcation at $\alpha_2^{cc} = 0.1023\ldots$ and the fold bifurcation of crossing cycles at $\alpha_2^{lp} = 0.1044\ldots$.

described by the equation

$$\ddot{x}_1 + x_1 = \alpha_1 \, \text{sgn}(1 - \dot{x}_1) - \alpha_2(1 - \dot{x}_1) + \alpha_3(1 - \dot{x}_1)^3 + \alpha_4 \cos(\alpha_5 t). \tag{42}$$

Here positive constants $\alpha_1, \alpha_2, \alpha_3$ are the coefficients of the kinematic friction characteristics, α_4 is the amplitude of the forcing and α_5 is its angular frequency. Since $\cos(\alpha_5 t)$ is the x_3-component of a stable periodic solution to the planar system

$$\begin{cases} \dot{x}_3 &= x_3 - \alpha_5 x_4 - x_3(x_3^2 + x_4^2), \\ \dot{x}_4 &= \alpha_5 x_3 + x_4 - x_4(x_3^2 + x_4^2), \end{cases}$$

equation (42) is equivalent to a 4-dimensional Filippov system (5) with

$$f^{(1,2)}(x,\alpha) = \begin{pmatrix} x_2 \\ -x_1 \pm \alpha_1 - \alpha_2(1-x_2) + \alpha_3(1-x_2)^3 + \alpha_4 x_3 \\ x_3 - \alpha_5 x_4 - x_3(x_3^2 + x_4^2) \\ \alpha_5 x_3 + x_4 - x_4(x_3^2 + x_4^2) \end{pmatrix}, \tag{43}$$

and

$$H(x,\alpha) = x_2 - 1. \tag{44}$$

It is known (see Bernardo di *et al.* (2003)) that at

$$\alpha_1 = \alpha_2 = 1.5,\ \alpha_3 = 0.45,\ \alpha_4 = 0.1,\ \alpha_5 = 1.7078\ldots$$

a $\frac{8\pi}{\alpha_5}$-cycle existing in region S_1 touches the discontinuity boundary Σ (see Figure 21). This is a touching

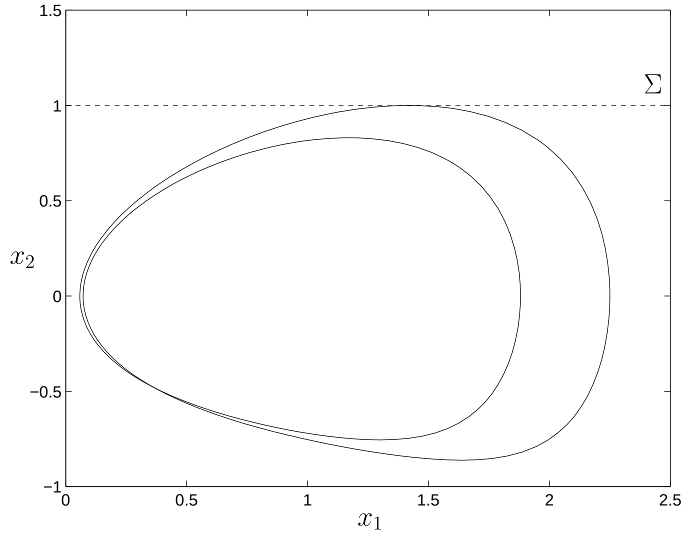

Figure 21: A touching (grazing) cycle of the periodically forced dry friction oscillator (42).

(grazing) bifurcation of the system. Starting with the numerical solution corresponding to Figure 21, we continue this bifurcation in two control parameters: α_4 and α_5 (see Section 5.20).

To execute the prepared computations simply enter `make` in the directory `$SC_DIR/examples/fdfo`. The list of all commands to be typed is reported in Table 3.

k	Command	Action
$1,2$	`cp sc.fdfo.`k `sc.fdfo`	get the constants file
	`cp fdfo.f.`k `fdfo.f`	get the equations file
	`cp fdfo.dat.`k `fdfo.dat`	get the data file
	`@scdat fdfo`	convert the data file into the starting solution file
	`cp q.fdfo q.dat.`k	save the starting solution file
	`@sc fdfo`	run SLIDECONT
	`@sv fdfo.`k	save output to `p.fdfo.`k, `q.fdfo.`k, `d.fdfo.`k
3	`cp sc.fdfo.3 sc.fdfo`	get the constants file
	`cp fdfo.f.0 fdfo.f`	get the equations file
	`cp q.dat.2 q.fdfo`	get the starting solution file
	`@sc fdfo`	run SLIDECONT
	`@sv fdfo.3`	save output to `p.fdfo.3`, `q.fdfo.3`, `d.fdfo.3`
4	`cp sc.fdfo.4 sc.fdfo`	get the constants file
	`cp fdfo.f.0 fdfo.f`	get the equations file
	`cp q.fdfo.2 q.fdfo`	get the starting solution file
	`@sc fdfo`	run SLIDECONT
	`@sv fdfo.4`	save output to `p.fdfo.4`, `q.fdfo.4`, `d.fdfo.4`
$5,6$	`cp sc.fdfo.`k `sc.fdfo`	get the constants file
	`cp fdfo.f.0 fdfo.f`	get the equations file
	`cp q.fdfo.4 q.fdfo`	get the starting solution file
	`@sc fdfo`	run SLIDECONT
	`@sv fdfo.`k	save output to `p.fdfo.`k, `q.fdfo.`k, `d.fdfo.`k

Table 3: Command list; `@sv` is an AUTO97 command.

We get the following output (computation 1):

```
 PT  TY LAB    PAR(4)     ...   MAX U(2)    ...   PAR(5)        PAR(61)
 30         2   1.46553E-01 ...  1.00000E+00 ...  1.79104E+00   1.40324E+01
 35  PD     3   1.59166E-01 ...  1.00000E+00 ...  1.80658E+00   1.39118E+01
 60         4   2.27099E-01 ...  1.00000E+00 ...  1.87385E+00   1.34123E+01
 90         5   3.18761E-01 ...  1.00000E+00 ...  1.94700E+00   1.29084E+01
120         6   4.28346E-01 ...  1.00000E+00 ...  2.02670E+00   1.24008E+01
150         7   5.62395E-01 ...  1.00000E+00 ...  2.12073E+00   1.18510E+01
180         8   7.22631E-01 ...  1.00000E+00 ...  2.22943E+00   1.12732E+01
210         9   9.05530E-01 ...  1.00000E+00 ...  2.34532E+00   1.07161E+01
236  PD    10   1.07114E+00 ...  1.00000E+00 ...  2.43853E+00   1.03065E+01
240        11   1.09846E+00 ...  1.00000E+00 ...  2.45252E+00   1.02477E+01
270        12   1.30693E+00 ...  1.00000E+00 ...  2.54231E+00   9.88578E+00
273  BP    13   1.32427E+00 ...  1.00249E+00 ...  2.54812E+00   9.86325E+00
300  EP    14   1.50283E+00 ...  1.18394E+00 ...  2.58419E+00   9.72559E+00
```

The computed family of $\frac{8\pi}{\alpha_5}$-periodic touching (grazing) cycles is shown in Figure 22, where the end solution with label `14` is omitted. The corresponding bifurcation curve TCH_2 in the (α_4, α_5)-plane is presented in Figure 23.

The solution at label 13 is close to a touching (grazing) solution with period $\frac{4\pi}{\alpha_5}$ that is traced twice: The continued touching (grazing) cycle undergoes a period-halving bifurcation that is detected as a branch point (BP). In other words, the periodically forced dry-friction oscillator exhibits a codimension 2 bifurcation, when a nonhyperbolic $\frac{4\pi}{\alpha_5}$-cycle touches the discontinuity boundary at $A_1 = (\alpha_4.\alpha_5) = (1.32\ldots, 2.55\ldots)$. Notice also two flip (PD) bifurcations of the $\frac{8\pi}{\alpha_5}$-periodic touching (grazing) cycle at labels 3 and 10. These are another codimension-2 flip-garzing points in the system. We skip their analysis.

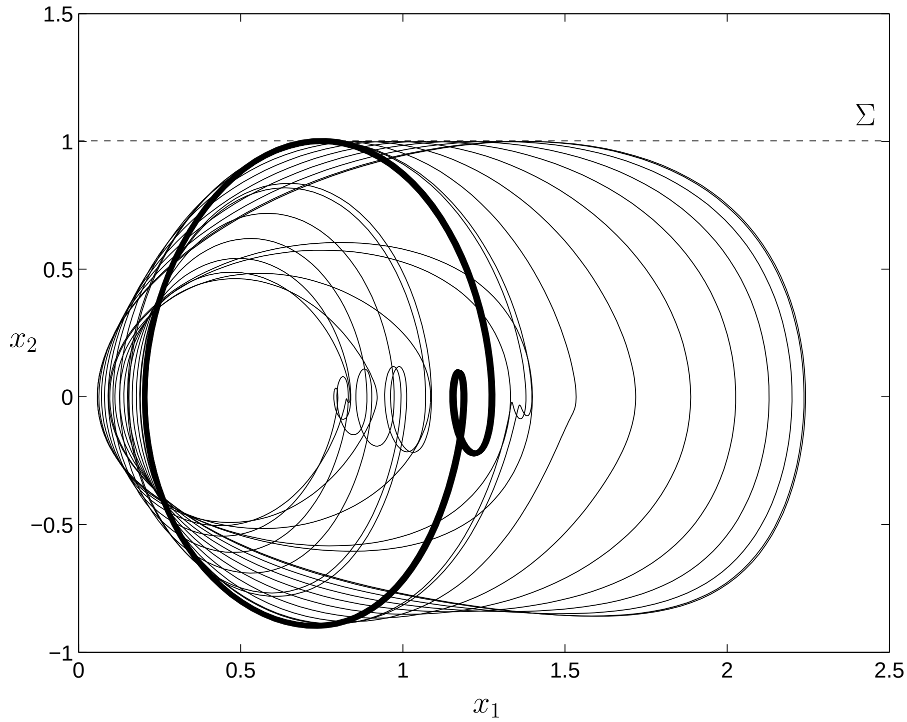

Figure 22: A family of $\frac{8\pi}{\alpha_5}$-periodic touching (grazing) cycles terminating in a period-halving bifurcation at $(\alpha_4, \alpha_5) = (1.3208\ldots, 2.5470\ldots)$. The thick orbit is traced twice.

To continue the touching (grazing) bifurcation of the $\frac{4\pi}{\alpha_5}$-cycle, the data file fdfo.dat.2 can be either constructed manually from the output at label 13 or obtained by simulations. The backward continuation of the touching (grazing) $\frac{4\pi}{\alpha_5}$-cycle gives (computation 2):

```
  PT   TY LAB    PAR(4)      ...   MAX U(2)    ...   PAR(5)        PAR(61)
   5   PD   2   1.32089E+00 ...  1.00000E+00 ...  2.54701E+00   4.93377E+00
  30        3   1.15631E+00 ...  1.00000E+00 ...  2.44812E+00   5.13307E+00
  60        4   9.61275E-01 ...  1.00000E+00 ...  2.31858E+00   5.41986E+00
  90        5   7.82608E-01 ...  1.00000E+00 ...  2.18948E+00   5.73942E+00
 120        6   6.22837E-01 ...  1.00000E+00 ...  2.06772E+00   6.07742E+00
```

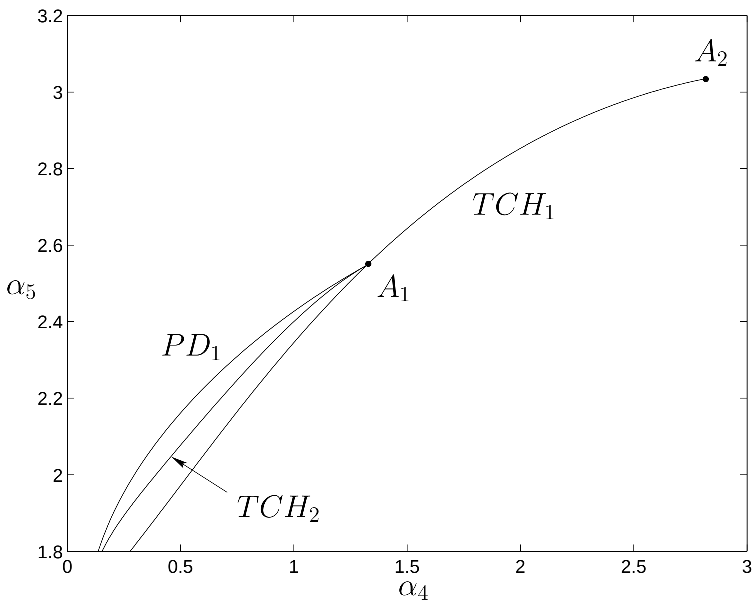

Figure 23: Bifurcation curves of (42): TCH_1 - touching (grazing) bifurcation of the $\frac{4\pi}{\alpha_5}$-cycle; PD_1 - flip (period-doubling) bifurcation of the $\frac{4\pi}{\alpha_5}$-cycle; TCH_2 - touching (grazing) bifurcation of the $\frac{8\pi}{\alpha_5}$-cycle; $A_{1,2}$ - codimension-2 flip-grazing points.

```
 150       7   4.83843E-01 ...  1.00000E+00 ...  1.95931E+00   6.41366E+00
 180       8   3.64053E-01 ...  1.00000E+00 ...  1.86614E+00   6.73388E+00
 210       9   2.57025E-01 ...  1.00000E+00 ...  1.78451E+00   7.04190E+00
 240      10   1.50801E-01 ...  1.00000E+00 ...  1.70534E+00   7.36883E+00
 255  PD  11   8.76020E-02 ...  1.00000E+00 ...  1.65844E+00   7.57721E+00
 270      12   3.85974E-02 ...  1.00000E+00 ...  1.62087E+00   7.75285E+00
 300  EP  13   2.17858E-02 ...  1.00000E+00 ...  1.60619E+00   7.82372E+00
```

while the forward continuation (computation 3) results in:

```
  PT  TY LAB    PAR(4)      ...   MAX U(2)    ...   PAR(5)        PAR(61)
  30       2   1.49773E+00 ...  1.00000E+00 ...  2.64225E+00   4.75593E+00
  60       3   1.72744E+00 ...  1.00000E+00 ...  2.74919E+00   4.57094E+00
  90       4   1.96735E+00 ...  1.00000E+00 ...  2.84171E+00   4.42212E+00
 120       5   2.21504E+00 ...  1.00000E+00 ...  2.91835E+00   4.30599E+00
 150       6   2.46787E+00 ...  1.00000E+00 ...  2.97864E+00   4.21883E+00
 180       7   2.72335E+00 ...  1.00000E+00 ...  3.02272E+00   4.15730E+00
 191  BP   8   2.81386E+00 ...  1.00000E+00 ...  3.03447E+00   4.14121E+00
 210       9   2.67440E+00 ...  1.00000E+00 ...  2.90491E+00   4.32591E+00
 240      10   2.45441E+00 ...  1.00000E+00 ...  2.70230E+00   4.65025E+00
 270      11   2.23336E+00 ...  1.00000E+00 ...  2.50128E+00   5.02397E+00
 300  EP  12   2.01106E+00 ...  1.00000E+00 ...  2.30227E+00   5.45826E+00
```

The computed family of $\frac{4\pi}{\alpha_5}$-periodic touching (grazing) cycles is shown in Figure 24, where solutions with labels 9–12 are omitted. The corresponding bifurcation curve TCH_1 in the (α_4, α_5)-plane can be seen in Figure 23. The flip-grazing point A_1 is more accurately detected in the backward continuation at

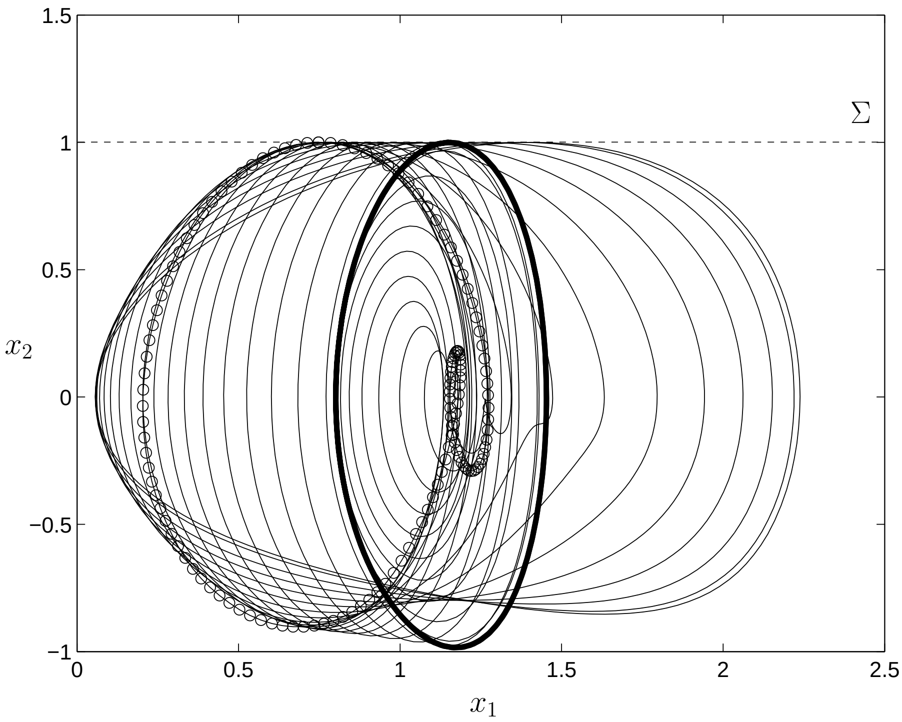

Figure 24: A family of $\frac{4\pi}{\alpha_5}$-periodic touching (grazing) cycles terminating in a period-halving bifurcation at $(\alpha_4.\alpha_5) = (2.814\ldots, 3.0345\ldots)$. The starting solution is o-marked, while the thick solution corresponding to another flip-grazing is traced twice.

label `2`, where the $\frac{4\pi}{\alpha_5}$-periodic touching (grazing) cycle exhibits the flip (`PD`) bifurcation. Notice that a branching point `BP` is detected at label `8`, where the touching (grazing) cycle undergoes a period-halving bifurcation at $A_2 = (\alpha_4.\alpha_5) = (2.8138\ldots, 3.0344\ldots)$.

Finally, we compute the flip bifurcation curve, where the standard $\frac{4\pi}{\alpha_5}$-cycle exhibits a period-doubling bifurcation. This curve can be obtained by switching at point A_1 detected in computation 2 as the `PD`-point at label `2`. A dummy continuation (computation 4):

```
 PT  TY LAB    PAR(4)      ...   MAX U(2)   ...   PAR(5)        PERIOD
  5  EP  14   1.32089E+00 ...  1.00000E+00 ...  2.54701E+00   4.93377E+00
```

generates the starting data to continue the flip bifurcation curve. Restarting fromthe end point at label `14`, we continue the flip curve forward (computation 5):

```
 PT  TY LAB    PAR(4)      ...   MAX U(2)   ...   PAR(5)        PERIOD
 30      15   1.52127E+00 ...  1.08642E+00 ...  2.60775E+00   4.81885E+00
 60      16   1.77224E+00 ...  1.20247E+00 ...  2.66918E+00   4.70796E+00
 90      17   2.03041E+00 ...  1.33688E+00 ...  2.71408E+00   4.63007E+00
120      18   2.28608E+00 ...  1.49759E+00 ...  2.73420E+00   4.59599E+00
```

150		19	2.50460E+00	...	1.69995E+00	...	2.70917E+00	4.63846E+00
173	LP	20	2.56957E+00	...	1.86351E+00	...	2.64333E+00	4.75398E+00
180		21	2.56423E+00	...	1.90582E+00	...	2.61787E+00	4.80023E+00
210		22	2.46371E+00	...	2.03333E+00	...	2.50886E+00	5.00880E+00
240		23	2.30919E+00	...	2.10546E+00	...	2.40856E+00	5.21738E+00
270		24	2.13677E+00	...	2.14713E+00	...	2.31599E+00	5.42592E+00
300	EP	25	1.95813E+00	...	2.16871E+00	...	2.22938E+00	5.63672E+00

and backward (computation 6):

PT	TY	LAB	PAR(4)	...	MAX U(2)	...	PAR(5)	PERIOD
30		15	1.13172E+00	...	9.21174E-01	...	2.47978E+00	5.06753E+00
60		16	9.23250E-01	...	8.38379E-01	...	2.39323E+00	5.25081E+00
90		17	7.37259E-01	...	7.67585E-01	...	2.30263E+00	5.45740E+00
120		18	5.76049E-01	...	7.10921E-01	...	2.21067E+00	5.68442E+00
150		19	4.40663E-01	...	6.72635E-01	...	2.11974E+00	5.92826E+00
180		20	3.30787E-01	...	6.57456E-01	...	2.03179E+00	6.18489E+00
210		21	2.44931E-01	...	6.71390E-01	...	1.94823E+00	6.45016E+00
240		22	1.80767E-01	...	7.17286E-01	...	1.87000E+00	6.71998E+00
270		23	1.35527E-01	...	7.89069E-01	...	1.79777E+00	6.98998E+00
300	EP	24	1.06293E-01	...	8.76882E-01	...	1.73204E+00	7.25523E+00

Since `MAX U(2)` is greater than `1.0` in the forward continuation, only the backward part of the flip bifurcation curve is meaningfull. It is depicted in Figure 23 as PD_1.

7.3 Harvesting a prey-predator community

This example illustrates a variety of one- and two-parameter calculations in a planar Filippov system.

System (5) with

$$f^{(1)}(x,\alpha) = \begin{pmatrix} x_1(1-x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - \alpha_3 x_2 \end{pmatrix}, \tag{45}$$

$$f^{(2)}(x,\alpha) = \begin{pmatrix} x_1(1-x_1) - \psi(x_1)x_2 \\ \psi(x_1)x_2 - \alpha_3 x_2 - \alpha_4 x_2 \end{pmatrix}, \tag{46}$$

$$\psi(x_1) = \frac{\alpha_1 x_1}{\alpha_2 + x_1}, \tag{47}$$

$$H(x,\alpha) = x_2 - \alpha_5. \tag{48}$$

models an harvested prey-predator community where x_1 and x_2 are prey and predator population densities and harvesting of the predator population, at constant effort (α_4), is allowed only if predator are sufficiently abundant (i.e. if $x_2 > \alpha_5$) (see Kuznetsov *et al.*, 2003; Dercole *et al.*, 2003, for more details). The analysis is performed with respect to α_2 and α_5 for the following other parameter values: $\alpha_1 = 0.3556$, $\alpha_3 = 0.0444$, $\alpha_4 = 0.2067$. The results are shown in Figures 25 and 26.

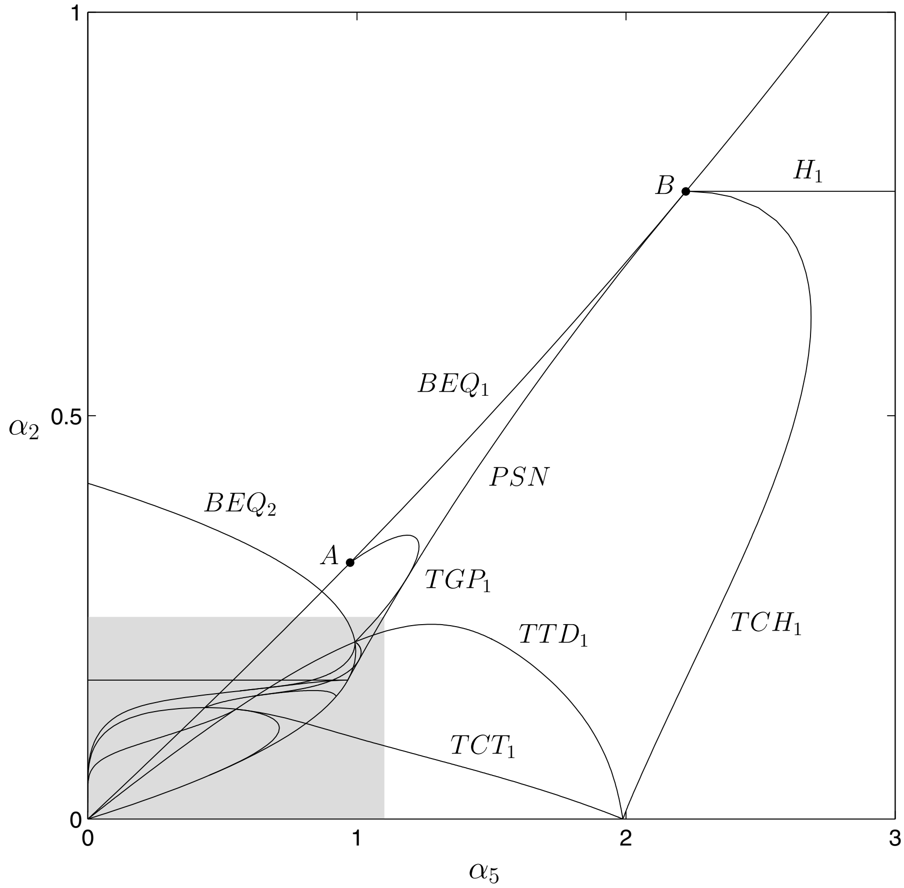

Figure 25: Bifurcation diagram of model (5,45-48) in the (α_5, α_2) plane. Bifurcation curves: $BEQ_{1,2}$ - boundary equilibrium of vector field $f^{(1,2)}$; PSN - pseudo-saddle-node bifurcation; TCH_1 - touching (grazing) bifurcation of vector field $f^{(1)}$; TCT_1 - crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting two tangent points of $f^{(1)}$; TTD_1 - orbit of vector field $f^{(1)}$ connecting a tangent point of $f^{(1)}$ with a tangent point of $f^{(2)}$; TGP_1 - orbit of vector field $f^{(1)}$ connecting a tangent point of $f^{(1)}$ with a pseudo-equilibrium; H_1 - Hopf bifurcation of vector field $f^{(1)}$; Points A and B are codimension-2 bifurcation points detected by SLIDECONT.

Notice that in order to avoid numerical problems when x_1 or x_2 are very small, new state variables (z_1, z_2) with $z_i = \ln(x_i)$, $i = 1, 2$, are introduced.

To execute all prepared computations simply enter `make` in the directory `$SC_DIR/examples/hppc`. In the following we execute all computations separately by means of SLIDECONT commands (see Section 6). The list of all commands to be typed is given in Table 4.

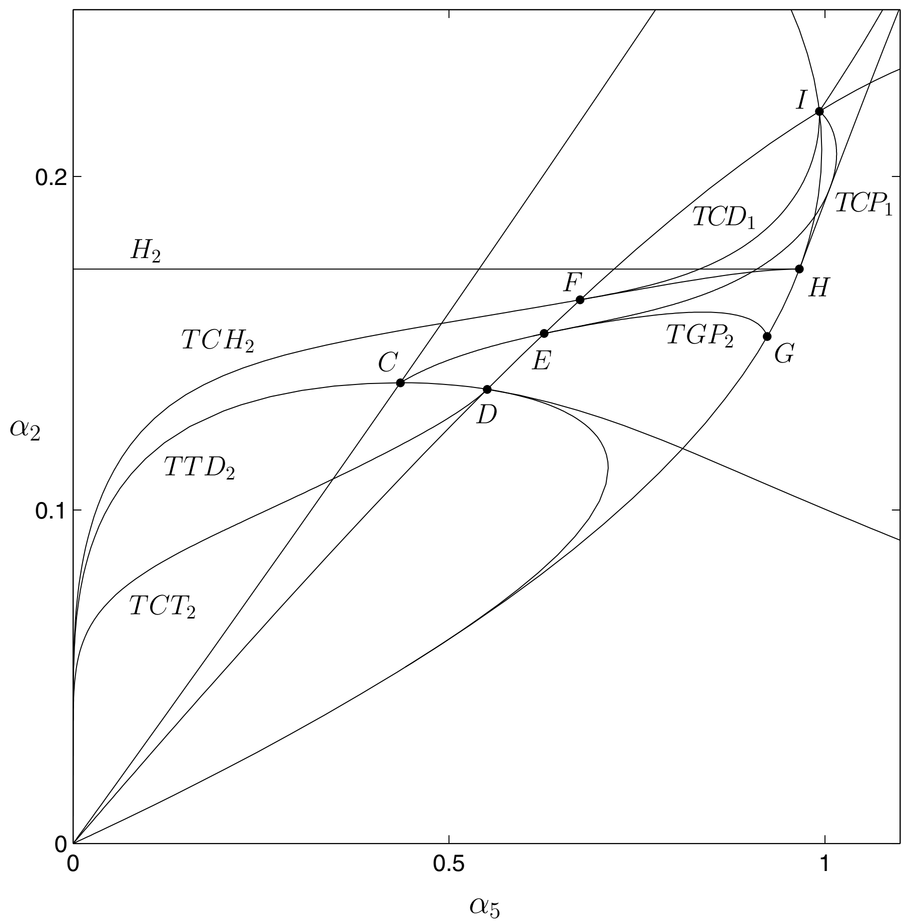

Figure 26: Magnified view of the shaded region in Figure 25. Bifurcation curves: TCH_2 - touching (grazing) bifurcation of vector field $f^{(2)}$; TCT_2 - crossing orbit of vector fields $f^{(2)}, f^{(1)}$ connecting two tangent points of $f^{(2)}$; TTD_2 - orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a tangent point of $f^{(1)}$; TCD_1 - crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a tangent point of $f^{(2)}$; TGP_2 - orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a pseudo-equilibrium; TCP_1 - crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a pseudo-equilibrium; H_2 - Hopf bifurcation of vector field $f^{(2)}$; Points $C - I$ are codimension-2 bifurcation points detected by SLIDECONT.

k	Command	Action

1	`cp sc.hppc.`k `sc.hppc`
	`cp hppc.f.`k `hppc.f`
	`@sc hppc`
	`@sv hppc.`k
		get the constants file
		get the equations file
		run SLIDECONT
		save output to `p.hppc.`k, `q.hppc.`k, `d.hppc.`k

2	`cp q.hppc.1 q.hppc`	get the starting solution file
3	`cp q.hppc.2 q.hppc`	get the starting solution file
4	`cp q.hppc.2 q.hppc`	get the starting solution file
5	`cp q.hppc.2 q.hppc`	get the starting solution file
6	`cp q.hppc.5 q.hppc`	get the starting solution file
7	`cp q.hppc.5 q.hppc`	get the starting solution file
8	`cp q.hppc.5 q.hppc`	get the starting solution file
9	`cp q.hppc.8 q.hppc`	get the starting solution file
10	`cp q.hppc.8 q.hppc`	get the starting solution file
11	`cp q.hppc.8 q.hppc`	get the starting solution file
12	`cp q.hppc.11 q.hppc`	get the starting solution file

13	`cp hppc.dat.13 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.13`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

14	`cp q.dat.13 q.hppc`	get the starting solution file
15	`cp q.dat.13 q.hppc`	get the starting solution file

16	`cp hppc.dat.16 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.16`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

17	`cp q.dat.16 q.hppc`	get the starting solution file
18	`cp q.hppc.15 q.hppc`	get the starting solution file
19	`cp q.hppc.15 q.hppc`	get the starting solution file

20	`cp hppc.dat.20 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.20`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

21	`cp q.hppc.20 q.hppc`	get the starting solution file
22	`cp q.hppc.20 q.hppc`	get the starting solution file
23	`cp q.hppc.1 q.hppc`	get the starting solution file
24	`cp q.hppc.23 q.hppc`	get the starting solution file

25	`cp hppc.dat.25 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.25`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

26	`cp q.dat.25 q.hppc`	get the starting solution file
27	`cp q.dat.25 q.hppc`	get the starting solution file

28	`cp hppc.dat.28 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.28`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

29	`cp q.dat.28 q.hppc`	get the starting solution file
30	`cp q.hppc.27 q.hppc`	get the starting solution file
31	`cp hppc.dat.31 hppc.dat`	get the data file

Table 4: (continue)

k	Command	Action
	`@scdat hppc` `cp q.hppc q.dat.31`	convert the data file into the starting solution file save the starting solution file
32	`cp q.dat.31 q.hppc`	get the starting solution file

33	`cp hppc.dat.33 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.33`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

34	`cp q.hppc.33 q.hppc`	get the starting solution file
35	`cp q.hppc.33 q.hppc`	get the starting solution file
36	`cp q.hppc.35 q.hppc`	get the starting solution file

37	`cp hppc.dat.37 hppc.dat`
	`@scdat hppc`
	`cp q.hppc q.dat.37`
		get the data file
		convert the data file into the starting solution file
		save the starting solution file

38	`cp q.dat.37 q.hppc`	get the starting solution file
39	`cp q.dat.36 q.hppc`	get the starting solution file
40	`cp q.dat.36 q.hppc`	get the starting solution file

Table 4: Command list: the first (last) two commands of computation 1 must precede (follow) the commands listed for $k = 2, \ldots, 40$; the equations file `hppc.f.`k, if not present, must be replaced with `hppc.f.0`; `@sv` is an AUTO97 command.

For $\alpha_2 = 1$ and $\alpha_5 = 3$ (top-right corner of Figure 25) vector field $f^{(1)}$ has a stable focus at $(z_1 = -1.9472\ldots, z_2 = 1.0134\ldots)$. Starting from this solution, we continue the equilibrium for decreasing values of α_2 (computation 1). We get the following output

```
PT  TY LAB    PAR(2)     ...      U(1)         U(2)        PAR(41)
 1  EP   1   1.00000E+00 ... -1.94720E+00   1.01338E+00  -2.45095E-01
10       2   9.64219E-01 ... -1.98363E+00   9.82883E-01  -3.27852E-01
20       3   8.28333E-01 ... -2.13554E+00   8.53210E-01  -6.52830E-01
23  HB   4   7.78000E-01 ... -2.19823E+00   7.98633E-01  -7.77500E-01
30       5   5.75949E-01 ... -2.49893E+00   5.29837E-01  -1.30134E+00
39  UZ   6   3.30000E-01 ... -3.05586E+00   1.04302E-02  -1.98952E+00
40       7   3.07518E-01 ... -3.12642E+00  -5.67696E-02  -2.05519E+00
50  EP   8   1.51267E-01 ... -3.83590E+00  -7.43205E-01  -2.52441E+00
```

where label `4` (`LAB=4`) indicates a Hopf bifurcation while label `6` is a user output point at $\alpha_2 = 0.33$ (see constants file `sc.hppc.1`). Since vector field $f^{(1)}$ does not depend on α_5, the Hopf bifurcation curve in the (α_5, α_2) plane is a straight line (see curve H_1 in Figure 25), therefore, we skip its numerical computation. By inspection of the output file `q.hppc.1`, one can check that after the Hopf bifurcation the equilibrium is an unstable focus.

Starting from the solution at the user output point (label `6`), we continue the equilibrium for decreasing values of α_5 (computation 2).

PT	TY	LAB	PAR(5)	...	U(1)	U(2)	PAR(41)
1	EP	9	3.00000E+00	...	-3.05586E+00	1.04302E-02	-1.98952E+00
10		10	2.67200E+00	...	-3.05586E+00	1.04302E-02	-1.66152E+00
20		11	1.67200E+00	...	-3.05586E+00	1.04302E-02	-6.61515E-01
27	UZ	12	1.01048E+00	...	-3.05586E+00	1.04302E-02	-3.84850E-09
30		13	7.10485E-01	...	-3.05586E+00	1.04302E-02	3.00000E-01
38	EP	14	-8.95154E-02	...	-3.05586E+00	1.04302E-02	1.10000E+00

Label 12 indicates a zero of test function 1, i.e. a boundary equilibrium bifurcation. Starting from this solution, we continue the bifurcation in the plane (α_5, α_2) forward (computation 3) and backward (computation 4), thus obtaining the following outputs (see curve BEQ_1 in Figure 25)

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(41)
1	EP	15	1.01048E+00	...	3.30000E-01	3.59492E-02
10		16	1.04966E+00	...	3.43488E-01	3.48669E-02
20		17	1.24872E+00	...	4.12928E-01	2.92948E-02
30		18	1.80786E+00	...	6.16903E-01	1.29270E-02
36	UZ	19	2.22250E+00	...	7.78000E-01	4.94230E-11
40		20	2.54049E+00	...	9.08309E-01	-1.04565E-02
43	UZ	21	2.75491E+00	...	1.00000E+00	-1.78141E-02
46	EP	22	3.00004E+00	...	1.10912E+00	-2.65706E-02

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(41)
1	EP	15	1.01048E+00	...	3.30000E-01	3.59492E-02
20		16	8.02944E-01	...	2.59482E-01	4.16078E-02
40		17	1.07293E-01	...	3.35501E-02	5.97375E-02
60	EP	18	6.15279E-04	...	1.91480E-04	6.24144E-02

where label 19 in the forward computation identifies a codimension-2 bifurcation, namely a boundary-Hopf bifurcation (see point B in Figure 25), as a zero of test function 1.

Starting again from the solution at label 12, we continue the pseudo-saddle colliding with the unstable focus at the boundary equilibrium bifurcation for increasing values of α_5 (computation 5). We get the following output

PT	TY	LAB	PAR(5)	...	PAR(41)	PAR(42)
1	EP	15	1.01048E+00	...	1.25638E-11	1.00000E+00
10		16	1.01568E+00	...	1.27664E-02	9.87234E-01
20		17	1.04423E+00	...	8.17597E-02	9.18240E-01
30		18	1.13988E+00	...	3.05189E-01	6.94811E-01
40	LP	19	1.24360E+00	...	6.51847E-01	3.48153E-01
50	EP	20	9.04329E-01	...	9.44763E-01	5.52371E-02

where label 19 indicates a pseudo-saddle-node bifurcation, whose forward and backward continuations (computations 6 and 7) are reported below (see curve PSN in Figure 25).

```
PT   TY LAB    PAR(5)      ...   PAR(2)        PAR(41)       PAR(42)
10       21   1.27118E+00 ...  3.44665E-01   6.23634E-01   3.76366E-01
20       22   1.38762E+00 ...  4.04902E-01   5.13921E-01   4.86079E-01
30       23   1.71214E+00 ...  5.60562E-01   2.69634E-01   7.30366E-01
40       24   2.21424E+00 ...  7.74693E-01   3.60651E-03   9.96393E-01
41   UZ  25   2.22250E+00 ...  7.78000E-01  -1.87790E-08   1.00000E+00
50   EP  26   2.53582E+00 ...  8.99199E-01  -1.23494E-01   1.12349E+00
```

```
PT   TY LAB    PAR(5)      ...   PAR(2)        PAR(41)       PAR(42)
10       21   1.21650E+00 ...  3.15428E-01   6.80505E-01   3.19495E-01
20       22   1.11167E+00 ...  2.57473E-01   8.01058E-01   1.98942E-01
28   UZ  23   9.66081E-01 ...  1.72243E-01   1.00000E+00   4.54661E-10
30       24   9.11820E-01 ...  1.38847E-01   1.08607E+00  -8.60733E-02
39   UZ  25   7.03037E-01 ... -1.99083E-10   1.50556E+00  -5.05564E-01
40       26   6.85973E-01 ... -1.22104E-02   1.54810E+00  -5.48096E-01
50   EP  27   5.35108E-01 ... -1.27568E-01   2.00867E+00  -1.00867E+00
```

Notice that point B of Figure 25 is detected again during the forward computation (zero of test function 1 at label `25`), while label `23` in the backward computation identifies another codimension-2 bifurcation, namely a boundary-pseudo-saddle-node bifurcation (see point H in Figure 26), as a zero of test function 2.

Starting from the solution at label `19` of computation 5, we now continue the pseudo-node colliding with the pseudo-saddle at the pseudo-saddle-node bifurcation for decreasing values of α_5 (computation 8)

```
PT   TY LAB    PAR(5)      ...   PAR(41)       PAR(42)
 1   EP  21   1.24360E+00 ...  6.51847E-01   3.48153E-01
10       22   1.24302E+00 ...  6.70003E-01   3.29997E-01
20       23   1.22931E+00 ...  7.34503E-01   2.65497E-01
30       24   1.11341E+00 ...  8.60547E-01   1.39453E-01
40       25   8.39853E-01 ...  9.61723E-01   3.82774E-02
45   UZ  26   6.53850E-01 ...  1.00000E+00  -2.99377E-13
50   EP  27   4.53636E-01 ...  1.03049E+00  -3.04875E-02
```

detecting a boundary equilibrium of vector field $f^{(2)}$ (zero of test function 2 at label `26`). The forward and backward continuations of the boundary equilibrium bifurcation (computations 9 and 10) give the following outputs (see curve BEQ_2 in Figure 25)

```
PT   TY LAB    PAR(5)      ...   PAR(2)        PAR(41)
 1   EP  28   6.53850E-01 ...  3.30000E-01   0.00000E+00
20       29   7.76769E-01 ...  3.05643E-01   0.00000E+00
34   LP  30   9.95619E-01 ...  2.08084E-01  -7.34674E-02
39   UZ  31   9.66081E-01 ...  1.72243E-01   1.93017E-08
40       32   9.54478E-01 ...  1.65785E-01   1.32372E-02
60       33   2.39419E-01 ...  2.67371E-02   0.00000E+00
80   EP  34   8.85139E-05 ...  9.24969E-06   0.00000E+00
```

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(41)
1	EP	28	6.53850E-01	...	3.30000E-01	0.00000E+00
20		29	5.41730E-01	...	3.48582E-01	0.00000E+00
40		30	1.12237E-01	...	4.04090E-01	0.00000E+00
60	EP	31	1.59648E-06	...	4.16169E-01	0.00000E+00

where label `31` in the forward computation identifies a second boundary-Hopf codimension-2 bifurcation (see again point H in Figure 26), as a zero of test function 1.

Restarting from the solution at label `26` of computation 8, we continue the stable focus of vector field $f^{(2)}$ for decreasing values of α_2 (computation 11)

PT	TY	LAB	PAR(2)	...	U(1)	U(2)	PAR(41)
1	EP	28	3.30000E-01	...	-2.31998E-01	-4.24877E-01	-5.73852E-12
10		29	3.25483E-01	...	-2.45781E-01	-3.87566E-01	2.48564E-02
20		30	3.08632E-01	...	-2.98943E-01	-2.70292E-01	1.09306E-01
30		31	2.57424E-01	...	-4.80366E-01	-6.22558E-02	2.85792E-01
40		32	1.92281E-01	...	-7.72133E-01	-1.01750E-02	3.36026E-01
44	HB	33	1.72243E-01	...	-8.82183E-01	-3.45071E-02	3.12231E-01
50	EP	34	1.32406E-01	...	-1.14522E+00	-1.46268E-01	2.10076E-01

detecting a Hopf bifurcation (label `33`). For the same arguments used before, the Hopf bifurcation curve of vector field $f^{(2)}$ is a straight line (see curve H_2 in Figure 26), therefore, we avoid its numerical computation.

We now continue the stable limit cycle originating from the Hopf bifurcation (which can be shown to be supercritical by means of suitable software, e.g. CONTENT (Kuznetsov & Levitin, 1995-1997); see Kuznetsov (1998) for an analytical proof) for decreasing values of α_2 (computation 12), obtaining the following output

PT	TY	LAB	PAR(2)	...	PERIOD	PAR(41)
10		35	1.72111E-01	...	3.02580E+01	2.90096E-01
20		36	1.70870E-01	...	3.07029E+01	2.30798E-01
30		37	1.67490E-01	...	3.20771E+01	1.33031E-01
40	UZ	38	1.62226E-01	...	3.48569E+01	-6.10623E-14
50	EP	39	1.57846E-01	...	3.79726E+01	-1.13806E-01

where label `38` indicates a touching (grazing) bifurcation (zero of test function 1). Notice that SLIDECONT does not support the automatic switch from cycle to touching (grazing) continuation. Thus, the user must provide a data file (`hppc.dat.13`) which specifies numerically the starting solution. This file can be easily constructed from the solution at label `38` of the output file (`q.hppc.12`) produced by cycle continuation. Though not explicitly pointed out in the following, a similar remark holds for each non-automatically supported switch between problems. The forward and backward continuations (computations 13 and 14) give the following outputs (see curve TCH_2 in Figure 26)

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(61)	PAR(41)
10		2	6.78992E-01	...	1.63219E-01	3.42591E+01	-1.17523E-01
20		3	7.87620E-01	...	1.67518E-01	3.20644E+01	-8.36044E-02
30		4	8.96762E-01	...	1.71199E-01	3.05823E+01	-3.69883E-02
38	UZ	5	9.66082E-01	...	1.72243E-01	3.02124E+01	1.00995E-08
40		6	9.82029E-01	...	1.72158E-01	3.02418E+01	9.58993E-03
50	EP	7	1.03897E+00	...	1.69287E-01	3.13145E+01	5.00037E-02

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(61)	PAR(41)
40		2	5.44655E-02	...	1.11966E-01	6.80716E+01	-3.70434E-02
80		3	1.28314E-05	...	3.66698E-02	1.75100E+02	-1.17004E-05
120		4	7.38073E-11	...	1.84162E-02	3.37676E+02	-7.05412E-11
160		5	3.09171E-17	...	1.14904E-02	5.36404E+02	-3.00635E-17
200	EP	6	3.46004E-24	...	8.15577E-03	7.53186E+02	-3.39223E-24

where point H of Figure 25 is detected again during the forward computation (zero of test function 1 at label 5). At point H the touching (grazing) cycle degenerates to a point (boundary-Hopf bifurcation). Thus, the solutions obtained in computation 13 after this point are spurious, because the touching (grazing) cycle of vector field $f^{(1)}$ lies in region S_2.

Starting from the touching (grazing) cycle at label 38, we continue the sliding cycle originating from the touching (grazing) bifurcation (i.e. an orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with the boundary Σ) for decreasing values of α_2 (computation 15).

PT	TY	LAB	PAR(2)	...	PAR(61)	PAR(44)	PAR(45)
10		2	1.62194E-01	...	3.44617E+01	1.31463E-01	-1.71899E-02
20		3	1.60021E-01	...	3.15805E+01	1.00953E-01	-1.38230E-02
26	UZ	4	1.54167E-01	...	2.85774E+01	6.56335E-02	-3.45955E-12
30		5	1.42164E-01	...	2.50970E+01	2.46167E-02	2.65791E-02
34	UZ	6	1.29442E-01	...	2.27310E+01	-8.43333E-11	5.21760E-02
40		7	9.76398E-02	...	1.99475E+01	-2.10734E-02	1.15167E-01
48	UZ	8	8.87051E-02	...	2.11274E+01	-1.50894E-09	1.15456E-01
50	EP	9	8.75947E-02	...	2.17676E+01	2.76574E-02	9.15012E-02

Label 4 indicates a pseudo-homoclinic bifurcation, i.e. the presence of an orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a pseudo-equilibrium (zero of test function 5), while labels 6 and 8 indicate buckling (switching) bifurcations, i.e. the presence of an orbit of vector field $f^{(2)}$ connecting a tangent point of $f^{(2)}$ with a tangent point of $f^{(1)}$ (zeros of test function 4).

Starting from the solutions at labels 4 and 6, we continue the corresponding bifurcations, forward and backward (computations 16-19), thus obtaining the outputs reported below (see curves TGP_2 and TTD_2 in Figure 26).

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(43)	PAR(44)
20		2	6.97557E-01	...	1.55978E-01	...	4.36976E-01	5.63024E-01
40		3	7.67487E-01	...	1.58273E-01	...	3.21617E-01	6.78383E-01
60		4	8.40062E-01	...	1.59322E-01	...	2.03457E-01	7.96543E-01
80		5	9.07660E-01	...	1.56286E-01	...	6.53144E-02	9.34686E-01
88	UZ	6	9.23286E-01	...	1.51997E-01	...	3.36276E-08	1.00000E+00
93	LP	7	9.25049E-01	...	1.49351E-01	...	-2.84730E-02	1.02847E+00
100	EP	8	9.12387E-01	...	1.40275E-01	...	-9.98941E-02	1.09989E+00

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(43)	PAR(44)
20		2	6.09661E-01	...	1.52064E-01	...	5.98773E-01	4.01227E-01
40		3	5.56162E-01	...	1.49051E-01	...	7.10961E-01	2.89039E-01
60		4	5.08334E-01	...	1.45688E-01	...	8.21017E-01	1.78983E-01
80		5	4.65764E-01	...	1.41796E-01	...	9.24967E-01	7.50331E-02
97	UZ	6	4.35242E-01	...	1.38171E-01	...	1.00000E+00	-9.17078E-10
100	EP	7	4.29923E-01	...	1.37443E-01	...	1.01281E+00	-1.28088E-02

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(61)	PAR(43)
19	LP	10	7.11638E-01	...	1.12554E-01	1.99485E+01	7.00017E-01
20		11	7.08720E-01	...	1.07769E-01	1.98371E+01	7.52608E-01
40		12	4.19217E-01	...	4.97571E-02	7.62366E+01	6.82921E-01
60		13	4.12679E-01	...	4.88618E-02	9.72040E+01	6.74863E-01
80		14	4.12442E-01	...	4.88294E-02	9.84043E+01	6.74569E-01
100	EP	15	4.12440E-01	...	4.88290E-02	9.84173E+01	6.74566E-01

PT	TY	LAB	PAR(5)	...	PAR(2)	PAR(61)	PAR(43)
20		10	5.23313E-01	...	1.37062E-01	2.84719E+01	1.08711E-01
25	UZ	11	4.35242E-01	...	1.38171E-01	3.26994E+01	-9.88708E-11
31	UZ	12	2.00000E-01	...	1.29693E-01	4.64692E+01	-1.00318E-01
40		13	2.85815E-02	...	9.18817E-02	7.16601E+01	-2.54400E-02
60		14	8.18660E-04	...	5.36179E-02	1.17558E+02	-8.08507E-04
80		15	1.43765E-05	...	3.60365E-02	1.71494E+02	-1.43008E-05
100	EP	16	1.41020E-07	...	2.62269E-02	2.33775E+02	-1.40492E-07

Labels `6` in computations 16 and 17 correspond to the codimension-2 bifurcation points G and C in Figure 26, respectively (zeros of test functions 3 and 4). Point C is also detected during computation 19 (zero of test function 3 at label `11`).

Starting from the user output point at label `12` of computation 19, we continue the sliding cycle originating from the buckling (switching) bifurcation (i.e. a crossing orbit of vector fields $f^{(2)}, f^{(1)}$ connecting a tangent point of $f^{(2)}$ with the boundary Σ) for decreasing values of α_2 (computation 20)

PT	TY	LAB	PAR(2)	...	PAR(61)	PAR(62)	PAR(45)
20		2	1.25296E-01	...	4.50573E+01	2.27843E+00	-3.63682E-02
40		3	9.43993E-02	...	3.80478E+01	1.64804E+01	-5.38328E-04
41	UZ	4	9.38293E-02	...	3.79463E+01	1.67780E+01	2.34665E-12
50	EP	5	5.93235E-02	...	3.28817E+01	4.36469E+01	1.62700E-02

detecting a crossing-crossing bifurcation (zero of test function 5 at label 4), whose forward and backward continuations (computations 21 and 22) give the following outputs (see curve TCT_2 in Figure 26)

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(44)
20		6	2.40485E-01	...	9.81182E-02	...	-1.05289E-02
40		7	3.36752E-01	...	1.08317E-01	...	-1.41588E-02
60		8	4.55907E-01	...	1.21933E-01	...	-1.48638E-02
78	UZ	9	5.50830E-01	...	1.36213E-01	...	1.81976E-11
80		10	5.57225E-01	...	1.37788E-01	...	3.09460E-03
87	LP	11	5.66527E-01	...	1.42303E-01	...	1.32836E-02
100	EP	12	5.42426E-01	...	1.46714E-01	...	2.38698E-02

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(44)
20		6	1.46530E-01	...	8.78584E-02	...	-6.48039E-03
40		7	1.05962E-02	...	5.87733E-02	...	-4.70094E-04
60		8	1.56275E-04	...	3.79024E-02	...	-6.93305E-06
80		9	1.23652E-06	...	2.68820E-02	...	-5.48539E-08
100	EP	10	6.39789E-09	...	2.04284E-02	...	-2.83807E-10

where label 9 of the forward computation corresponds to the codimension-2 bifurcation points D of Figure 26 (zero of test function 4).

We now restart from the (supercritical) Hopf bifurcation of vector field $f^{(1)}$ (label 4 of computation 1) and continue the stable limit cycle originating from it for decreasing values of α_2 (computation 23)

PT	TY	LAB	PAR(2)	...	PERIOD	PAR(41)
10		9	7.77537E-01	...	3.38224E+01	-7.30472E-01
20		10	7.58064E-01	...	3.45201E+01	-5.10637E-01
30		11	6.63927E-01	...	3.86395E+01	-3.22890E-01
40		12	4.91392E-01	...	5.07445E+01	-3.76301E-01
49	UZ	13	3.30000E-01	...	7.18753E+01	-5.59307E-01
50	EP	14	3.14964E-01	...	7.47881E+01	-5.79500E-01

and, starting from the user output at label 13, for decreasing values of α_5 (computation 24)

PT	TY	LAB	PAR(5)	...	PERIOD	PAR(41)
10		15	2.48900E+00	...	7.18753E+01	-4.82933E-02
11	UZ	16	2.44071E+00	...	7.18753E+01	-4.63707E-08
14	EP	17	-5.59294E-01	...	7.18753E+01	3.00000E+00

thus detecting, at label 16, a touching (grazing) bifurcation (zero of test function 1), whose forward and backward continuations (computations 25 and 26) are reported below (see curve TCH_1 in Figure 25).

```
  PT   TY LAB    PAR(5)      ...    PAR(2)        PAR(61)       PAR(41)
  20        2   2.46383E+00 ...  3.47550E-01   6.87607E+01   3.09391E+00
  40        3   2.53907E+00 ...  4.08450E-01   5.97995E+01   2.52081E+00
  60        4   2.62710E+00 ...  4.95379E-01   5.03757E+01   1.89425E+00
  80   LP   5   2.68762E+00 ...  6.21707E-01   4.09735E+01   1.16644E+00
  97   UZ   6   2.22250E+00 ...  7.78000E-01   3.38064E+01   5.06351E-08
 100   EP   7   1.88932E+00 ...  7.60450E-01   3.44321E+01  -2.23574E-01
```

```
  PT   TY LAB    PAR(5)      ...    PAR(2)        PAR(61)       PAR(41)
  20        2   2.40026E+00 ...  3.00123E-01   7.79182E+01   3.67640E+00
  40        3   2.18753E+00 ...  1.49210E-01   1.41240E+02   7.83949E+00
  60        4   2.03316E+00 ...  3.68003E-02   5.12072E+02   3.29342E+01
  80        5   1.99537E+00 ...  6.28300E-03   2.91247E+03   1.95212E+02
 100   EP   6   1.99139E+00 ...  2.55826E-03   7.12930E+03   4.80409E+02
```

where label `6` of the forward computation corresponds to the codimension-2 bifurcation points B of Figure 25 (zero of test function 1). After this point the solution has no sense.

As already done for vector field $f^{(2)}$, we continue the sliding cycle originating from the touching (grazing) bifurcation (label `16` of computation 24) for decreasing values of α_5 (computation 27)

```
  PT   TY LAB    PAR(5)      ...    PAR(41)       PAR(45)
  40        2   2.28039E+00 ...  2.73091E+00  -7.65388E-01
  80        3   1.81794E+00 ...  1.38427E+00  -1.91760E-01
 103   UZ   4   1.50000E+00 ...  6.92441E-01  -6.24439E-02
 119   UZ   5   1.22753E+00 ...  2.51248E-01   2.39698E-13
 120        6   1.20857E+00 ...  2.25765E-01   1.60261E-03
 130   UZ   7   1.02940E+00 ...  1.83584E-02  -4.03145E-13
 132   UZ   8   1.01048E+00 ... -2.30242E-09   2.47076E-11
 160        9   8.11376E-01 ... -1.52349E-01   1.73077E-02
 200   EP  10   6.69562E-01 ... -2.15265E-01   2.58086E-02
```

thus detecting a pseudo-homoclinic bifurcation (zero of test function 5 at labels `5` and `7`), whose forward and backward continuations (computations 28 and 29) are reported below (see curve TGP_1 in Figure 25)

```
  PT   TY LAB    PAR(5)      ...    PAR(2)      ...    PAR(43)       PAR(44)
  10        2   1.22947E+00 ...  3.33549E-01 ...  5.17583E-01   4.82417E-01
  14   LP   3   1.23038E+00 ...  3.37747E-01 ...  4.85347E-01   5.14653E-01
  20        4   1.21954E+00 ...  3.48173E-01 ...  3.75685E-01   6.24315E-01
  30        5   1.13889E+00 ...  3.48593E-01 ...  1.76992E-01   8.23008E-01
  40        6   9.78219E-01 ...  3.18569E-01 ...  2.65354E-03   9.97346E-01
  41   UZ   7   9.74779E-01 ...  3.17757E-01 ...  3.38507E-09   1.00000E+00
  50   EP   8   8.52887E-01 ...  2.86163E-01 ... -7.73273E-02   1.07733E+00
```

```
  PT   TY LAB    PAR(5)      ...    PAR(2)      ...    PAR(43)       PAR(44)
  10        2   1.22462E+00 ...  3.26290E-01 ...  5.66373E-01   4.33627E-01
  20        3   1.19145E+00 ...  3.01936E-01 ...  6.96532E-01   3.03468E-01
  30        4   1.07533E+00 ...  2.49321E-01 ...  9.04037E-01   9.59632E-02
  36   UZ   5   9.92599E-01 ...  2.19545E-01 ...  1.00000E+00  -2.20176E-09
  40        6   9.35376E-01 ...  2.00814E-01 ...  1.05534E+00  -5.53443E-02
  50   EP   7   7.98581E-01 ...  1.60300E-01 ...  1.16489E+00  -1.64888E-01
```

and identify the codimension-2 bifurcation points A and I in Figures 25 and 26 (zeros of test functions 3 and 4 at labels `7` and `5` in the forward and backward computations, respectively).

Continuing again the sliding cycle of computation 27, starting from the user output point at label `4` for decreasing values of α_2 (computation 30)

```
  PT   TY LAB    PAR(2)      ...   PAR(61)       PAR(44)
  10       11   3.27180E-01 ...  3.79075E+01  -6.52592E-02
  20       12   3.00261E-01 ...  4.23003E+01  -4.28902E-02
  30       13   2.51521E-01 ...  5.29961E+01  -1.36618E-02
  35   UZ  14   2.23157E-01 ...  6.12665E+01   5.25195E-13
  40       15   1.84635E-01 ...  7.60351E+01   1.78660E-02
  50   EP  16   1.40280E-01 ...  1.01650E+02   4.03611E-02
```

we detect a buckling (switching) bifurcation (zero of test function 4 at label `14`), whose forward and backward continuations (computations 31 and 32) give the following outputs (see curve TTD_1 in Figure 26)

```
  PT   TY LAB    PAR(5)      ...   PAR(2)        PAR(61)       PAR(43)
  20       17   1.57430E+00 ...  2.09054E-01   7.00127E+01   4.55407E-01
  40       18   1.83312E+00 ...  1.29146E-01   1.35543E+02   1.45478E+00
  60       19   1.96568E+00 ...  3.64351E-02   4.99926E+02   9.28855E+00
  80       20   1.98489E+00 ...  6.36240E-03   2.86109E+03   6.27551E+01
 100   EP  21   1.98719E+00 ...  2.57161E-03   7.07785E+03   1.58493E+02
```

```
  PT   TY LAB    PAR(5)      ...   PAR(2)        PAR(61)       PAR(43)
  40       17   1.35673E+00 ...  2.39149E-01   4.95065E+01   2.27234E-01
  68   UZ  18   9.92599E-01 ...  2.19545E-01   3.62614E+01   2.91412E-11
  80       19   4.65709E-01 ...  1.16683E-01   3.01776E+01  -1.40896E-01
 120       20   2.42453E-02 ...  6.36225E-03   2.79175E+01  -1.42195E-02
 160       21   1.42387E-03 ...  3.74265E-04   2.78305E+01  -8.56510E-04
 200   EP  22   8.41273E-05 ...  2.21150E-05   2.78255E+01  -5.06800E-05
```

where label `18` in the backward computation corresponds again to the codimension-2 bifurcation point I in Figure 26 (zero of test function 3).

We now continue the sliding cycle originating from the buckling (switching) bifurcation (label `14` of computation 30) for decreasing values of α_2 (computation 33)

```
  PT   TY LAB    PAR(2)      ...   PAR(61)       PAR(62)       PAR(45)
  20        2   2.18770E-01 ...  6.27164E+01   1.51874E-01   3.08010E-01
  40        3   1.88132E-01 ...  7.44769E+01   1.06016E+00   2.92309E-01
  60        4   1.03584E-01 ...  1.37591E+02   2.02648E+00   1.81107E-01
  72   UZ   5   5.45143E-02 ...  2.57363E+02   1.84682E+00  -8.49024E-11
  80        6   2.33203E-02 ...  5.94075E+02   1.61535E+00  -6.61825E-02
 100   EP   7   3.95110E-03 ...  3.47969E+03   1.47370E+00  -6.66000E-02
```

thus detecting a crossing-crossing bifurcation (zero of test function 5 at label `5`), whose forward and backward continuations (computations 34 and 35) are reported below (see curve TCT_1 in Figure 26).

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(41)	PAR(44)
20		8	1.52984E+00	...	5.16540E-02	...	1.25818E+01	1.01595E-01
40		9	1.73297E+00	...	3.08930E-02	...	2.85271E+01	1.19799E-01
60		10	1.92386E+00	...	8.44640E-03	...	1.34447E+02	1.38984E-01
80		11	1.96698E+00	...	2.88164E-03	...	4.15863E+02	1.43693E-01
100	EP	12	1.97580E+00	...	1.72740E-03	...	7.01312E+02	1.44674E-01

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(41)	PAR(44)
80		8	1.41617E+00	...	6.23651E-02	...	8.60407E+00	9.20870E-02
160		9	1.27960E+00	...	7.47643E-02	...	5.54947E+00	8.11215E-02
240		10	1.06399E+00	...	9.42596E-02	...	2.68788E+00	6.45378E-02
279	UZ	11	9.00000E-01	...	1.09860E-01	...	1.40860E+00	5.23570E-02
319	UZ	12	5.50830E-01	...	1.36213E-01	...	1.53073E-01	1.33521E-10
320		13	5.32750E-01	...	1.36814E-01	...	1.23238E-01	-1.05770E-02
327	UZ	14	4.35266E-01	...	1.38179E-01	...	-2.68822E-09	-8.99695E-02
400	EP	15	3.09084E-01	...	1.36026E-01	...	-8.45254E-02	-7.76109E-02

Labels `12` and `14` of the backward computation correspond again to the codimension-2 bifurcation points D and C of Figure 26, respectively (zeros of test functions 4 and 1).

Starting from the user output point at label `10`, we continue the sliding cycle originating from the crossing-crossing bifurcation (i.e. a crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with the boundary Σ) for increasing values of α_2 (computation 36).

PT	TY	LAB	PAR(2)	...	PAR(46)	PAR(47)
20		16	1.14806E-01	...	-1.76448E-01	-2.30574E-01
40		17	1.29605E-01	...	-1.42232E-01	-1.45501E-01
60		18	1.48027E-01	...	-9.28119E-02	-6.15887E-02
80		19	1.69601E-01	...	-3.54030E-02	-2.17204E-03
82	UZ	20	1.70986E-01	...	-3.18574E-02	5.04256E-12
99	LP	21	1.80346E-01	...	1.74496E-13	7.56008E-03
100	EP	22	1.80301E-01	...	1.49712E-03	7.28495E-03

Label `20` indicates a pseudo-homoclinic bifurcation, i.e. the presence of a crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a pseudo-equilibrium (zero of test function 7), while label `21` indicates the presence of a crossing orbit of vector fields $f^{(1)}, f^{(2)}$ connecting a tangent point of $f^{(1)}$ with a tangent point of $f^{(2)}$ (zero of test function 6). The forward and backward continuations of these bifurcations (computations 37-40) give the following outputs (see curves TCP_1 and TCD_1 in Figure 26)

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(46)
20		2	7.36964E-01	...	1.58109E-01	...	3.76354E-01
40		3	1.01536E+00	...	2.05810E-01	...	3.73121E-02
41	LP	4	1.01545E+00	...	2.06746E-01	...	3.51155E-02
46	UZ	5	9.92599E-01	...	2.19545E-01	...	3.03505E-10
60		6	1.93557E-01	...	5.85610E-02	...	-4.03693E-01
80		7	8.80653E-03	...	2.84106E-03	...	-5.00675E-01
100	EP	8	4.73232E-04	...	1.59320E-04	...	-5.05290E-01

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(44)
20		2	6.64025E-01	...	1.54660E-01	...	9.66445E-03
25	UZ	3	6.26627E-01	...	1.52906E-01	...	4.79962E-15
40		4	4.92656E-01	...	1.44396E-01	...	-6.56560E-02
60		5	3.90105E-01	...	1.30804E-01	...	-9.25824E-02
80		6	1.27499E-01	...	4.53387E-02	...	-3.19934E-02
100	EP	7	5.46504E-02	...	1.94337E-02	...	-1.37203E-02

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(44)	PAR(45)
40		23	9.80627E-01	...	2.03284E-01	...	1.34864E-02	-7.29041E-03
78	LP	24	9.92657E-01	...	2.18461E-01	...	1.13581E-03	-2.31160E-04
80		25	9.92640E-01	...	2.19054E-01	...	5.19980E-04	-1.00907E-04
82	UZ	26	9.92599E-01	...	2.19545E-01	...	-1.17031E-10	2.18153E-11
120		27	9.88472E-01	...	2.27873E-01	...	-1.03617E-02	8.41806E-04
160		28	9.82712E-01	...	2.33226E-01	...	-1.89340E-02	7.16653E-04
200	EP	29	9.77730E-01	...	2.36848E-01	...	-2.57838E-02	4.89891E-04

PT	TY	LAB	PAR(5)	...	PAR(2)	...	PAR(44)	PAR(45)
20		23	8.59208E-01	...	1.74863E-01	...	2.32613E-02	-5.70129E-02
40		24	7.58033E-01	...	1.66882E-01	...	1.63950E-02	-9.42467E-02
60		25	6.74568E-01	...	1.63044E-01	...	4.81413E-05	-1.18639E-01
61	UZ	26	6.74401E-01	...	1.63037E-01	...	1.01363E-09	-1.18681E-01
80		27	6.10048E-01	...	1.60579E-01	...	-2.55614E-02	-1.32471E-01
100	EP	28	5.64458E-01	...	1.58824E-01	...	-5.34391E-02	-1.39407E-01

and identify the codimension-2 bifurcation points E, I and F of Figure 26 (zeros of test functions 4, 5, and 4 at labels `3`, `26`, and `26` of computations 38, 39, and 40, respectively).

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