Homoclinic Bifurcations to Equilibria

II. Numerical continuation

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Outline: Lecture 2

1. HomCont: continuation of homoclinic orbits in AUTO
   - projection boundary conditions for homoclinics
   - heteroclinic orbits
   - 4 ways start
   - conservative systems
2. Codimension-two homoclinic bifurcations
   - eigenvalue degeneracies
   - non-hyperbolic equilibria
   - orientation flips
   - other cases
3. HomCont implementation of codim-2 cases

Homoclinic orbit continuation

C. Kuznetsov, Sandstede 95

Consider the BVP on an infinite interval

$$\dot{x}(t) = f(x(t), \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^p, \quad f \in C^r$$
$$f(x_0, \alpha) = 0,$$
$$x(t) \to x_0 \quad \text{as} \quad t \to \pm\infty.$$

Plus phase condition $\int_{-\infty}^{\infty} \dot{\tilde{x}}^{\mathsf{T}}(t)[x(t) - \tilde{x}(t)]dt = 0$ which minimises L_2 distance from reference solution $\tilde{x}$ (previous solution on branch).

Well posed provided equilibrium $x_0(\alpha)$ is hyperbolic:

$A := D_x f(x_0, \alpha)$ has n_s stable and n_u unstable eigs: $n_s + n_u = n$.

University of BRISTOL

Basic idea

Beyn 1990, Friedman & Doedel 91

- Truncate to a finite interval $t \in [-T, T]$, $2T =$`par(11)`
- pose projection boundary conditions:

$$L_s(x_0,\alpha)(x(-T)-x_0)=0, \quad L_u(x_0,\alpha)(x(+T)-x_0)=0.$$

where rows of $L_{s,u}(x_0,\alpha) \in \mathbb{R}^{n_{s,u}\times n}$ forms basis of stable/unstable eigenspace of A^T
& truncated phase condition $\int_{-\infty}^{\infty} \dot{\tilde{x}}^{\mathsf{T}}(t)[x(t)-\tilde{x}(t)]dt = 0$

- $n_s + n_u = n+1$ side conditions for n unknowns $x(t) \Rightarrow$ continuation problem for two pars: α_1, α_2
- Convergence as $T \to \infty \sim e^{-2T|\lambda_d|}$, where λ_d = determining eigenvalue (Sandstede 95).
[Other linear (e.g. periodic) B.C.'s converge $\sim e^{-T|\lambda_d|}$]

Heteroclinic case

$$x(t) \to x_0 \text{ as } t \to -\infty, \qquad x(t) \to x_1 \text{ as } t \to +\infty$$

Truncate to $[-T, T]$. Can treat x_0 ,x_1 as unknowns in $\mathbb{R}^n$ defined by $2n$ algebraic conditions

$$f(x_0, \alpha) = 0, \qquad f(x_1, \alpha) = 0.$$

Well posed provided $Df(x_0)$ and $Df(x_1)$ hyperbolic. Pose

$$\begin{aligned} L_s(x_0, \alpha)(x(-T) - x_0) &= 0, \\ L_u(x_1, \alpha)(x(+T) - x_1) &= 0. \end{aligned}$$

+ same integral phase condition.
Need $p = n_s + n_u - n + 2$ free parameters. where $n_s = \text{dim}(W^s(x_0))$ and $n_u = \text{dim}(W^u(x_1))$.

AUTO implementation

Re-scale to $\tau \in [0,1]$ $\tau = (t-T)/2T \in [0,1]$. Solve

$$\dot{x} = Tf(x,\alpha), \qquad \text{WLOG} \quad \dot{} = d/d\tau$$

subject to boundary conditions

$$\begin{aligned} L_s(x_0,\alpha)(x(0)-x_0) &= 0, \\ L_u(x_1,\alpha)(x(1)-x_1) &= 0. \\ f(x_0,\alpha) &= 0 \\ f(x_1,\alpha) &= 0 \end{aligned}$$

and integral condition $\int_0^1 \dot{\tilde{x}}^{\mathsf{T}}(t)[x(t)-\tilde{x}(t)]dt = 0$

for $3n$ unknowns $x(t)$, x_0, x_1, $\quad n_s + n_u + 2n + 1$ side conditions and p continuation parameters

HomCont

- AUTO sets up these boundary conditions (and rescaling to $\tau \in [0, 1]$) automatically
- User should specify n_s=`NSTAB`, n_u= `NUNSTAB`, and `IEQUIB = 0 or -1` for explicit equilibria OR `IEQUIB = 1 or -2` for continued equilibria
- Actually `NSTAB` and `NUNSTAB` can be automatically detected using eigenvalues of $Df(x_0)$ in homoclinic case.
- Several ways to start, using `ISTART` $\ldots$

Ways to start

ISTART=1 Data from a previous numerical integration is read into AUTO using the `@fc` command. This data should be in multicolumn format $T, [U(i), i = 1 \ldots n]$. See e.g. demo `cir`.

ISTART=2 An explicit homoclinic solution is stored in STPNT. See e.g. demo `san`.

ISTART=3 The “homotopy method” $\ldots$ see e.g. demo `kpr`

ISTART=4 Data from a large-period periodic orbit. AUTO first performs a computation to “rotate” the data so that the equilibrium is at $\tau = 0$ and $\tau = 1$.

Starting using homotopy

(for homoclinic case with real eigenvalues λ_i, eigenvects v_i)

- Start with small solution tangent to $W^u(x_0)$:

$$x(0) = x_0 + \epsilon_0 \sum_{i=1}^{n_u} \xi_i v_i e^{\lambda_i T \tau}, \quad T \ll 1, \quad \sum_{i=1}^{n_u} \xi_i^2 = 1$$

- Continue in $T = PAR(11)$, and one ξ_i. monitor test functions $\omega_i = w_i^T(x(1) - x(0))$, $i = 1, \ldots n_u$ where $Df^T w_i = \lambda_i w_i$.
- Freeze T, successively free up ξ_i and α to look for zeros of $\omega_j = 0$.
- Continue in T again, freeing up a parameter but freezing all $\omega_i = 0$ until $x(1) - x_0 = O(\epsilon_0)$
- Recommend to use only in case $n_u = 1$.

Conservative case

(including Hamiltonian)

- Suppose $\dot{x} = f(x, \alpha)$ conserves a integral $H(x, \alpha)$.
- Then homoclinic orbits are codim 0, since $W^u(x_0)$ and $W^s(x_0)$ are contained in level set $H(x) =$ const.
- Use the conserved quantity $H = const.$ to include a dummy free parameter,

$$\dot{x} = f(x, \alpha) + \alpha_0 \nabla H(x)$$

  and use regular algorithm to continue in two free parameters α_1, α_0 where α_1 is a regular problem parameter. True solution has $\alpha_0 = 0 \Rightarrow$ accuracy test.
- See Doedel's lectures for extensions ...

2. Codim 2 homoclinic bifurcations

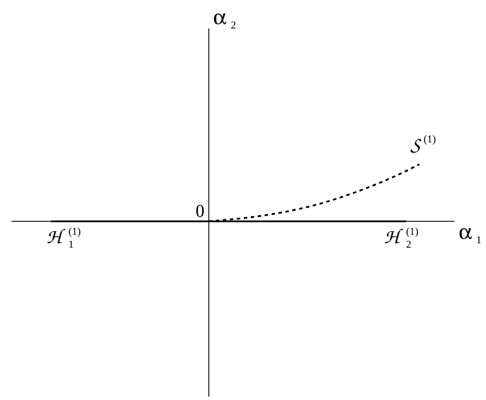
Interesting as ‘organising centres’ for parameter space; birth of multi-pulse homoclinics etc.

sources of degeneracy in codim 1 bifurcation:

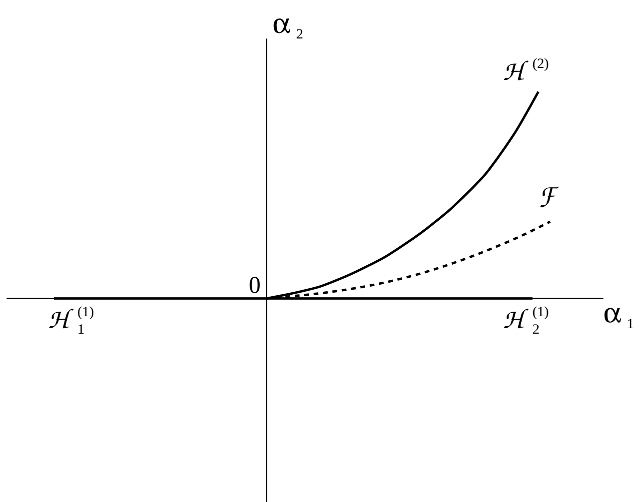
- eigenvalue degeneracy (hyperbolic cases)
- non-hyperbolic equilibrium (saddle-node, Hopf)
- (for real saddle) orientable $\rightarrow$ twisted transition
- other cases

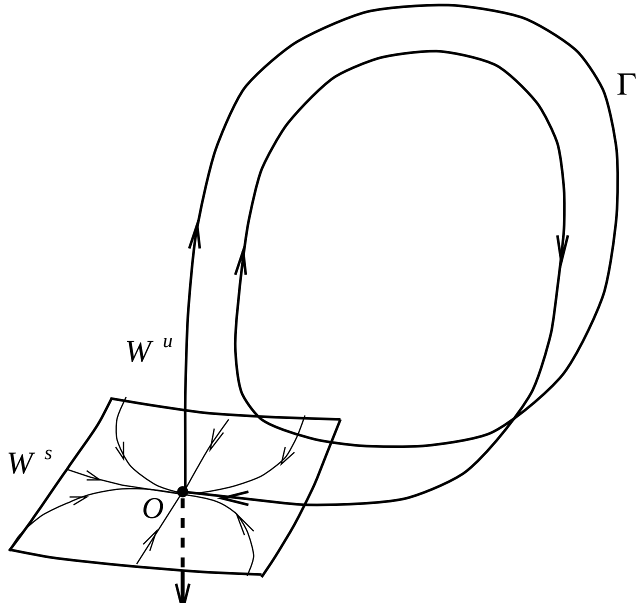
2.1 Eigenvalue transitions; Belyakov cases

A. Resonant eigenvalues: $\lambda_1 = \mu_1$
$\Rightarrow$ 'side-switching' of periodic orbit + saddle-node



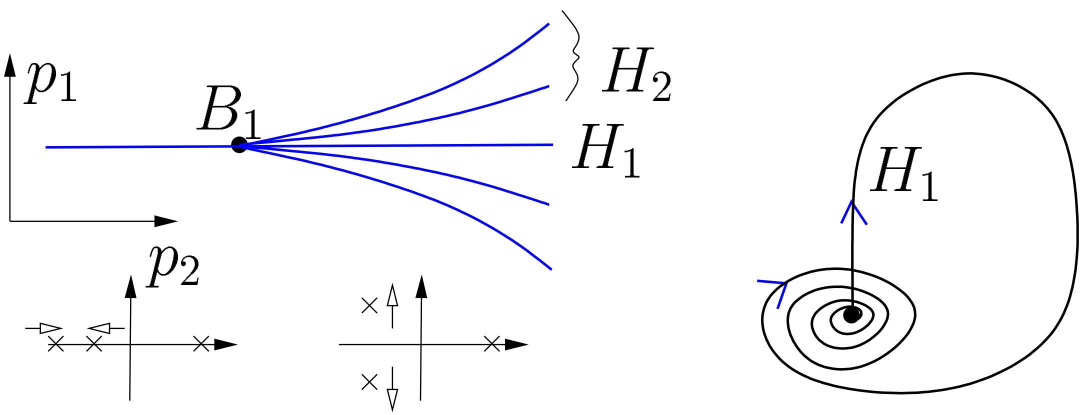
OR (if twisted) bifurcation of a 2-pulse homoclinic orbit

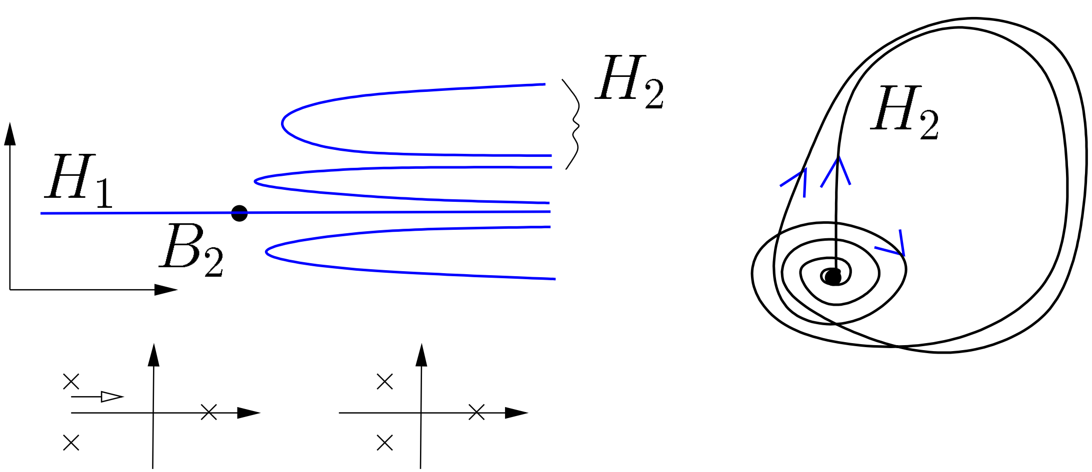




B_1 Double real determining eigenvalue: e.g. $\mu_1 = \mu_2 \Rightarrow$ 'broom handle' bifurcation

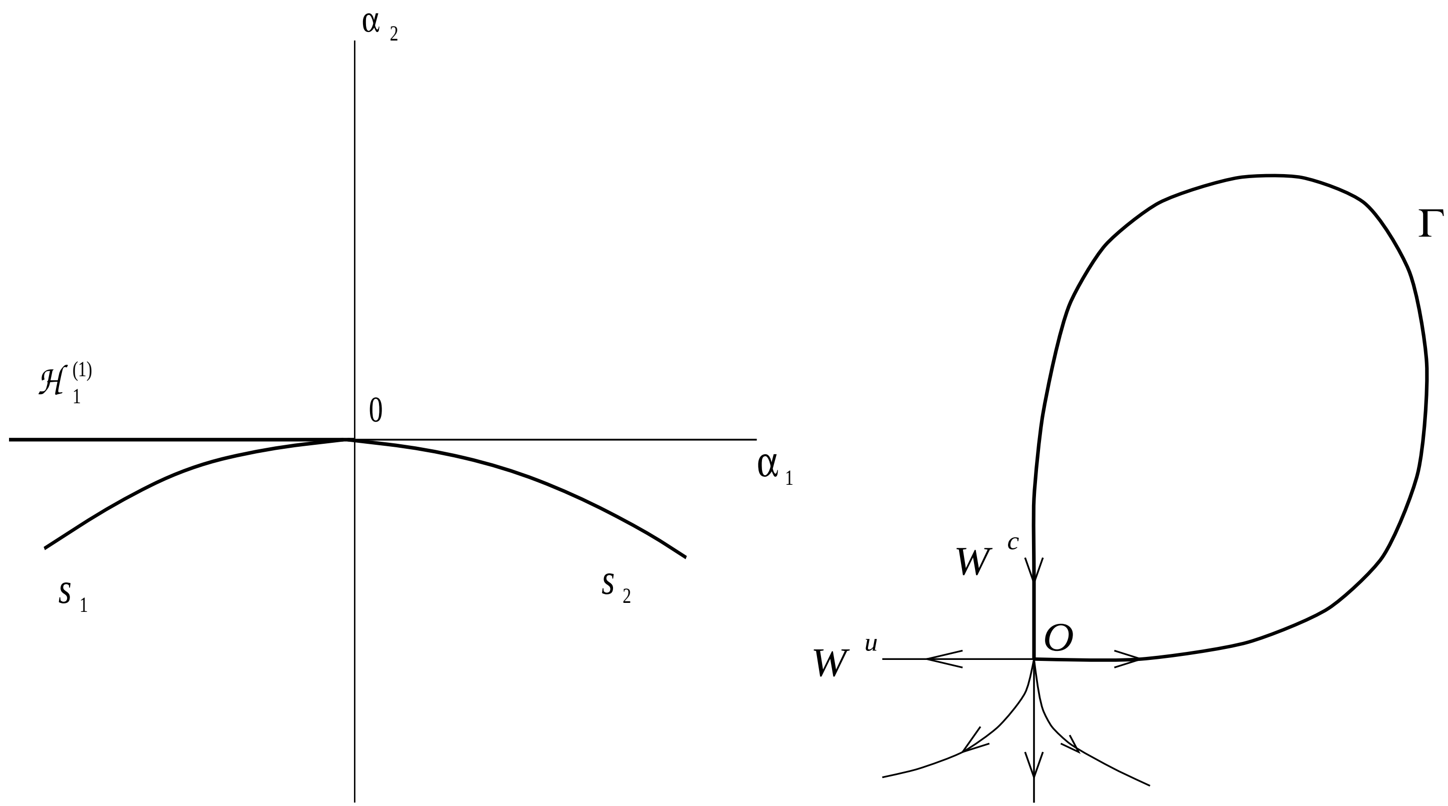
B_2. Resonant saddle-focus $\Rightarrow$ 'geological fold' bifurcation





2.2 Non-hyperbolic equilibria

A. Saddle-node homoclinic (Deng, Schecter)
$\Rightarrow$ hom H gets 'glued' to saddle-node curve S:



$\Rightarrow$ homoclinic to saddle-node along S_2

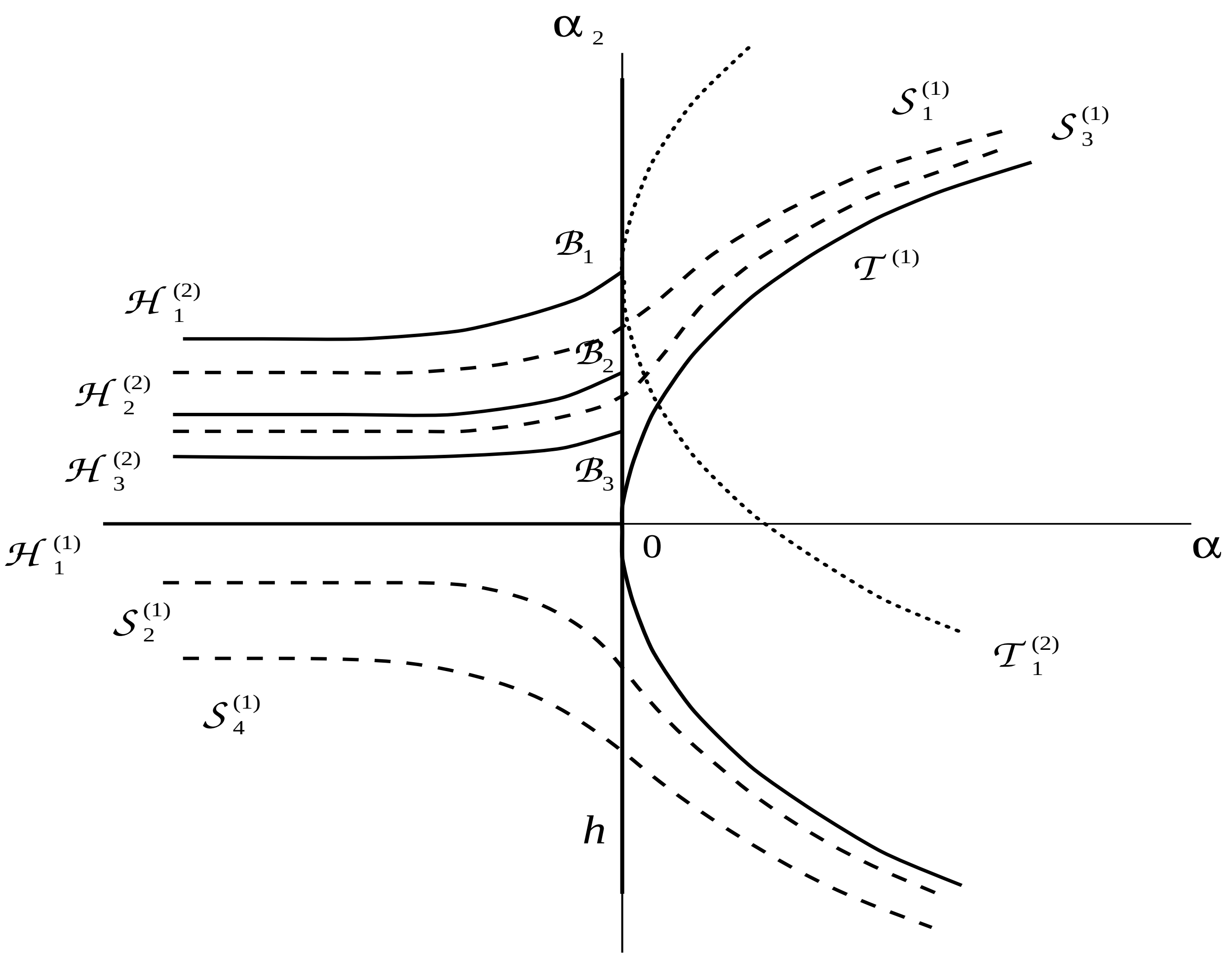
[no extra codimension to add 2 or more homs to saddle-node $\Rightarrow$ horseshoe dynamics; see Kuznetsov 2004]

B. Shil’nikov-Hopf bifurcation

(Belyakov, Hirschberg & Knobloch)

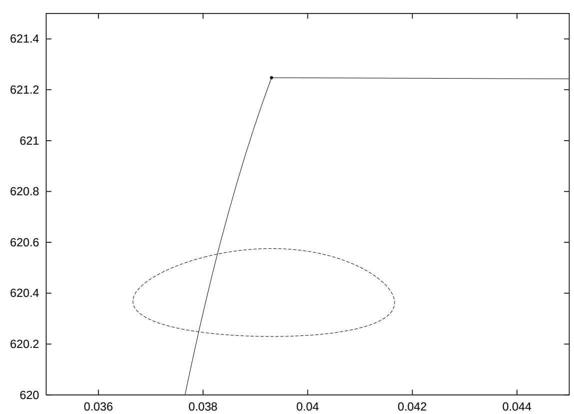
$\Rightarrow$ ‘wine glass bifurcation’ homoclinic to 0

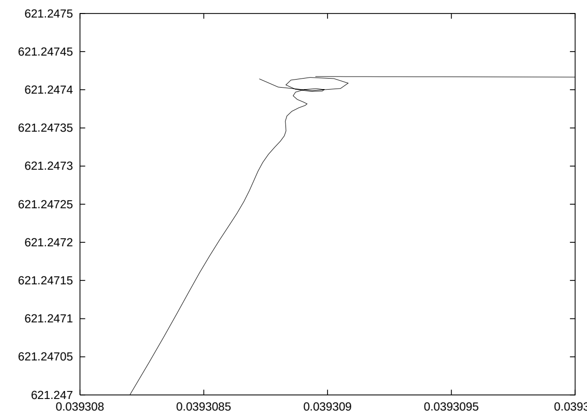
$\longrightarrow$ homoclinic tangency to periodic orbit

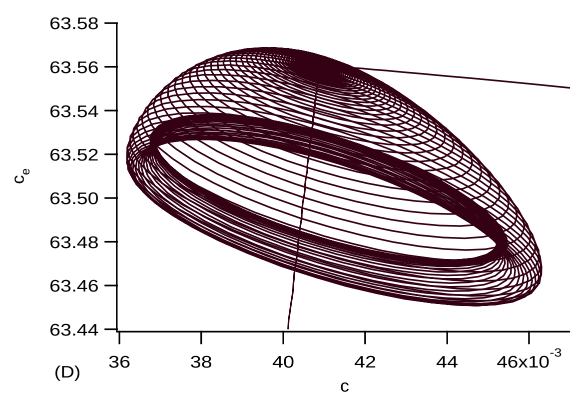


example: ‘anomolous’ Shil’nikov-Hopf

return to 8-variable **Ca**$^{2+}$ **model** (from lecture 1)
why does homoclinic “not see” the Hopf bifurcation?
In fact: numerical artifact due to slow-fast nature of system.
zoom of homoclinic + periodic for `PAR(11)` = 10^2, 10^4, 10^6

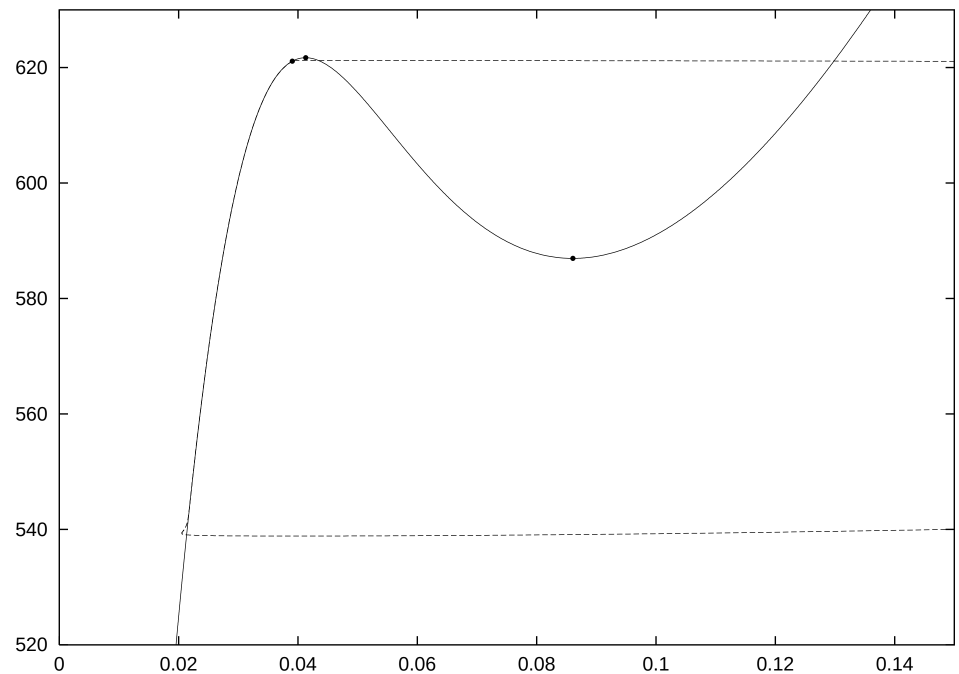






Explanation by slow-fast analysis

- Slow manifold + ‘homoclinic’ orbit



- No true homoclinic exists beyond the Hopf bifucation $\Rightarrow$, just regular Shil’nikov-Hopf scenario.
- BUT, this is region where pulse is stable. So could see pulse with ‘flat’ tail for exponentially long time.

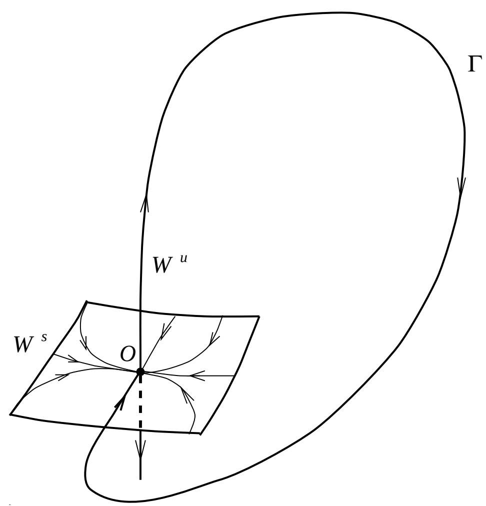
2.3 Orientation flips

Deng, Homberg, Kokubu, Sandstede . . .

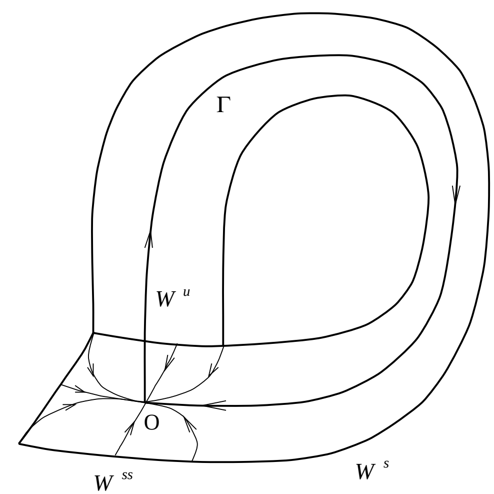
e.g. $\mathbb{R}^3$ x_0 is real saddle.

How can $W^s(x_0)$ change from orientable (cylinder) to twisted (Möbius strip)?

A. Inclination flip



B. Orbit flip

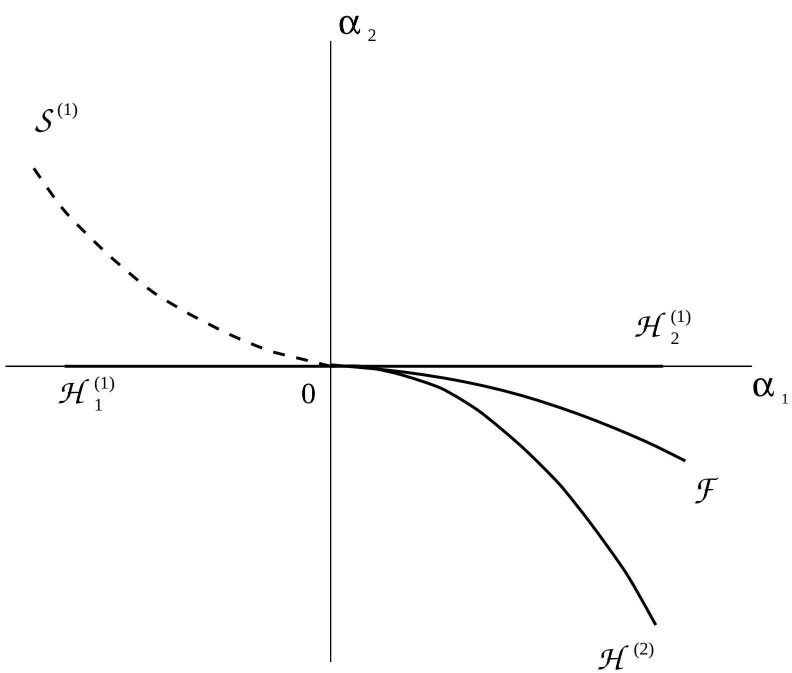


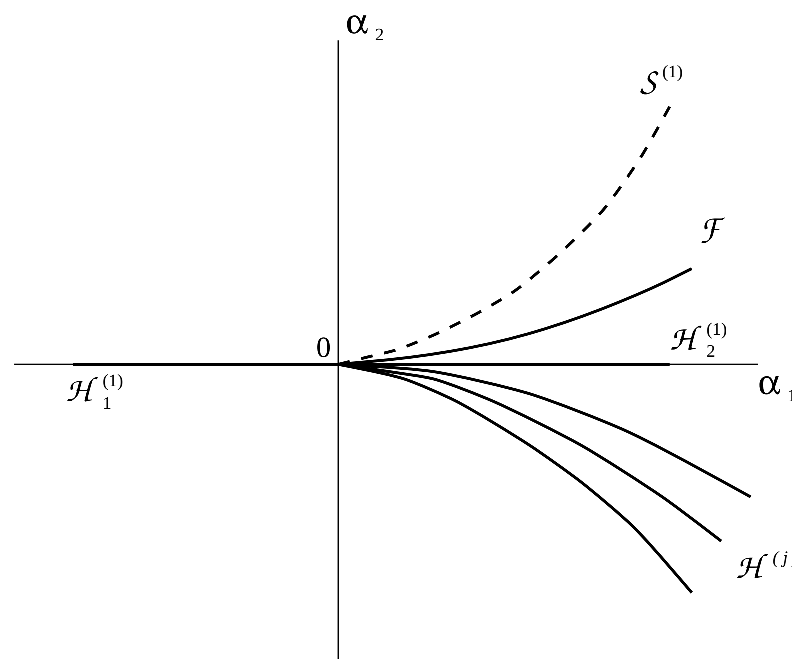
extensions to $\mathbb{R}^n$ if leading eigenvalues real.

Unfolding inclination and orbit flips

Three cases (all found in applications)

- No change
- homoclinic doubling (left plot)
- 'broom handle' (finite N-pulses $\forall N$) (right plot)

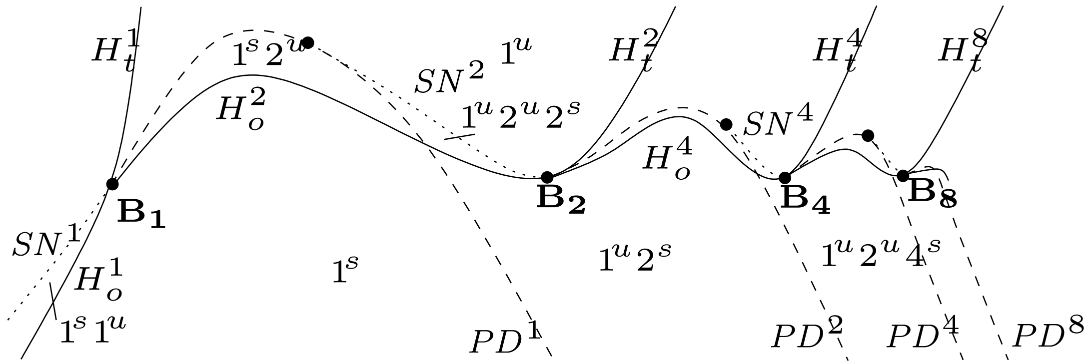




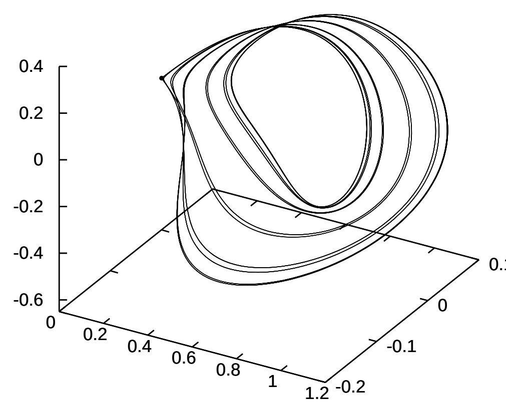
‘Death of period doubling’

Oldeman, Krauskopf, C. 2000

Accumulation of homoclinic doublings



Renormalisation $\Rightarrow$ different Feigenbaum constant



hom$_{32}$:

Other kinds of codimension-two homoclin

Not covered in these lectures

- Local birth of homoclinic orbits e.g.
  - from Takens-Bogdanov (implemented in MatCont!)
  - or Saddle-node Hopf (see e.g. C. & Kirk 2004)
- Forming heteroclinic cycle with other equilibrium (T-points)
- Forming a heteroclinic cycle with a periodic orbit (see e.g. C., Kirk et al 2009)
- . . .

3. Codim 2 homoclinics in HomCont

Concept: find a test function $\psi(x(t), \alpha)$

- Defined and smooth in a neighbourhood of the true curve of codim 1 homoclinic orbits $\mathcal{H}$ in function and parameter space.
- Has a regular zero in theory at the codim 2 point in question
- Has a regular zero for the truncated problem $\mathcal{H}_\mathcal{T}$ for sufficiently large $T > 0$
- its zero tends to the true one as $T \to \infty$
- can append $\psi = 0$ to continuation problem to continue codim 2 points in three pars.

3.1 Eigenvalue degeneracies

Compute eigenvalues of $A(x_0, \alpha) = D_x f(x_0, \alpha)$ and order according to real part.

Negative real part: $\mu_i, \ i = 1, 2, \ldots, n_s$

zero real part: $\gamma_j, \ j = 1, 2, \ldots, n_0$

positive real part: $\lambda_k, \ k = 1, 2, \ldots, n_u$

$$\text{Re } \mu_{n_s} \le \cdots \le \text{Re } \mu_1 < 0 < \text{Re } \lambda_1 \le \cdots \le \text{Re } \lambda_{n_u}.$$

real leading eigenvalue cases

Resonant eigenvalues :

$$\psi_1 = \mu_1 + \lambda_1$$

Double leading eigenvalues (node to focus transition):

$$\psi_2 = \begin{cases} (\mathrm{Re}\{\mu_1\} - \mathrm{Re}\{\mu_2\})^2, & \mathrm{Im}\{\mu_1\} = 0, \\ -(\mathrm{Im}\{\mu_1\} - \mathrm{Im}\{\mu_2\})^2, & \mathrm{Im}\{\mu_1\} \neq 0. \end{cases}$$

$$\psi_3 = \begin{cases} (\mathrm{Re}\{\lambda_1\} - \mathrm{Re}\{\lambda_2\})^2, & \mathrm{Im}\{\lambda_1\} = 0, \\ -(\mathrm{Im}\{\lambda_1\} - \mathrm{Im}\{\lambda_2\})^2, & \mathrm{Im}\{\lambda_1\} \neq 0. \end{cases}$$

Complex leading eigenvalue cases

neutral saddle-focus or bi-focus ($\delta = 1$ from lecture 1)

$$\psi_4 = \text{Re}\{\mu_1\} + \text{Re}\{\lambda_1\}.$$

neutrally divergent saddle-focus (stability change of dynamics)

$$\psi_5 = \text{Re}\{\mu_1\} + \text{Re}\{\mu_2\} + \text{Re}\{\lambda_1\},$$

$$\psi_6 = \text{Re}\{\lambda_1\} + \text{Re}\{\lambda_2\} + \text{Re}\{\mu_1\}.$$

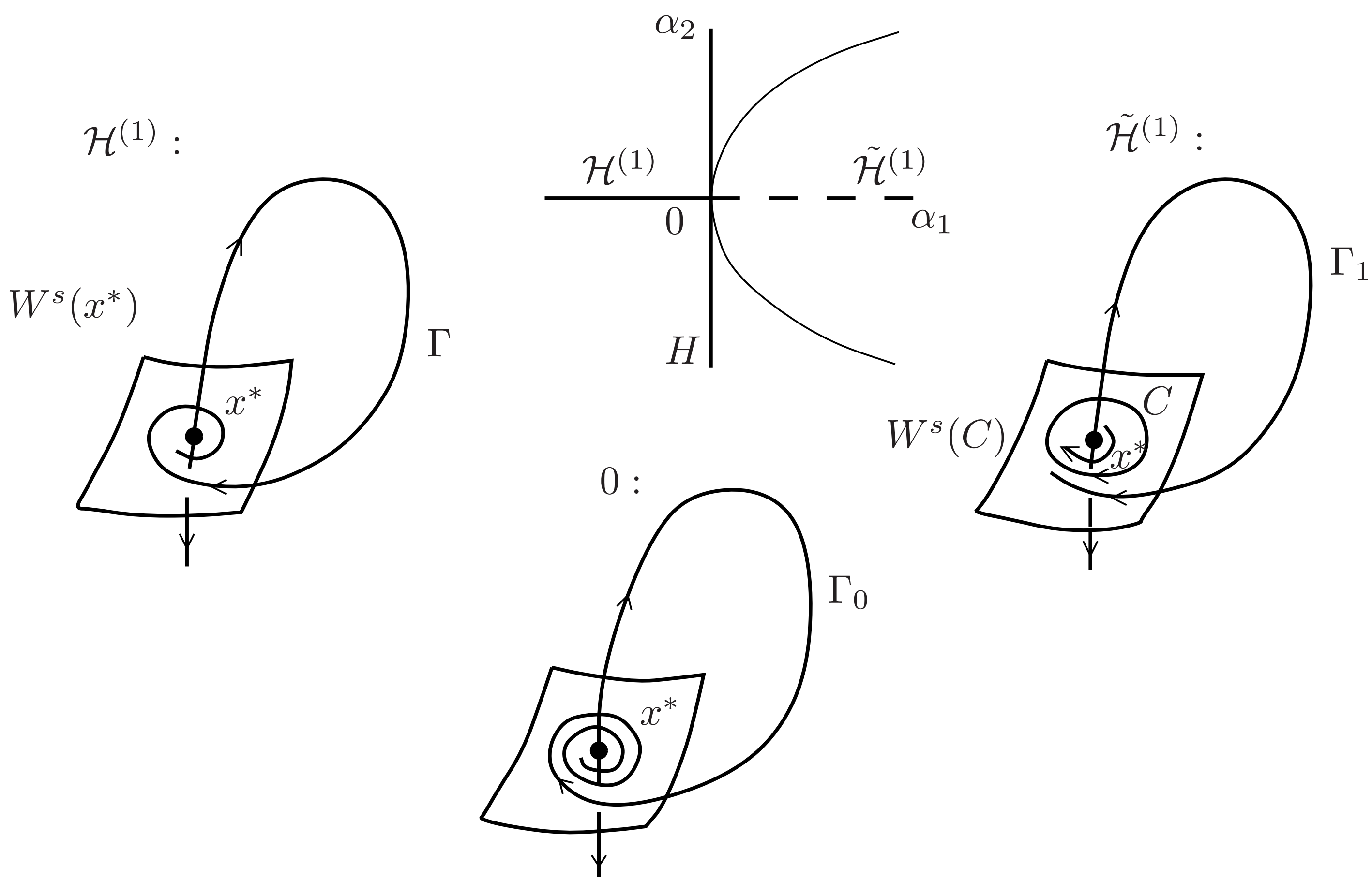
3.2 Non-hyperbolic equilibria

- Suppose that as we follow a homoclinic orbit along a path we find the equilibrium x_0 ceases to be hyperbolic.
- Relabel such that μ_i are the n_s leftmost eigenvalues and λ_i the n_u rightmost eigenvalues.
- Then a good test functions for either Hopf or fold bifurcation is

$$\psi_9 = \text{Re}\{\mu_1\}, \qquad \psi_{10} = \text{Re}\{\lambda_1\}$$

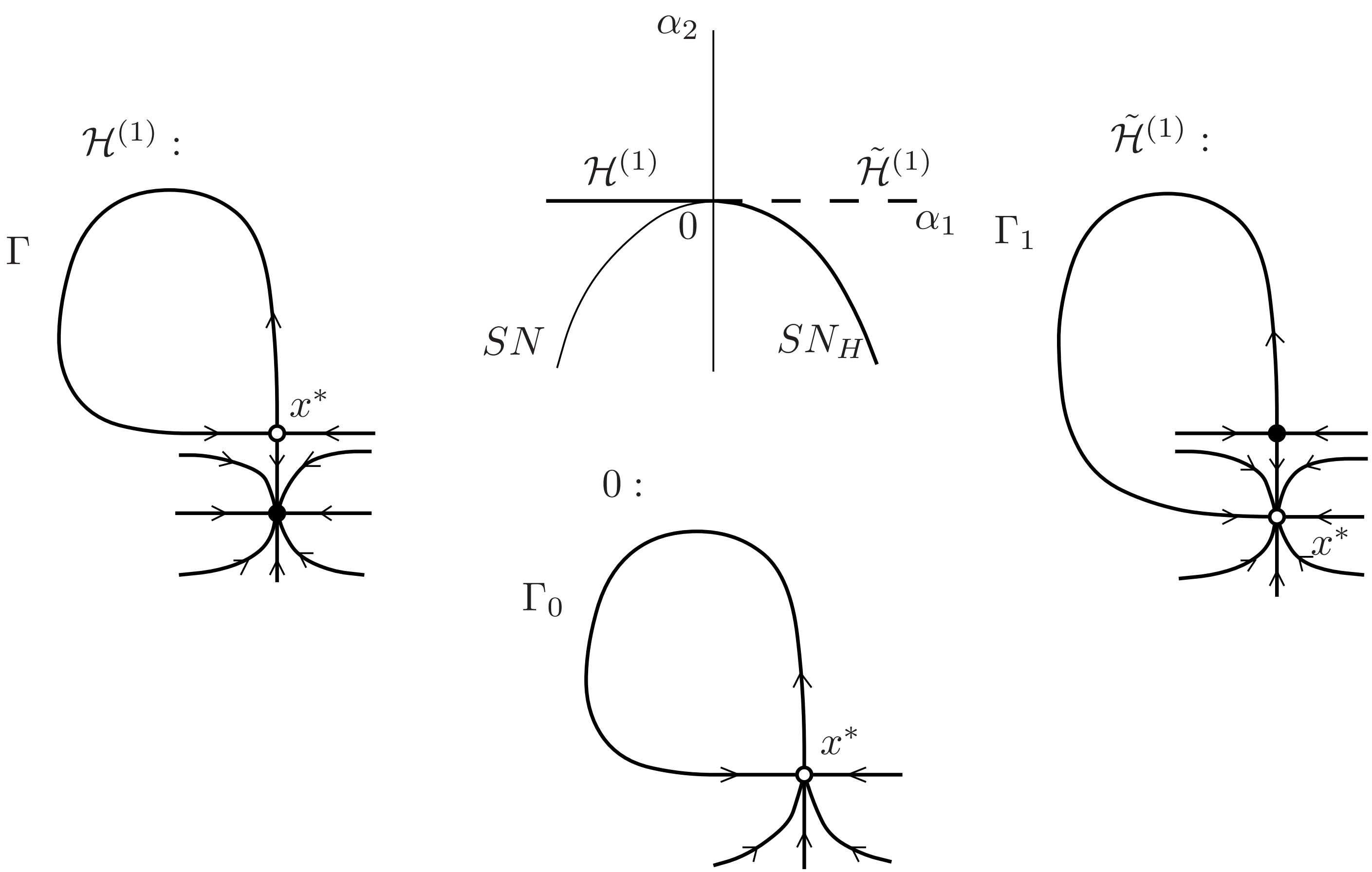
- Important to show truncated problem well-defined through the bifurcation . . .

Shil'nikov-Hopf bifurcation



Numerical problem approximates $\mathcal{H}^{(1)} \cup \tilde{\mathcal{H}}^{(1)}$:
$\mathcal{H}^{(1)}$ is curve of homoclinics
$\tilde{\mathcal{H}}^{(1)}$ is curve of point-periodic connections

Non-central saddle-node homoclinics



Truncated problem computes $\mathcal{H}^{(1)} \cup \tilde{\mathcal{H}}^{(1)}$:

$\tilde{\mathcal{H}}^{(1)}$ is a non-central heteroclininc orbit

In HomCont can switch branches to central saddle-node homoclinic SN_H at this point.

Continuation of saddle-node homoclinics

(central case - tangent to centre eigenspace as $t \to \pm\infty$)

- Consider equilibrium x_0 precisely at fold point with 1D centre manifold: $n_c = 1$, $n_u + n_s = n - 1$
- At fold point homoclinic is in intersection of W^{cs} and W^{cu} $\Rightarrow$ no extra codimension.
- The usual projection B.C.'s now give $n-1$ equations

$$L_s(x_0, \alpha)(x(0) - x_0) = 0, \qquad L_u(x_0, \alpha)(x(1) - x_0) = 0.$$

- + phase condition, but need constraint to stay on fold curve:

$$\det A(x_0, \alpha) = 0$$

3.3. Orientation flips

- Suppose leading eigenvalues are real $\mu_1 < 0 < \lambda_1$
- let $w_1^s(\alpha)$ and $w_1^u(\alpha)$ be normalised adjoint eigenvectors

$$A^{\mathrm{T}}(x_0,\alpha)\,w_1^s = \mu_1\,w_1^s \qquad A^{\mathrm{T}}(x_0,\alpha)\,w_1^u = \lambda_1\,w_1^u.$$

which are chosen to depend smoothly on α

- similarly let $v_1^s(\alpha)$ and $v_1^u(\alpha)$ be normalised eigenvectors

$$A(x_0,\alpha)\,v_1^s = \mu_1\,v_1^s \qquad A(x_0,\alpha)\,v_1^u = \lambda_1\,v_1^u.$$

chosen to vary smoothly with α

Orbit flip

- Generically homoclinic orbit $x(t) - x_0 \approx K v_1 e^{\mu t}$ as $t \to \infty$. Where

$$K = \lim_{t\to\infty} e^{-\mu_1 t} \langle w_1^s, x(t) - x_0 \rangle$$

- Orbit flip w.r.t. W^s occurs if $K = 0$ at codim two point.
- Similarly, orbit flip w.r.t. W^u occurs if

$$\lim_{t\to-\infty} e^{\lambda_1 t} \langle w_1^u, x(t) - x_0 \rangle = 0$$

- Therefore test functions for orbit flip:

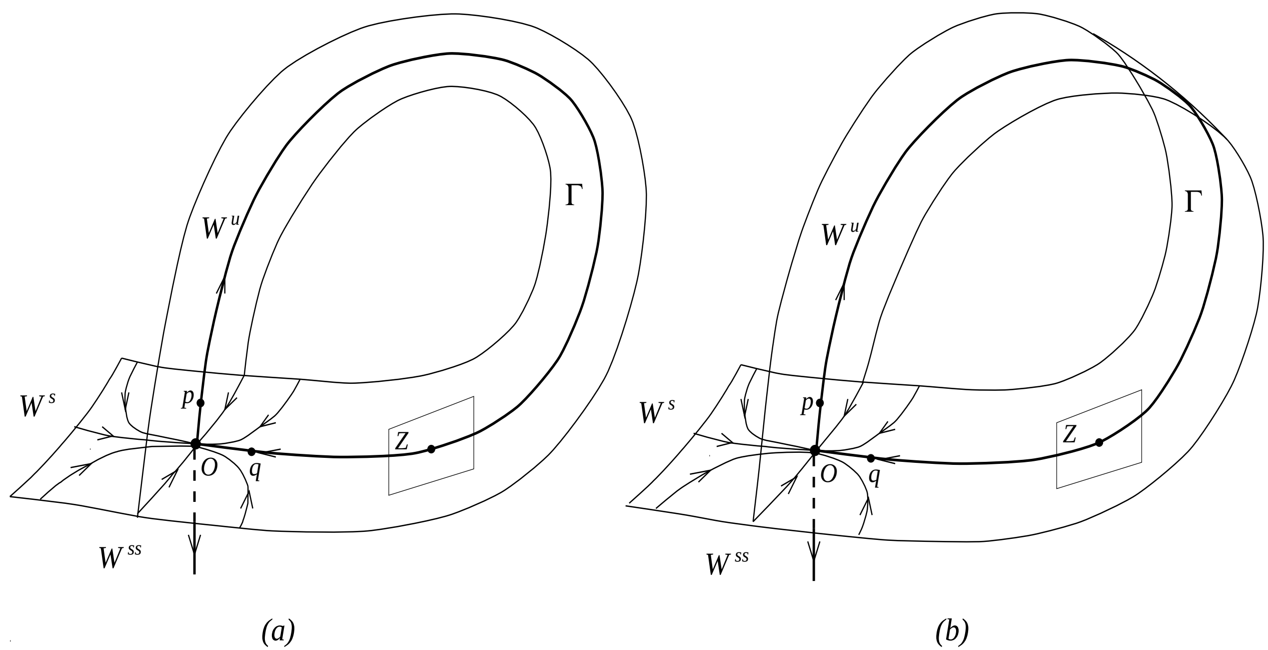
$$\begin{aligned} \psi_{11} &= e^{-\mu_1 T} \langle w_1^s, x(+T) - x_0 \rangle \\ \psi_{12} &= e^{\lambda_1 T} \langle w_1^u, x(-T) - x_0 \rangle \end{aligned}$$

Inclination flip

At each point $x(t)$ along homoclinic orbit define:

$$X(t) = T_{x(t)} W^s(x_0), \quad Y(t) = T_{x(t)} W^u(x_0)$$

Generically $X(t) \cap Y(t)$ is 1D $=$ span$\{\dot{x}(t)\}$.
The *twistedness* is the orientability of $Z(t) = X(t) + Y(t)$



At inclination flip twistedness changes without orbit flip

- Consider adjoint variational problem:

$$\dot{\varphi} = -(D_x f)^{\mathrm{T}}(x(t), \alpha)\,\varphi,$$

- Nondegenerate homoclinic $\Rightarrow$ unique (up to scale) bounded solution $\varphi(t) \to 0$ as $t \to \pm\infty$
- $\varphi(t)$ is normal vector to $Z(t)$.
- Generically, $\varphi(t) \approx K w_1^s e^{-\mu_1 t}$ as $t \to \infty$, where

$$K = \lim_{t \to -\infty} e^{\mu_1 t} \langle v_1^s, \varphi(t) \rangle$$

Inclination flip w.r.t. stable manifold occurs when $K = 0$ at codim 2 point along homoclinic branch.

How to compute $\varphi(t)$

- truncate to $[-T,T]$ and compute (along with $x(t)$)

$$\begin{aligned}
&\dot{\varphi} = -(D_x f)^{\mathrm{T}}(x(t),\alpha)\,\varphi + \varepsilon\, f(x(t),\alpha),\\
&P_s(x_0,\alpha)\,\varphi(+T) = 0,\\
&P_u(x_0,\alpha)\,\varphi(-T) = 0,\\
&\int_{-T}^{T} \tilde{\varphi}^{\mathrm{T}}(t)[\varphi(t) - \tilde{\varphi}(t)]dt = 0
\end{aligned}$$

- $P_{s,u}(x_0,\alpha) \in \mathbb{R}^{n_{s,u}\times n}$ forms basis of stable/unstable eigenspace of A.
- ε is regularising parameter ($= 0$ in theory) because of amplitude degeneracy. Amplitude fixed by 'phase condition'.
- $\Rightarrow n+1$ unknowns and $n+1$ side conditions.

- HomCont has flag `ITWIST=1` which turns on the computation of $\varphi(t)$
- Since the adjoint variational equations (with $\varepsilon = 0$) are linear, AUTO converges in one Newton step)
- Test functions for inclination flips:

$$\begin{aligned} \psi_{13} &= e^{-\mu_1 T} \langle v_1^s, \varphi(-T) \rangle \\ \psi_{14} &= e^{\lambda_1 T} \langle v_1^u, \varphi(+T) \rangle. \end{aligned}$$

- examples in demos `san` and `kpr`
- Finally, test function for non-central saddle-node homoclinic while continuing central saddle-node homoclinic

$$\psi_{15} = \frac{1}{T} \langle w_1, x(+T) - x_0 \rangle,$$

What we have learnt

- How to continue homo/heteroclinic orbits directly in AUTO
- Special cases for homoclinics to saddle-node equilibria
- Certain kinds of codimension-two homoclinic orbits
- How to detect and continue them in AUTO using HomCont
- Important in applications for tame to chaotic transitions and bifurcation of multi-pulses.

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