ON CLASSIFICATION OF SINGULAR MEASURES AND FRACTAL PROPERTIES OF QUASI-SELF-AFFINE MEASURES IN R^2

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Abstract. A multidimensional classification of singularly continuous (w.r.t. the Lebesgue measure) probability measures in R^2 is introduced and a theorem on canonical representation of such measures is proven. A class of random elements on the unit square which is defined by a system of partitions generated by the Q^*-representation of real numbers is introduced and studied in details. Conditions for the discreteness, absolute continuity resp. singular continuity (w.r.t. Lebesgue measure) of the corresponding probability measures are found. Metric, topological and fractal properties of the spectra of the corresponding probability distributions are investigated. A class of transformations preserving the Hausdorff-Besicovitch dimension of any subset of the unit square (DP-transformations) is also studied.

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1. Introduction

The class of probability measures in R^2 is rather rich and diverse. It is, in general, of a more complicated structure than the class of measures in R^1 (it is clear, e.g., that even the uniform probability distribution on a smooth curve is singularly continuous w.r.t. the Lebesgue measure on R^2). Their local geometrical properties are often hard to investigate even in the case of measures which are products of one-dimensional measures.

The study of geometrical properties of continuous probability measures has several aspects. One of them is the study of the Lebesgue structure of a measure, i.e. the content of the absolutely continuous and singular (with respect to the Lebesgue measure) components. Somewhat easier to study are pure measures (pure absolutely continuous and pure singular measures). The question of the purity of a measure and the description of the classes of pure measures is a separate problem. In the one dimensional case by the Jessen-Wintner theorem the sum of a series of random variables which is convergent with probability one with independent discrete terms is of pure type. A geometrical interpretation of this theorem allows to find some generalizations and analogous statements for the case of measures on R^2. In fact in this paper we present a two-dimensional analogue of this theorem for one class of random elements in R^2 and give criteria for the corresponding probability measures to be of pure type.

Singular continuous measures are quite specific in their topological and metric properties. One of the important characteristics of a measure is its spectrum (the smallest closed support of a measure). There exists a classification of singular continuous measures on R^1 according to the properties of their spectrum. By that classification there are three pure types of measures and all others are mixtures of those three. A measure supported by the union of intervals containing in the spectrum is of pure GS-type. A measure is of pure GC-type if it is supported by a nowhere dense set, every point of which has a neighborhood whose intersection with the spectrum is of zero Lebesgue measure. If a measure is supported by a nowhere dense set E, and the spectrum has an intersection of positive Lebesgue measure with every neighborhood of any point from E, then such a measure is of pure GP-type. In this paper we generalize this classification to measures on R^2 and prove that every singularly continuous measure is a linear combination of the above mentioned tree types.

Another problem of the study of the geometry of singularly continuous probability measures are the fractal properties of measures (their spectra and other important supports). The problem of computation of the Hausdorff dimension of subsets of R^2 is much more complicated than for subsets of R^1. There is indeed a growing amount of papers appearing on the subject which solve the problem in specific cases. Let us mention few achievement in this study.

The first approach is to compute the Hausdorff dimension of the sets generated by affine transformation of some particular type. McMullen [21] (and independently Bedford [7]) gave an explicit formula for the dimension of self-affine sets generated by quite a narrow set of affinities. Later in the works Gatzouras and Lalley [20] and Baranski [6] there where given generalizations to wider classes of sets. The formulas in these papers seem however hard to use in practise because they require the finding of the maxima of some expressions over an infinite set of probability vectors. In the work by Yuval Peres [22] it was shown that for a 'typical' McMullen's carpet the Hausdorff measure of the order of its dimension is infinite. In several sequential papers Peres with collaborators generalized the results to higher dimensions and different notions of dimension. There are also several works which made some progress in the computation of the dimension for self-affine sets which are graphs of continuous functions [8, 18]

Another approach is devoted to the computation of the dimension for a self-affine set generated by a 'typical' family of affinities, and give an 'almost sure' value of the dimension, although it seems that the expressions are also very hard to use in practice.

A transformation of the space R^2 (bijective mapping of the space onto itself) is said to be a DP-transformation (dimension preserving Hausdorff transformation) if for any set $E \subset R^2$ its dimension coincides with the dimension of its image $E' = f(E)$.

In the one dimensional case the authors of the present paper had already got some results for DR-transformations which we use in this paper and obtain analogies in two dimensions. A set is called quasi-self-affine if there exists a DP-transformation which sends it to a self-affine set.

In this paper we consider the problems mentioned above for a class of probability measures in R^2, mainly singularly continuous with respect to the two-dimensional Lebesgue measure.

2. Q^*-REPRESENTATION OF REAL NUMBERS AND THE HAUSDORFF DIMENSION

2.1. Q^*-representation of real numbers.

Let s be a fixed integer, $s \geq 2$, and let $Q^* = \|q_{ik}\|$ be a fixed infinite matrix with the following properties:

$$q_{ik} > 0, i \in A = \{0, 1, \ldots, s-1\}, k = 1, 2, \ldots, \tag{1}$$

$$q_{0k} + q_{1k} + \ldots + q_{s-1k} = 1, \tag{2}$$

$$\prod_{k=1}^{\infty} q_{i_k k} = 0 \tag{3}$$

for any sequence $\{i_k\}$, $i_k \in A$,

$$q_* = \inf_{ik}\{q_{ik}\} > 0. \tag{4}$$

It is easy to prove (see, e.g., [4], [23]) that for any $x \in [0,1]$ there exists a sequence $\{\alpha_k\}$, $\alpha_k \in A$ such that

$$x = \beta_{\alpha_1 1} + \sum_{k=2}^{\infty}\left(\beta_{\alpha_k k}\prod_{i=1}^{k-1} q_{\alpha_i i}\right), \tag{5}$$

where $\beta_{0k} = 0$, $\beta_{\alpha_k k} = \sum\limits_{j=0}^{\alpha_k - 1} q_{jk}$, $k = 1, 2, \ldots$. For any sequence $\{\alpha_k\}$, $\alpha_k \in A$ the series (5) converges and the sum belongs to $[0,1]$.

The representation of the real number x in the form (5) is said to be the Q^*-representation of x, and we shall use the notation

$$x = \Delta^{Q^*}_{\alpha_1\ldots\alpha_k\ldots}. \tag{6}$$

The number $\alpha_k = \alpha_k(x)$ is said to be the k-th Q^*-symbol of x.

Let $\Delta^{Q^*}_{c_1\ldots c_m}$ be the set of real numbers x such that $\alpha_1(x) = c_1, \ldots, \alpha_m(x) = c_m$. It is easy to see that $\Delta^{Q^*}_{c_1\ldots c_m}$ is a closed interval. This interval is said to be a cylindrical set (cylinder) of rank m with the base $c_1, c_2, \ldots, c_m$. The set of all cylindrical sets of all ranks for a given matrix Q^* will be denoted by V_{Q^*}. Let $\nabla^{Q^*}_{c_1\ldots c_m}$ be the interior of $\Delta^{Q^*}_{c_1\ldots c_m}$.

The following properties of cylindrical sets are obvious:

(1) $|\Delta^{Q^*}_{c_1\ldots c_m}| = \prod\limits_{i=1}^{m} q_{c_i i}$.

(2) $\Delta^{Q^*}_{c_1\ldots c_m} = \bigcup\limits_{c=0}^{s-1} \Delta^{Q^*}_{c_1\ldots c_m c}$,
$\nabla^{Q^*}_{c_1\ldots c_m c} \bigcap \nabla^{Q^*}_{c_1\ldots c_m c'} = \varnothing$, iff $c \neq c'$.

(3) $\bigcap\limits_{m=1}^{\infty} \Delta^{Q^*}_{c_1\ldots c_m} = x \equiv \Delta^{Q^*}_{c_1\ldots c_m\ldots}$.

There exists a countable set of real numbers having two different Q^*-expansions

$$x = \Delta^{Q^*}_{\alpha_1\ldots\alpha_{k-1}\alpha_k 0\ldots 0\ldots} = \Delta^{Q^*}_{\alpha_1\ldots\alpha_{k-1}(\alpha_k - 1)(s-1)\ldots(s-1)\ldots} = x'.$$

We shall call them Q^*-rational. All other numbers have a unique Q^*-representation.

If $q_{ik} = q_i$ for all $k \in N$ (i.e., if all columns of the matrix $Q^* = \|q_{ik}\|$ are the same) then the Q^*-representation is called the Q-representation. It is easy to see that the Q^*-expansion is also a generalization of the s-adic expansion.

Lemma 1. *Let* $q_* = \inf\{q_{ik}\} > 0$, $q_{**} = \sup\{q_{ik}\}$, *and let* γ *be the smallest integer such that*

$$q_{**}^{\gamma} \leq q_*. \tag{7}$$

For any interval $u \in [0,1]$ *there exists at most*

$$l = 2(s^{\gamma} + s - 1)$$

cylinders from V_{Q^*} *which covers* u *and whose lengths are at most* $|u|$.

Proof. Let k be the smallest number such that u contains a cylinder of rank k and u does not contain any whole cylinder of rank $k-1$. Then there exist cylinders v_0 and v_1 from $V^{k-1}_{Q^*}$ such that $u \subset v_0 \bigcup v_1$, $\sup v_0 = \inf v_1$. Let us set $u_0 = [\inf u, \sup v_0]$, $u_1 = [\sup v_0, \sup u]$.

Firstly, let us consider the closed interval u_1 (it is clear that $|u_1| < |u|$).

Let $v_1 = v_{10} \bigcup v_{11} \bigcup \ldots \bigcup v_{1(s-1)}$, where $v_{1i} \in V^k_{Q^*}$ and $\sup v_{1i} = \inf v_{1(i+1)}$, and let

$$j = \min \left\{ m : \sum_{i=0}^{m} |v_{0i}| \geq |u_1| \right\}.$$

It is clear that $j \leq s-1$. Let us consider all possible cases.

I. If $|v_{1j}| \leq |u|$, then $j+1$ cylinders $v_{10}, v_{11}, ..., v_{1j}$ cover u_1 and their lengths do not exceed $|u|$.

II. Now let us consider the case where $|v_{1j}| > |u|$.

II.1 If $j > 0$, then we consider all s^γ cylinders l_i of rank $k+\gamma$ contained in v_{1j}. It is obvious that

$$|l_i| \leq q^\gamma_{**}|v_{1j}| \leq q_*|v_{1j}| \leq q_*|v_1| \leq |v_{10}| \leq |u_1| \leq |u|.$$

Hence, the cylinders $v_{10}, v_{11}, ..., v_{1(j-1)}$ and the above defined s^γ cylinders from $V^{k+\gamma}_{Q^*}$ cover u_1 and their lengths do not exceed $|u|$. The total number of such cylinders is less then $s^\gamma + s$.

II.2 Finally, let us consider the case where $j = 0$. It is clear that in such a case $|v_{10}| > |u_1|$. Let n be the minimal positive integer such that there exists a cylinder of rank $k+n$, which completely belongs to u_1. Then there exists a cylinder $l \in V^{k+n-1}_{Q^*}$ such that $u_1 \subset l$. Let $l = l_0 \cup l_1 \ldots \cup l_{s-1}$, $l_i \in V^{k+n}_{Q^*}$, $\sup l_i = \inf l_{i+1}$ and let

$j^* = \min \left\{ m : \sum_{i=0}^{m} |l_i| \geq |u_1| \right\}$. From the definition of the number n it follows that $l_0 \subset l$. Therefore, $j^* > 0$ and all further considerations are the same as in the case **II.1.**

So, in all cases for the interval u_1 there exist at most $s^\gamma + s - 1$ cylinders from V_{Q^*} which cover u and whose lengths do not exceed $|u|$.

In the completely similar way we can apply the above described covering procedure for the interval u_0. Therefore, the interval $u = u_0 \bigcup u_1$ can be covered by at most $2(s^\gamma + s - 1)$ intervals whose lengths do not exceed $|u|$. ☐

2.2. Q^***-representation and the Hausdorff dimension.** Let $M \subset R^n$ be a fixed bounded subset. A family Φ_M of **subsets of** R^n is said to be a *fine (Vitali) covering family* for M if for any subset $E \subset M$, and for any $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_j\}$ of E, $E_j \in \Phi_M$, i.e., $\forall E \subset M$, $\forall \varepsilon > 0$ $\exists \{E_j\}$ $(E_j \in \Phi_M,\ |E_j| \leq \varepsilon)$: $E \subset \bigcup_j E_j$, where $|E_j|$ denotes the diameter of E_j.

For any subset $E \subset M$ and for any $\alpha > 0$ and $\varepsilon > 0$ one can define

$$m^\alpha_\varepsilon(E, \Phi_M) = \inf_{|E_j| \leq \varepsilon} \left\{ \sum_j |E_j|^\alpha \right\},$$

where the infimum is taken over all at most countable ε-coverings of the subset E, $E_j \in \Phi_M$.

The number

$$H^\alpha(E, \Phi_M) = \lim_{\varepsilon \to 0} m^\alpha_\varepsilon(E, \Phi_M) = \sup_{\varepsilon > 0} m^\alpha_\varepsilon(E, \Phi_M) \tag{8}$$

is said to be the α-dimensional Hausdorff measure of the subset E w.r.t. a given family of coverings Φ_M.

Definition. *The non-negative number*

$$\alpha_0(E, \Phi_M) = \inf\{\alpha : \quad H^\alpha(E, \Phi_M) = 0\} \tag{9}$$

is said to be the Hausdorff (Hausdorff-Besicovitch) dimension of $E \subset M$ *w.r.t. a given family of coverings* Φ_M.

If $M = [0,1]$ and Φ_M is the family of all subsets of $[0,1]$, or Φ_M coincides with the family of all closed subintervals of $[0,1]$, then $\alpha_0(E, \Phi_M)$ coincides with the classical Hausdorff dimension $\alpha_0(E)$ of $E \subset [0,1]$.

Definition. *A fine covering family Φ_M is said to be* $\underline{\textit{fractal}}$ *if for the determination of the Hausdorff dimension of any subset $E \subset M$ it is enough to consider only coverings from Φ_M, i.e., if*

$$\alpha_0(E, \Phi_M) = \alpha_0(E).$$

It is well known (see, e.g., ([9]), that the family of all s-adic cylinders of the unit interval is fractal. A similar assertion is true for the family V_{Q^*} of cylinders generating by the Q^*-partitions.

Theorem 1. *For the α-dimensional Hausdorff measure H^α and α-dimensional measure $H^\alpha(E, V_{Q^*})$ the following inequality holds:*

$$H^\alpha(E) \leq H^\alpha(E, V_{Q^*}) \leq l \cdot H^\alpha(E),\ \forall E \subset [0,1], \tag{8}$$

where $l = 2(s^\gamma + s - 1)$, $q_{**}^\gamma \leq q_* = \inf\{q_{ik}\} > 0$.

Proof. The left part of the inequality (8) is obvious. So, it is enough to prove the right part. According to lemma 1, for any ε-covering $\{u_k\}$ of the set E there exists an ε-covering $\{v_i^k\}$, $\{v_i^k\} \in V_{Q^*}$, $i = \overline{1, m_k}$, $m_k \leq 2(s^\gamma + s - 1)$, such that $|v_i^k| \leq |u_k|$ for any $i \in \{1, 2, \dots, m_k\}$. Hence

$$\sum |v_i^k|^\alpha = \sum_k \sum_{i=1}^{m_k} |v_i^k|^\alpha \leq 2(s^\gamma + s - 1) \sum |u_k|^\alpha$$

and

$$m_\varepsilon^\alpha(E) \leq m_\varepsilon^\alpha(E, V_{Q^*}) \leq l \cdot m_\varepsilon^\alpha(E).$$

Taking the limit as $\varepsilon \to 0$, we get inequalities (8). □

Corollary 1. *The family V_{Q^*} is fractal, i.e., $\alpha_0(E, V_{Q^*}) = \alpha_0(E)$ for any subset E of the unit interval.*

2.3. Q^*-representation and the Hausdorff-Billingsley dimension.

Let M be a fixed subset of R^n, let Φ_M be a fine covering family for M, let α be a positive number and let ν be a continuous probability measure. The ν-α-Hausdorff–Billingsley measure of a subset $E \subset M$ w.r.t. Φ_M is defined as follows:

$$H_\alpha(E, \nu, \Phi_M) = \lim_{\varepsilon \to 0} \left\{ \inf_{\nu(E_j) \leq \varepsilon} \sum_j (\nu(E_j))^\alpha \right\},$$

where $E_j \in \Phi_M$ and $\bigcup_j E_j \supset E$.

Definition. *The non-negative number $\alpha_\nu(E, \Phi_M) = \inf\{\alpha : H_\alpha(E, \nu, \Phi_M) = 0\}$ is said to be the* Hausdorff–Billingsley dimension of the set E with respect to the measure ν and the family of coverings Φ_M.

Remark. *1) If M is a bounded subset of the real line and if Φ_M is the family of all closed subintervals of the minimal closed interval $[a, b]$ containing M, then for any $E \subset M$ the number $\alpha_\nu(E, \Phi_M)$ coincides with the classical Hausdorff–Billingsley dimension $\alpha_\nu(E)$ of the subset E w.r.t. the measure ν.*

2) If $M = [0,1]$, ν is the Lebesgue measure on $[0,1]$ and Φ_M is a fractal family of coverings, then for any $E \subset M$ the number $\alpha_\nu(E, \Phi_M)$ coincides with the classical Hausdorff dimension $\alpha_0(E)$ of the subset E.

Let $M \subset R^1$ and let Φ_M^ν be the image of a fine covering family under the distribution function of a one-dimensional probability measure ν, i.e., $\Phi_M^\nu = \{E' : E' = F_\nu(E), E \in \Phi_M\}$.

Lemma 2. *A fine covering family Φ_M can be used for the equivalent definition of the Hausdorff–Billingsley dimension w.r.t. a measure ν of any subset $E \subset M$ if and only if the covering family Φ^{ν}_M can be used for the equivalent definition of the classical Hausdorff dimension of any subset $E' = F_\nu(E)$, $E \subset M$, i.e.,*

$$\alpha_\nu(E, \Phi_M) = \alpha_\nu(E) \quad \text{for any } E \subset M$$

if and only if the covering family Φ^{ν}_M is fractal.

Proof. Because of the countable stability of the above mentioned dimensions, without loss of generality we may assume that the set M is bounded. Let Φ be the family of all closed subintervals from the minimal closed interval $[a, b]$ containing M and let Φ_M be a fine covering family. Since $\nu(E_j) = |E'_j|$ and $\bigcup_j E'_j \supset E'$, we have

$$H^\alpha(E, \nu, \Phi_M) = \lim_{\varepsilon \to 0} \left\{ \inf_{\nu(E_j) \le \varepsilon} \sum_j (\nu(E_j))^\alpha \right\} =$$

$$= \lim_{\varepsilon \to 0} \left\{ \inf_{|E'_j| \le \varepsilon} \sum_j |E'_j|^\alpha \right\} = H^\alpha(E', \Phi^{\nu}_M).$$

Therefore,

$$\alpha_\nu(E, \Phi_M) = \alpha_0(E', \Phi^{\nu}_M).$$

The family $\Phi^\nu = F_\nu(\Phi)$ coincides with the family of all closed subintervals of $[a', b'] \supset M'$. Therefore,

$$H^\alpha(E, \nu) = H^\alpha(E, \nu, \Phi) = H^\alpha(E', \Phi^\nu),$$

and

$$\alpha_\nu(E) = \alpha_\nu(E, \Phi) = \alpha_0(E', \Phi^\nu) = \alpha_0(E').$$

If the family Φ^{ν}_M is fractal, then $\alpha_\nu(E, \Phi_M) = \alpha_0(E', \Phi^{\nu}_M) = \alpha_0(E') = \alpha_\nu(E)$.

If the family Φ^{ν}_M is not fractal, then there exists a subset $E' \subset M'$ such that $\alpha_0(E') < \alpha_0(E', \Phi^{\nu}_M)$. In such a case we have

$$\alpha_\nu(E) = \alpha_0(E') < \alpha_0(E', \Phi^{\nu}_M) = \alpha_\nu(E, \Phi_M),$$

which proves the Lemma. □

The following theorem can easily be proven by using arguments which are quite similar to the Billingsley arguments in [11].

Theorem 2. *Let μ_1 and μ_2 be continuous probability measures on Borel subsets of $[0, 1]$, and let $\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)}$ be a cylinder of rank k containing the point x.*

If

$$E \subset E_0 = \left\{ x : \lim_{k \to \infty} \frac{\ln \mu_1(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \mu_2(\Delta^{Q^*}_{\alpha_1(x)\ldots\alpha_k(x)})} = \delta \right\},$$

then

$$\alpha_{\mu_2}(E, V_{Q^*}) = \delta \alpha_{\mu_1}(E, V_{Q^*}).$$

Corollary 2. *If μ_1 is a probability measure with $\mu_1(E) > 0$ and μ_2 is the Lebesgue measure, then $\alpha_0(E) = \alpha_{\mu_2}(E) = \delta$.*

Let us consider an example of the application of the latter theorem. For a given stochastic vector $(\tau_0, \ldots, \tau_{s-1})$, let $M \equiv M[Q; \tau_0, \ldots, \tau_{s-1}]$ be the set of all real numbers from $[0, 1]$ with the prescribed asymptotic frequencies of their digits $0, 1 \ldots, s-1$ in the Q-representation, i.e.,

$$M[Q; \tau_0, \ldots, \tau_{s-1}] = \{x : x = \Delta^{Q}_{\alpha_1(x)\ldots\alpha_k(x)\ldots},$$

$$\lim_{k\to\infty}\frac{N_i(x,k)}{k}\equiv \nu_i(x)=\tau_i, i=\overline{0,s-1}\},$$

where $N_i(x,k)$ is the number of symbols i in the Q-representation of x until the k-th position.

Let us show that

$$\alpha_0(M[Q,\tau_0,\dots,\tau_{s-1}])=\frac{\ln\tau_0^{\tau_0}\tau_1^{\tau_1}\dots\tau_{s-1}^{\tau_{s-1}}}{\ln q_0^{\tau_0}q_1^{\tau_1}\dots q_{s-1}^{\tau_{s-1}}}$$

Let μ_2 be the Lebesgue measure and let μ_1 be the probability measure on Borel subsets of $[0,1]$ with respect to which $\{\alpha_i(x)\}$ is a sequence of independent random variables with distributions $\mu_1\{x:\alpha_j(x)=i\}=\tau_i,\quad i=0,\dots,s-1$. Then

$$\mu_1(\Delta^Q_{\alpha_1(x)\dots\alpha_k(x)})=\tau_0^{N_0(x,k)}\tau_1^{N_1(x,k)}\dots\tau_{s-1}^{N_{s-1}(x,k)}=\left(\prod_{i=0}^{s-1}\tau_i^{\frac{N_i(x,k)}{k}}\right)^{1/k},$$

$$\mu_2(\Delta^Q_{\alpha_1(x)\dots\alpha_k(x)})=q_0^{N_0(x,k)}q_1^{N_1(x,k)}\dots q_{s-1}^{N_{s-1}(x,k)}=\left(\prod_{i=0}^{s-1}q_i^{\frac{N_i(x,k)}{k}}\right)^{1/k},$$

and the following limit exists

$$\lim_{k\to\infty}\frac{\ln\mu_1(\Delta^{Q^*}_{\alpha_1(x)\dots\alpha_k(x)})}{\ln\mu_2(\Delta^{Q^*}_{\alpha_1(x)\dots\alpha_k(x)})}=\frac{\ln\tau_0^{\tau_0}\tau_1^{\tau_1}\dots\tau_{s-1}^{\tau_{s-1}}}{\ln q_0^{\tau_0}q_1^{\tau_1}\dots q_{s-1}^{\tau_{s-1}}}\equiv\delta.$$

Since $\mu_1(M[Q,\tau_0,\dots,\tau_{s-1}])=1$, we have $\alpha_{\mu_1}(M)=1$, and , in according to Theorem 1 and Theorem 2, we get:

$$\alpha_0(M,V_{Q^*})=\alpha_0(M)=\delta=\frac{\ln\tau_0^{\tau_0}\tau_1^{\tau_1}\dots\tau_{s-1}^{\tau_{s-1}}}{\ln q_0^{\tau_0}q_1^{\tau_1}\dots q_{s-1}^{\tau_{s-1}}}.$$

2.4. **Q^*-representation and transformations of $[0,1]$ preserving the Hausdorff dimension.** A transformation f of R^n (in the sense of a bijective mapping of the space into itself) is said to be a DP-transformation (dimension preserving transformation) if the condition

$$\alpha_0(E)=\alpha_0(f(E))$$

holds for any $E\subset R^n$.

Let $Q_1^*=\|q_{ik}\|$ and $Q_2^*=\|q'_{jk}\|$ be fixed matrices which satisfy conditions (1) - (4).

Theorem 3. *If for any $x\in[0,1]$ the following equality*

$$\lim_{k\to\infty}\frac{\ln|\Delta^{Q_2^*}_{\alpha_1(x)\dots\alpha_k(x)}|}{\ln|\Delta^{Q_1^*}_{\alpha_1(x)\dots\alpha_k(x)}|}=1$$

holds, then the function F defined by

$$F(x)=F\left(\Delta^{Q_1^*}_{\alpha_1(x)\dots\alpha_k(x)\dots}\right)=\Delta^{Q_2^*}_{\alpha_1(x)\dots\alpha_k(x)\dots}\tag{9}$$

is a strictly increasing probability distribution function which preserves the Hausdorff dimension.

Proof. Let us consider probability measures ν and μ such that for any cylindrical set $\Delta^{Q_1^*}_{\alpha_1\dots\alpha_k}$ we have:

$$\nu\left(\Delta^{Q_1^*}_{\alpha_1\dots\alpha_k}\right)=|\Delta^{Q_1^*}_{\alpha_1\dots\alpha_k}|$$

and

$$\mu\left(\Delta^{Q_1^*}_{\alpha_1\dots\alpha_k}\right)=|\Delta^{Q_2^*}_{\alpha_1\dots\alpha_k}|.$$

(See, e.g., [4] for the method of construction of such measures). It is clear that ν coincides with the Lebesgue measure on $[0,1]$. Then for any $x \in [0,1]$

$$\lim_{k\to\infty} \frac{\ln |\Delta^{Q_2^*}_{\alpha_1(x)\ldots\alpha_k(x)}|}{\ln |\Delta^{Q_1^*}_{\alpha_1(x)\ldots\alpha_k(x)}|} = \lim_{k\to\infty} \frac{\ln \mu(\Delta^{Q_1^*}_{\alpha_1(x)\ldots\alpha_k(x)})}{\ln \nu(\Delta^{Q_1^*}_{\alpha_1(x)\ldots\alpha_k(x)})} = 1,$$

and from Theorem 2 it follows that $\alpha_\mu(E, V_{Q_1^*}) = \alpha_\nu(E, V_{Q_1^*})\ \forall E \subset [0,1]$. Since the families $V_{Q_1^*}$ and $V_{Q_2^*}$ of coverings are fractal, and $F_\mu(V_{Q_1^*}) = V_{Q_2^*}$, we have:

$$\alpha_\nu(E, V_{Q_1^*}) = \alpha_0(E, V_{Q_1^*}) = \alpha_0(E);$$

and

$$\alpha_\mu(E, V_{Q_1^*}) = \alpha_0(f(E), V_{Q_2^*}) = \alpha_0(f(E)).$$

Hence $\alpha_0(E) = \alpha_0(f(E))$. □

Let now $Q_1^* = Q = \|q_{ik}\|$ with $q_{ik} = q_i$ (i.e., all columns of the matrix Q_1^* are the same). In such a case the above defined function F coincides with the probability distribution function of the random variable

$$\xi = \Delta^Q_{\eta_1\ldots\eta_k\ldots}$$

with independent symbols η_k of its Q-representation, $P\{\eta_k = i\} = q'_{ik} = p_{ik}$, $i \in A = \{0, \ldots, s-1\}$, $k \in N$.

It is known (see, e.g., [4, 23]) that the distribution of the random variable ξ is:

1. absolutely continuous (w.r.t. the Lebesgue measure) iff
$$L = \sum_{k=1}^{\infty} \sum_{i=0}^{s-1} (1 - q_i^{-1} p_{ik})^2 < \infty;$$
2. singularly continuous (w.r.t. the Lebesgue measure) iff $L = +\infty$ and
$$M = \prod_{k=1}^{\infty} \max_i \{p_{ik}\} = 0.$$

Theorem 4. *If*

$$p_{ik} \to q_i\ (k \to \infty),\ i = \overline{0, s-1},$$

then the distribution function F_ξ of the random variable ξ preserves the Hausdorff dimension of all subsets of the unit interval.

Proof. From the definition of the random variable ξ with independent $Q-$symbols it follows that $P_\xi(\Delta^{Q_1^*}_{\alpha_1\alpha_2\ldots\alpha_k}) = p_{\alpha_1 1} p_{\alpha_2 2} \ldots p_{\alpha_k k} = |\Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_k}|$ for any sequence $\{\alpha_n\}$. So,

$$F_\xi(\Delta^{Q_1^*}_{\alpha_1\alpha_2\ldots\alpha_k}) = \Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_k}.$$

The following equalities of limits hold:

$$\lim_{k\to\infty} \frac{\ln |\Delta^{Q_2^*}_{\alpha_1\alpha_2\ldots\alpha_k}|}{\ln |\Delta^{Q_1^*}_{\alpha_1\alpha_2\ldots\alpha_k}|} = \lim_{k\to\infty} \frac{\ln p_{\alpha_1 1} p_{\alpha_2 2} \ldots p_{\alpha_k k}}{\ln q_{\alpha_1}\ q_{\alpha_2}\ \ldots\ q_{\alpha_k}} = \lim_{k\to\infty} \frac{\sum\limits_{j=1}^{k} \ln p_{\alpha_j j}}{\sum\limits_{j=1}^{k} \ln q_{\alpha_j}} =$$

$$= \lim_{k\to\infty} \frac{\sum\limits_{j=1}^{k} \ln \left(q_{\alpha_j} \cdot \frac{p_{\alpha_j j}}{q_{\alpha_j}} \right)}{\sum\limits_{j=1}^{k} \ln q_{\alpha_j}} = 1 + \lim_{k\to\infty} \frac{\sum\limits_{j=1}^{k} \ln \frac{p_{\alpha_j j}}{q_{\alpha_j}}}{\sum\limits_{j=1}^{k} \ln q_{\alpha_j}} = 1 + \lim_{k\to\infty} \frac{\frac{1}{k}\sum\limits_{j=1}^{k} \ln \frac{p_{\alpha_j j}}{q_{\alpha_j}}}{\frac{1}{k}\sum\limits_{j=1}^{k} \ln q_{\alpha_j}} = 1$$

for any sequence $\{\alpha_n\}$, because $\lim\limits_{j\to\infty} \frac{p_{\alpha_j j}}{q_{\alpha_j}} = 1$, and $q_i \leq 1 - c_0$ for some positive constant c_0.

So, from Theorem 3 it follows that the distribution function F_ξ preserves the Hausdorff dimension of every subset of the unit interval. □

Remark 1. All strictly increasing *absolutely continuous* distribution functions of random variables with independent $Q-$symbols are DP-functions.

Remark 2. There exist strictly increasing absolutely continuous distribution functions F_ξ of random variables with independent $Q-$symbols such that both the set $N_0(F) = \{x : F'_\xi(x) = 0\}$ and the set $N_\infty(F) = \{x : \underline{\lim}_{\varepsilon\to 0} \frac{F_\xi(x+\varepsilon) - F_\xi(x-\varepsilon)}{\varepsilon} = +\infty\}$ are everywhere dense sets of full Hausdorff dimension (see, e.g. [1]).

Remark 3. There exist strictly increasing *singularly continuous* distribution functions which preserve the Hausdorff dimension. One can construct an example of such a function taking $s = 2, q_0 = 1/3, q_1 = 2/3, p_{0k} = 1/3 + \frac{1}{3\sqrt{k}}$, $p_{1k} = 2/3 - \frac{1}{3\sqrt{k}}$.

Remark 4. Theorem 4 gives us only sufficient conditions for the dimension preservation under F_ξ. For further studies of this problem see, e.g., [24].

3. W^*-REPRESENTATION OF POINTS OF $[0,1]^2 \subset R^2$ AND SELF-AFFINE FRACTALS

Let $s > 1$ and $r > 1$ be fixed positive integers, let $m = sr$, $A_1 = \{0, 1, \dots, s-1\}$, $A_2 = \{0, 1, \dots, r-1\}$, $\overline{A} = A_1 \times A_2$, and let $Q_1^* = \|q_{ik}\|$, $Q_2^* = \|q'_{jk}\|$ be fixed matrices which satisfy conditions (1) - (4), $i \in A_1$, $j \in A_2$, and, finally, let $\alpha_k(x)$ be the k-th Q_1^* symbol of x, and $\beta_k(y)$ be k-th Q_2^* symbol of y.

3.1. $Q_1^* \times Q_2^*$-cylinders and their properties.

The set

$$\Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k} \equiv \Delta^{Q_1^*}_{\alpha_1\dots\alpha_k} \times \Delta^{Q_2^*}_{\beta_1\dots\beta_m} =$$

$$= \left\{(x,y) : x \in \Delta^{Q_1^*}_{\alpha_1\dots\alpha_k}, y \in \Delta^{Q_2^*}_{\beta_1\dots\beta_m}\right\}$$

is said to be the cylinder (cylindrical set) of rank $(k \times m)$, which corresponds to the couple of Q_1^*- and Q_2^*-representations.

The following properties of cylindrical sets are obvious:

(1) $\Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k} = \bigcup\limits_{\alpha=0}^{s-1} \Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k\alpha} = \bigcup\limits_{\beta=0}^{r-1} \Delta^{\beta_1\dots\beta_m\beta}_{\alpha_1\dots\alpha_k} = \bigcup\limits_{\beta=0}^{r-1}\bigcup\limits_{\alpha=0}^{s-1} \Delta^{\beta_1\dots\beta_m\beta}_{\alpha_1\dots\alpha_k\alpha}$.

(2) $\nabla^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k} \bigcap \nabla^{\varphi_1\dots\varphi_n}_{\gamma_1\dots\gamma_l} \neq \varnothing \Leftrightarrow \begin{cases} \alpha_i = \gamma_i, i = \overline{1,a},\ a = \min\{k,l\}, \\ \beta_j = \varphi_j, j = \overline{1,c},\ c = \min\{m,n\}. \end{cases}$

(3) $\Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k} \bigcap \Delta^{\varphi_1\dots\varphi_n}_{\gamma_1\dots\gamma_l} = \Delta^{\varphi_1\dots\varphi_n}_{\alpha_1\dots\alpha_k}$, if $\begin{cases} k = \max\{k,l\}, \\ n = \max\{m,n\}, \\ \alpha_i = \gamma_i,\ i = \overline{1,l}, \\ \beta_j = \varphi_j,\ j = \overline{1,m}. \end{cases}$

(4) $\left|\Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k}\right| = \left(\prod\limits_{i=1}^{k} q^2_{\alpha_i i} + \prod\limits_{j=1}^{m} q'^2_{\beta_j j}\right)^{\frac{1}{2}} \to 0$

as $k \to \infty$ and $m \to \infty$. Where $|\cdot|$ stands for the diameter of the set.

(5) $\bigcap\limits_{m=1}^{\infty}\bigcap\limits_{k=1}^{\infty} \Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k} \equiv \Delta^{\beta_1\dots\beta_m\dots}_{\alpha_1\dots\alpha_k\dots} = M(\Delta^{Q_1^*}_{\alpha_1\dots\alpha_k\dots}, \Delta^{Q_2^*}_{\beta_1\dots\beta_m\dots})$.

(6) $\Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_k\dots} = \bigcap_{i=0}^{\infty} \Delta^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_{k+i}} = \{(x,y) : x = \Delta^{Q_1^*}_{\alpha_1\dots\alpha_k\dots}, y \in \Delta^{Q_2^*}_{\beta_1\dots\beta_m}\}$,

$$\Delta^{\beta_1\dots\beta_m\dots}_{\alpha_1\dots\alpha_k} = \bigcap\nolimits_{j=0}^{\infty} \Delta^{\beta_1\dots\beta_{m+j}}_{\alpha_1\dots\alpha_k} = \{(x,y) : x \in \Delta^{Q_1^*}_{\alpha_1\dots\alpha_k}, y \in \Delta^{Q_2^*}_{\beta_1\dots\beta_m\dots}\}.$$

$$(7)\quad \lambda_1\left(\Delta^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_k\ldots}\right)=|\Delta^{Q_2^*}_{\beta_1\ldots\beta_m}|=\prod_{j=1}^{m} q'_{\beta_j j},$$
$$\lambda_1\left(\Delta^{\beta_1\ldots\beta_m\ldots}_{\alpha_1\ldots\alpha_k}\right)=|\Delta^{Q_1^*}_{\alpha_1\ldots\alpha_k}|=\prod_{i=1}^{k} q_{\alpha_i i}.$$

3.2. $Q_1^*\times Q_2^*$**-cylinders and the Hausdorff dimension.** For the family of cylinders $\Delta^{\beta_1\ldots\beta_m}_{\alpha_1\ldots\alpha_k}$ we shall use the following notation:
$W^0_{km}=W^0_{km}(Q_1^*\times Q_2^*)$ is the class of all cylinders of the rank $(k\times m)$;
$W^0=W^0(Q_1^*\times\times Q_2^*)$ is the union of the classes W^0_{km} over all finite k and m;
$W^*=W^*(Q_1^*\times\times Q_2^*)$ is the union of the classes W^0_{kk} over all finite k.

Lemma 3. *If* $q_{**}=\sup\{q_{ik}\}$, $q'_{**}=\sup\{q'_{ik}\}$, $q_*=\inf\{q_{ik}\}>0$ *and* $q_{**}=\inf\{q'_{ik}\}>0$ *then for any set* $E\subset[0,1]^2$ *there exist no more than* $c=c_1c_2$ $(c_1=2s^{n_1}(s_1^\gamma+s-1)$, $c_2=2r^{n_2}(r_2^\gamma+r-1))$ *cylinders from* W^0 *which cover* E *and whose diameters are smaller than* $|E|$, *where* $q_{**}^{\gamma_1}\leq q_*$, $q_{**}^{n_1}\leq\frac{1}{\sqrt{2}}$, $q_{**}^{\gamma_2}\leq q'_*$, $q_{**}^{n_2}\leq\frac{1}{\sqrt{2}}$.

Proof. Let F be the smallest rectangle which contains E and whose sides are parallel to the axes Ox and Oy, and let F_x be the smallest interval from the axe Ox which contains the projection of the E and let F_y be the smallest interval which contains the projection of E onto the axis Oy.

According to the lemma 1 there exist no more than $l_1=2(s_1^\gamma+s-1)$ cylinders from the family $V_{Q_1^*}$ which cover F_x and whose lengths are not greater than $|F_x|$. Let $\Delta^{Q_1^*}_{e_1\ldots e_k}$ be one of those intervals. Then each of the cylinders

$$\Delta^{Q_1^*}_{e_1\ldots e_k\alpha_{k+1}\ldots\alpha_{k+n_1}},\ \alpha_j\in A_1=\{0,1,\ldots,s-1\}$$

have the length

$$|\Delta^{Q_1^*}_{e_1\ldots e_k\alpha_{k+1}\ldots\alpha_{k+n_1}}|\leq|\Delta_{e_1\ldots e_k}|q_{**}^{n_1}\leq\frac{1}{\sqrt{2}}|\Delta_{e_1\ldots e_k}|\leq\frac{1}{\sqrt{2}}|F_x|.$$

The union of c_1 such cylinders forms the covering of F_x. Similarly, there exist c_2 cylinders from $V_{Q_2^*}$ which cover F_y and whose lengths are not greater then $\frac{1}{\sqrt{2}}|F_y|$.

The Cartesian product of the above mentioned cylinders from $V_{Q_1^*}$ and $V_{Q_2^*}$ gives the $c=c_1c_2$ cylinders from W^0 which cover E and whose diameters are not greater than $|E|$. Indeed, if u and v are the lengths of the sides of one of those cylinders then its diameter d satisfies

$$d=\sqrt{u^2+v^2}\leq\sqrt{\frac{1}{2}|F_x|^2+\frac{1}{2}|F_y|^2}\leq\frac{1}{\sqrt{2}}|F|\leq|E|.$$

□

Let $H^\alpha(E,W^0)$ be the Hausdorff measure of a given subset E w.r.t. the family of coverings W^0, and let $\alpha_0(E,W^0)$ be the corresponding Hausdorff dimension.

Theorem 5. *For any subset* E *of the unit square we have*

$$(10)\qquad H^\alpha(E)\leq H^\alpha(E,W^0)\leq cH^\alpha(E),$$

and, therefore,

$$\alpha_0(E)=\alpha_0(E,W^0).$$

Proof. The left hand side of inequality (10) is obvious. Let us prove the right hand side. According to the lemma 3 for any ε-covering $\{E_k\}$ of E there exists an ε-covering $\{\omega_k^i\}$, $\omega_k^i \in W^0$, $\bigcup_i \omega_k^i \supset E_k$, $i = \overline{1, m_k}$, $m_k \leq c$, such that $|\omega_k^i| \leq |E_k|$ for any $i \in \{1, 2, \dots, m_k\}$. Hence

$$\sum_k \sum_{i=1}^{m_k} |\omega_k^i|^\alpha \leq c \sum_k |E_k|^\alpha$$

and

$$m_\varepsilon^\alpha(E, W^0) \leq c m_\varepsilon^\alpha(E).$$

Taking the limit as $\varepsilon \to 0$, we get (10).

From the inequality (10) it follows that $H^\alpha(E)$ and $H^\alpha(E, W^0)$ take values 0 or ∞ simultaneously, and, therefore, $\alpha_0(E) = \alpha_0(E, W^0)$, which completes the proof. □

3.3. **$Q_1^* \times Q_2^*$-representation and W^*-representation of points from** $[0,1]^2$**.** As before, let Q_1^* and Q_2^* be fixed matrices satisfying conditions (1)- (4). The existence of the family of cylinders W^0 (and W^*) allows us (according to the property 5 of the cylindrical sets) to identify every point $M \in [0,1]^2$ with an ordered pair of infinite sequences $(\alpha_1, \alpha_2, \dots, \alpha_k, \dots)$ and $(\beta_1, \beta_2, \dots, \beta_k, \dots)$ of symbols from the alphabets $A_1 = \{0, \dots, s-1\}$ and $A_2 = \{0, \dots, r-1\}$ respectively. The representation of the point $M \in [0,1]^2$ as an intersection of cylindrical sets

$$M = \bigcap_{k=1}^{\infty} \Delta_{\alpha_1 \dots \alpha_k}^{\beta_1 \dots \beta_k} \equiv$$

$$\equiv \Delta_{\alpha_1 \dots \alpha_k \dots}^{\beta_1 \dots \beta_k \dots} \tag{11}$$

is said to be the $Q_1^* \times Q_2^*$-expansion of M. We shall call the α_k the k-th Q_1^*-symbol and β_m the m-th Q_2^*-symbol of the point M: $\alpha_k = \alpha_k(M)$, $\beta_m = \beta_m(M)$. It is clear that

$$\Delta_{\alpha_1 \dots \alpha_k \dots}^{\beta_1 \dots \beta_k \dots} = M(x, y) \Leftrightarrow \begin{cases} x = \Delta_{\alpha_1 \dots \alpha_k \dots}^{Q_1^*}, \\ y = \Delta_{\beta_1 \dots \beta_k \dots}^{Q_2^*}. \end{cases}$$

Almost all (w.r.t. the Lebesgue measure λ_2) points from $[0,1]^2$ have a unique $Q_1^* \times Q_2^*$-representation since almost all (w.r.t. the Lebesgue measure λ_1) points from $[0,1]$ have a unique Q^*-representation. We shall call such points $Q_1^* \times Q_2^*$-irrational. Their Cartesian coordinates x and y are Q_1^*- resp. Q_2^*-irrational numbers.

If x is a Q_1^*-rational number and y is a Q_2^*-irrational number or vice versa then the point $M(x, y)$ have two $Q_1^* \times Q_2^*$-representations:

$$\Delta_{\alpha_1 \dots \alpha_k 00 \dots}^{\beta_1 \dots \beta_k \beta_{k+1} \beta_{k+2} \dots} = \Delta_{\alpha_1 \dots \alpha_{k-1} (\alpha_k - 1)(s-1)(s-1) \dots}^{\beta_1 \dots \beta_{k-1} \beta_k \beta_{k+1} \beta_{k+2} \dots},$$

$$\Delta_{\alpha_1 \dots \alpha_k \alpha_{k+1} \alpha_{k+2} \dots}^{\beta_1 \dots \beta_k 00 \dots} = \Delta_{\alpha_1 \dots \alpha_{k-1} \alpha_k \alpha_{k+1} \alpha_{k+2} \dots}^{\beta_1 \dots \beta_{k-1} (\beta_k - 1)(r-1)(r-1) \dots}.$$

If x is a Q_1^*-rational number and y is a Q_2^*-rational number then the point $M(x, y)$ have four $Q_1^* \times Q_2^*$-representations:

$$\Delta_{\alpha_1 \dots \alpha_k 00 \dots}^{\beta_1 \dots \beta_m 00 \dots} = \Delta_{\alpha_1 \dots \alpha_{k-1} (\alpha_k - 1)(s-1)(s-1) \dots}^{\beta_1 \dots \beta_{m-1} (\beta_m - 1)(r-1)(r-1) \dots}$$

$$= \Delta_{\alpha_1 \dots \alpha_{k-1} (\alpha_k - 1)(s-1)(s-1) \dots}^{\beta_1 \dots \beta_{m-1} \beta_m 00 \dots} = \Delta_{\alpha_1 \dots \alpha_{k-1} \alpha_k 00 \dots}^{\beta_1 \dots \beta_{m-1} (\beta_m - 1)(r-1)(r-1) \dots}.$$

We shall redenote the elements of the set $\overline{A} = A_1 \times A_2$ in the following way:

$$\begin{array}{llll}
(0,0) \equiv \overline{0}, & (1,0) \equiv \overline{r}, & \dots & (s-1,0) \equiv \overline{(s-1)r}, \\
(0,1) \equiv \overline{1}, & (1,1) \equiv \overline{r+1}, & \dots & (s-1,1) \equiv \overline{(s-1)r+1}, \\
\dots & \dots & \dots & \dots \\
(0,r-1) \equiv \overline{r-1}, & (1,r-1) \equiv \overline{2r-1}, & \dots & (s-1,r-1) \equiv \overline{sr-1}.
\end{array}$$

We get a new alphabet $\overline{A} = \{\overline{0}, \overline{1}, \dots, \overline{m-1}\}$, where $m = sr$. Let

$$\square_{\gamma_1 \dots \gamma_k} \equiv \left\{ M(x,y) : x \in \Delta^{Q_1^*}_{\alpha_1 \dots \alpha_k}, y \in \Delta^{Q_2^*}_{\beta_1 \dots \beta_k} \right\},$$

where $\overline{\gamma_i} = (\alpha_i, \beta_i)$, hence $\square_{\gamma_1 \dots \gamma_k} = \Delta^{\beta_1 \dots \beta_k}_{\alpha_1 \dots \alpha_k}$. Then for any point $M \in [0,1]^2$ there exists a sequence $\{\gamma_k\}$, $\overline{\gamma_k} \in \overline{A}$, such that

$$M = \square_{\gamma_1 \dots \gamma_k \dots} \equiv \bigcap_{k=1}^{\infty} \square_{\gamma_1 \dots \gamma_k} \tag{12}$$

and for any sequence $\{\gamma_k\}$, $\overline{\gamma_k} \in \overline{A}$: $\square_{\gamma_1 \dots \gamma_k \dots} = M \in [0,1]^2$. The representation of the point M in the form (12) is said to be the W^*-representation of M, and γ_k is said to be the k-th W^*-symbol of the point M: $\gamma_k = \gamma_k(M)$.

Remark. As it was mentioned above a point of the plane can have one, two or four W_*-representations.

3.4. **Some sets defined by the properties of their W^*-representations.** Let us consider a fixed W^*-representation of the points $[0,1]^2$ with the alphabet $A = \{0, 1, \dots, m-1\}$, $m = sr$.

Let $\{V_k\}$ be the sequence of non-empty subsets of A, and let us consider the set $C[W^*, \{V_k\}]$ consisting of points from $[0,1]^2$ whose k-th W^*-symbols belong to V_k, i.e.,

$$C[W^*, \{V_k\}] \equiv \{M : M = \square_{\gamma_1(M) \dots \gamma_k(M) \dots}, \ \gamma_k(M) \in V_k, k = 1, 2, \dots\}.$$

If $V_k = V$ for any $k \in N$ then the $C[W^*, \{V_k\}]$ will also be denoted by $C[W^*, \{V\}]$. It is obvious that $C[W^*, \{V_k\}]$ is a finite set if V_k consists of only 1 element for all $k > k_0$ and it is uncountable if the cardinality of V_k is large than 1 for infinitely many values of the index k. It is not hard to see that the set $C[W^*, \{V_k\}]$ is:

(1) a union of a finite number of cylinders of the rank k_0 if $V_k = A$ for all $k > k_0$;
(2) nowhere dense if $V_k \neq A$ for an infinite number of k;
(3) perfect (i.e., it is a closed set without isolated points);
(4) self-affine if $W^* = W$ and $V_k = V$, $k = 1, 2, \dots$.

Theorem 6. *The Lebesgue measure* λ_2 *of the set* $C = C[W^*, \{V_k\}]$ *is equal to*

$$\lambda_2(C) = \prod_{k=1}^{\infty} (1 - G_k)$$

where

$$G_k = \sum_{a \in A \setminus V_k} g_{ak}, \quad g_{ak} = q_{ik} q'_{jk}, (i,j) = \overline{a}, a \in A = \{0, 1, \dots, sr-1\}.$$

Proof. Let D_k be the union of all rank k cylinders whose interiors have non-empty intersections with C. Let U_{k+1} be the union of all cylinders of the rank $k+1$ which belongs to D_k and have no interior points from C. It is clear that $D_{k+1} = D_k \setminus U_{k+1}$, and $C = \bigcap\limits_{k=1}^{\infty} D_k = \lim\limits_{k \to \infty} D_k$ with

$$\lambda_2(C) = \lim_{k \to \infty} \lambda_2(D_k).$$

Taking into account that $\lambda_2(D_1) = 1 - G_1$ and $\lambda_2(U_{k+1}) = G_k \lambda_2(D_k)$ we have

$$\lambda_2(D_{k+1}) = \lambda_2(D_k) - G_k \lambda_2(D_k) = (1 - G_k)\lambda_2(D_k) =$$

$$= (1 - G_k)(1 - G_{k-1})\lambda_2(D_{k-1}) = \dots = \prod_{i=1}^{k} (1 - G_i)$$

which gives the desired equality. □

Corollary. The set $C[W^*, \{V_k\}]$ is of zero Lebesgue measure if and only if $\sum\limits_{k=1}^{\infty} G_k = +\infty$.

3.5. **$Q_1^* \times Q_1^*$-representation of points and transformations preserving dimension.** Let $Q_1^* = ||q_{ik}||$, $Q_2^* = ||q'_{jk}||$, $Q_1 = ||p_{ik}||$ with $p_{ik} = q_i$ and $Q_2 = ||p'_{jk}||$ with $p'_{jk} = q_j$, $i \in A_1 = \{0, 1, ..., s-1\}$, $j \in A_2 = \{0, 1, ..., r-1\}$. Let $q_i > 0$, $\forall i \in A_1$; and $q'_j > 0$, $\forall j \in A_2$.

Theorem 7. *If* $\forall i \in A_1$ *and* $\forall j \in A_2$

$$\begin{cases} q_{ik} \to q_i & (k \to \infty), \\ q'_{jk} \to q'_j & (k \to \infty), \end{cases} \tag{13}$$

then the transformation of $[0,1]^2$*, defined by the equality*

$$f(M) = f(\square^*_{\gamma_1(M)\ldots\gamma_k(M)\ldots}) = \square_{\gamma_1(M)\ldots\gamma_k(M)\ldots} = M', \tag{14}$$

preserves the Hausdorff dimension of every subset of the unit square.

Proof. From (14) it follows that for any cylinder $\Delta^{*\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_k} \in W^0(Q_1^* \times Q_2^*)$ we have

$$f(\Delta^{*\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_k}) = \Delta^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_k} \in W^0(Q_1 \times Q_2).$$

To obtain a contradiction, let us assume that f does not preserve the Hausdorff dimension. Therefore, there exists a set $E^* \subset [0,1]^2$ such that $\alpha_0(E^*) \neq \alpha_0(f(E^*))$. Without loss of generality we may assume that $\alpha' := \alpha_0(E) < \alpha_0(E^*) =: \alpha_0$, where $E = f(E^*)$.

So, there exist real numbers α and β such that $\alpha' < \alpha < \beta < \alpha_0$, and

$$H^\alpha(E) = H^\beta(E) = 0, \tag{15}$$

and

$$H^\alpha(E^*) = H^\beta(E^*) = +\infty. \tag{16}$$

Let us recall that $\alpha_0(E) = \alpha_0(E, W^0(Q_1 \times Q_2))$ and $\alpha_0(E^*) = \alpha_0(E^*, W^0(Q_1^* \times Q_2^*))$.

From (15) it follows that $m_\varepsilon^\alpha(E, W^0(Q_1 \times Q_2)) = 0$, $\forall \varepsilon > 0$. Therefore, for any $\gamma > 0$ and for any $\varepsilon > 0$ there exists an ε-covering of the set E by cylinders $\{\Delta_m\} \in W^0(Q_1 \times Q_2)$ ($\Delta_m := \Delta^{\beta_1\ldots\beta_{n_m}}_{\alpha_1\ldots\alpha_{k_m}} \in W^0(Q_1 \times Q_2)$) such that $\sum\limits_m |\Delta_m|^\alpha \leq \gamma$.

The transformation f is continuous and, therefore, it is uniformly continuous on the unit square. f^{-1} is also uniformly continuous. Hence, for any $\delta > 0$ there exists $\varepsilon(\delta) > 0$ such that any $\varepsilon(\delta)$-covering of E is mapped by f^{-1} to a δ-covering of the set $E^* = f^{-1}(E)$.

From (16) it follows that $\lim\limits_{\delta \downarrow 0} m_\delta^\beta(E^*, W^0(Q_1^* \times Q_2^*)) = +\infty$. So, for any $A > 0$ there exists $\delta_A > 0$ such that for any $\delta \leq \delta_A$ we have $m_\delta^\beta(E^*, W^0(Q_1^* \times Q_2^*)) \geq A$. Therefore, for any δ_A-covering $\{\Delta_m^*\}$ of E^* by cylinders from $W^0(Q_1^* \times Q_2^*)$ one has $\sum\limits_m |\Delta_m^*|^\beta \geq A$.

Let us fix a positive integer A, and let δ_A and $\varepsilon(\delta_A)$ be the above defined values.

For a given $\gamma > 0$, let us choose an $\varepsilon(\delta_A)$-covering $\{\Delta_m\}$ ($\Delta_m \in W^0(Q_1 \times Q_2)$) of the set E such that $\sum\limits_m |\Delta_m|^\alpha \leq \gamma$.

Let us consider the covering $\{\Delta_m^*\} = \{f^{-1}(\Delta_m)\}$ of the set E^*.

From $|\Delta_m| \leq \varepsilon(\delta_A)$ it follows that $|\Delta_m^*| \leq \delta_A$, and, therefore,

$$\sum_m |\Delta_m^*|^\beta \geq A. \tag{17}$$

On the other hand,

$$\sum_m |\Delta_m^*|^\beta = \sum_m |\Delta_m|^\alpha \cdot \frac{|\Delta_m^*|^\beta}{|\Delta_m|^\alpha} = \sum_m |\Delta_m|^\alpha \cdot \left(\frac{|\Delta_m^*|^{\frac{\beta}{\alpha}}}{|\Delta_m|} \right)^\alpha.$$

Since $\frac{\beta}{\alpha} > 1$ and $q_{ik} \to q_i$, $q'_{jk} \to q'_j$, we have:

$$\frac{(q_{\alpha_1 1} \cdot \ldots \cdot q_{\alpha_k k})^c}{q_{\alpha_1} \cdot \ldots \cdot q_{\alpha_k}} \to 0 \quad (k \to \infty)$$

for any $c > 1$ and for any $\{\alpha_k\} \in A_1$.

Similarly, we have

$$\frac{(q'_{\beta_1 1} \cdot \ldots \cdot q'_{\beta_n n})^c}{q'_{\beta_1} \cdot \ldots \cdot q'_{\beta_n}} \to 0 \quad (n \to \infty)$$

for any $c > 1$ and for any $\{\beta_n\} \in A_2$.

Therefore,

$$\frac{|\Delta^*_m|^{\frac{\beta}{\alpha}}}{|\Delta_m|} = \frac{\left|\Delta^{*\beta_1 \ldots \beta_{n_m}}_{\alpha_1 \ldots \alpha_{k_m}}\right|^{\frac{\beta}{\alpha}}}{|\Delta^{\beta_1 \ldots \beta_{n_m}}_{\alpha_1 \ldots \alpha_{k_m}}|} \to 0 \quad \text{for any} \quad \{n_m\} \to \infty \quad \text{and} \quad \{k_m\} \to \infty,$$

and, hence, for any $\varepsilon_1 > 0$ and for any $c > 1$ there exists $N_0(\varepsilon_1, c) \in N$ such that $\forall k \geq N_0(\varepsilon_1, c)$, $\forall n \geq N_0(\varepsilon_1, c)$

$$\frac{|\Delta^{*\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_k}|^c}{|\Delta^{\beta_1 \ldots \beta_n}_{\alpha_1 \ldots \alpha_k}|} \leq \varepsilon_1.$$

For given numbers $A > 0$, $\alpha \in (\alpha', \alpha_0)$ and $\gamma > 0$, let us choose $\varepsilon_1 > 0$ such that $\gamma \cdot \varepsilon_1^\alpha \leq \frac{1}{2}A$.

Finally, for a given $\gamma > 0$, let us choose a covering of the set E by cylinders $\{\Delta_m\}$ whose ranks are at least $N_0\left(\varepsilon_1, \frac{\beta}{\alpha}\right)$ and whose diameters are less than $\varepsilon(\delta_A)$, such that $\sum\limits_m |\Delta_m|^\alpha < \gamma$.

In such a case we have

$$\sum_m |\Delta^*_m|^\beta = \sum_m |\Delta_m|^\alpha \cdot \left(\frac{|\Delta^*_m|^{\frac{\beta}{\alpha}}}{|\Delta_m|}\right)^\alpha \leq \sum_m |\Delta_m|^\alpha \cdot \varepsilon_1^\alpha \leq \gamma \cdot \varepsilon_1^\alpha \leq \frac{1}{2}A,$$

which contradicts (17). So, f is a DP-transformation on $[0,1]^2$. □

4. RANDOM ELEMENTS OF $[0,1]^2$ WITH INDEPENDENT W^*-SYMBOLS

Let η_k be a sequence of independent discrete random variables taking values $0, 1, 2, \ldots, m-1$ $(m = sr)$ with probabilities

$$P\{\eta_k = i\} = p_{ik} \geq 0,\ p_{0k} + p_{1k} + \ldots + p_{(m-1)k} = 1.$$

The random element

$$\xi = \Box_{\eta_1 \eta_2 \ldots \eta_k \ldots} \in [0,1]^2$$

is said to be a random element with independent W^*-symbols. Properties of the distribution of ξ are defined by the matrices Q^*_1, Q^*_2 and the infinite stochastic matrix $\|p_{ik}\|$, $(i = \overline{0, m-1}, k \in N)$.

Let η and τ be two probability measures defined on the same measurable space $(\Omega, \mathcal{F})$. Let us recall that a probability measure is said to be *pure discrete* if it is supported by an at most countable set. A measure η is called *absolutely continuous* with respect to τ (symbolically $\eta \ll \tau$) if $\tau(E) = 0$ implies $\eta(E) = 0$. Measures η and τ are said to be *mutually orthogonal (=singular)* (symbolically $\eta \perp \tau$) if there exists a set $B \in \mathcal{F}$ such that $\eta(B) = 0$ and $\tau(B) = 1$.

4.1. On the Lebesgue structure of probability measures with independent W^*-symbols.

Let μ_ξ be the probability measure corresponding to the distribution of the random element ξ with independent W^*-symbols.

Theorem 8. *The probability measure μ_ξ with independent W^*-symbols is of pure type, and*

1. *it is pure discrete iff*
$$M = \prod_{k=1}^{\infty} \max_i \{p_{ik}\} > 0;$$
2. *it is pure absolutely continuous iff*
$$L = \prod_{k=1}^{\infty} \sum_{a=0}^{m-1} \sqrt{p_{ak} g_{ak}} > 0,$$
where $g_{ak} = q_{ik} q'_{jk}$, $(i,j) = \overline{a}$, $a \in A = \{0, \ldots, m-1\}$, $m = sr$;
3. *it is pure singularly continuous in all other cases, i.e., iff*
$$M = L = 0.$$

Proof. Let $\Omega_k = A = \{0, 1, \ldots, m-1\}$ with $m = rs$, and let $\mathcal{F}_k = 2^{\Omega_k}$. For any $k \in N$ we define probability measures μ_k and ν_k by $\mu_k(a) = p_{ak} \geq 0, \nu_k(a) = g_{ak} > 0$ for any $a \in A$. Let

$$(\Omega, \mathcal{F}, \mu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{F}_k, \mu_k), \quad (\Omega, \mathcal{F}, \nu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{F}_k, \nu_k)$$

be the infinite products of probability spaces.

Since $\mu_k \ll \nu_k, \forall k \in N$, from the well-known Kakutani's theorem (see, e.g., [16]) it follows that the measure μ is either absolutely continuous w.r.t. ν or it is orthogonal to ν. More precisely: $\mu \ll \nu$ if and only if $d = \prod\limits_{k=1}^{\infty} \rho(\mu_k, \nu_k) > 0$, and $\mu \perp \nu$ if and only if $d = 0$, where $\rho(\mu_k, \nu_k)$ is the corresponding Hellinger's integral, i.e.,

$$\rho(\mu_k, \nu_k) = \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} \, d\nu_k.$$

Let us prove that the measure μ is pure discrete if and only if $M = 0$.

If $M = \prod\limits_{k=1}^{\infty} \max\limits_{\omega_k \in \Omega_k} \mu_k(\omega_k) = 0$, then for any point $\omega \in \Omega$ we have $\mu(\omega) \leq \prod\limits_{k=1}^{\infty} \max\limits_{i \in \Omega_k} p_{ik} = 0$. Therefore, the condition $M > 0$ is necessary for the discreteness of the measure μ.

To prove the sufficiency we consider a subset $D_\mu \subset \Omega$:

$$D_\mu = \left\{ \omega : \, \mu_k(\omega_k) > 0 \text{ and } \prod_{k=1}^{\infty} \mu_k(\omega_k) > 0 \right\}.$$

The set D_μ consists of the points $\omega = (\omega_1, \omega_2, ..., \omega_k, ...)$ such that $\mu_k(\omega_k) > 0$ and the condition $\mu_k(\omega_k) \neq \max\limits_{\omega_k \in \Omega_k} \mu_k(\omega_k)$ holds only for a finite number of indices k. It is easy to see that the set D_μ is at most countable and the event "$\omega \in D_\mu$" does not depend on any finite coordinates of ω. Therefore, by using the "0 and 1" theorem of Kolmogorov, we conclude that $\mu(D_\mu) = 0$ or $\mu(D_\mu) = 1$. Since the set D_μ contains the point ω^* such that $\mu_k(\omega_k^*) = \max\limits_{\omega_k \in \Omega_k} \mu_k(\omega_k)$, we have $\mu(D_\mu) \geq \mu(\omega^*) > 0$. Thus, $\mu(D_\mu) = 1$, which proves the discreteness of the measure μ.

Now let us consider a measurable mapping $\phi : \Omega = A^\infty \to [0,1]^2$ defined as follows:

$$\phi(\omega) = \square_{\omega_1 \omega_2 \ldots \omega_k \ldots},$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_k, \ldots) \in \Omega$.

The mapping ϕ is not bijective (it is a surjection, but it is not an injection, because some points from $[0,1]^2$ have several (at most 4) different W^*-representations). The Lebesgue measure λ_2 of the set B of points with non-unique representations equals to zero (as the measure of an at most countable union of sets of zero measure). The images of the cylinders from Ω are $Q_1^* \times Q_2^*$-cylinders from $[0,1]^2$. So, ϕ^{-1} is also measurable.

Let μ^* and ν^* be the images of the measures μ and ν under the mapping ϕ:

$$\mu^*(E^*) := \mu^*(\phi^{-1}(E^*)), \quad \nu^*(E^*) := \nu^*(\phi^{-1}(E^*))$$

for any Borel subset E^* from the unit square. From the definition of the measures μ^* and ν^* it follows that ν^* is the Lebesgue measure λ_2 on the unit square and μ^* coincides with the probability measure μ_ξ with independent W^*-symbols.

If $M = 0$, then the measure μ is continuous. Therefore, $\mu(\omega) = 0$ for any $\omega \in \Omega$. So, for any point $K \in [0,1]^2$ we have $\mu^*(K) = \mu(\phi^{-1}(K)) = 0$, because $\phi^{-1}(K)$ consists of at most 4 points from Ω and each of them is of zero measure μ.

If $M > 0$, then the measure μ is pure discrete, and the above defined set D_μ consists of all atoms of this measure. Let $D_\mu^* = \phi(D_\mu)$. The set D_μ^* is at most countable, because ϕ is surjective and D_μ is at most countable. Since $\phi^{-1}(\phi(D_\mu)) \supset D_\mu$, we have

$$\mu^*(D_\mu^*) = \mu(\phi^{-1}(D_\mu^*)) \geq \mu(D_\mu) = 1.$$

Hence, in such a case μ^* is a pure discrete probability measure, and the set D_μ^* consists of all atoms of μ^*. So, μ^* is pure discrete if and only if $M > 0$.

It is well known (see, e.g., [5]) that for any pair of probability measures η and τ defined on the same measurable space and for any measurable mapping f, the absolute continuity of η w.r.t. τ implies the absolute continuity of the corresponding image measure $\eta^* = \eta(f^{-1})$ w.r.t. the corresponding image measure $\tau^* = \tau(f^{-1})$. So, from $\mu \ll \nu$ it follows that $\mu^* \ll \nu^*$.

Let us prove that in our case $\mu^* \ll \nu^*$ implies $\mu \ll \nu$. First of all let us remind that ϕ is a bi-measurable mapping (i.e., ϕ and ϕ^{-1} are measurable). Let E be an arbitrary set from $\mathcal{F}$ such that $\nu(E) = 0$, and let $E^* = \phi(E)$. We can represent E^* in the following form

$$E^* = (E^* \bigcap B) \bigcup (E^* \bigcap \overline{B}),$$

where B is the set of points having several different W^*-representations. Then

$$\lambda_2(E^*) \equiv \nu^*(E^*) = \nu^*(E^* \bigcap B) + \nu^*(E^* \bigcap \overline{B}) = 0,$$

because $\lambda_2(B) \equiv \nu^*(B) = 0$ and $\phi^{-1}(E^* \bigcap \overline{B}) \subset E$ with $\nu(E) = 0$.

Since $\mu^* \ll \nu^*$, we deduce that $\mu^*(E^*) = 0$, and, therefore, $\mu(E) \leq \mu^*(E^*) = 0$. Hence $\mu \ll \nu$. So, from

$$\int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\mu_k = \sum_{a=0}^{m-1} \sqrt{p_{ak} g_{ak}},$$

and the Kakutani theorem we have

$$\mu_\xi \ll \lambda_2 \Leftrightarrow \mu^* \ll \nu^* \Leftrightarrow \mu \ll \nu \Leftrightarrow d > 0 \Leftrightarrow L > 0.$$

It is also well known that for any pair of probability measures η and τ defined on the same measurable space and for any measurable mapping f, the singularity of the corresponding image measures $\eta^* = \eta(f^{-1})$ and $\tau^* = \tau(f^{-1})$ implies the mutual singularity of η and τ. So, from $\mu^* \perp \nu^*$ it follows that $\mu \perp \nu$.

Let us show that in our case $\mu \perp \nu$ implies $\mu^* \perp \nu^*$. If $\mu \perp \nu$ then there exists a set $G \in \mathcal{F}$ such that $\mu(G) = 1$ and $\nu(G) = 0$. Then $\mu^*(G^*) = 1$, because $\mu^*(G^*) \geq \mu(G) = 1$. From $\nu^*(B) = 0$ it follows that $\nu^*(G^* \bigcap B) = 0$ and

$$\nu^*(G^* \bigcap \overline{B}) \equiv \nu(\phi^{-1}(G^* \bigcap \overline{B})) \leq \nu(G) = 0,$$

and, therefore,

$$\nu^*(G^*) = \nu^*(G^* \bigcap B) + \nu^*(G^* \bigcap \overline{B}) = 0.$$

Hence $\mu^* \perp \nu^*$.

So, from this fact and from the Kakutani theorem it follows that

$$\mu_\xi \perp \lambda_2 \Leftrightarrow \mu^* \perp \nu^* \Leftrightarrow \mu \perp \nu \Leftrightarrow d = 0 \Leftrightarrow L = 0.$$

Finally, it is clear that the measure μ_ξ is singularly continuous w.r.t. the two-dimensional Lebesgue measure if and only if $M = 0$ and $L = 0$. □

Remark. From the latter theorem it follows that the continuous singularity plays a generic role in the family of probability measures with independent W^*-symbols. Indeed, the discreteness of the measure μ_ξ means that $\max\limits_a p_{ak} \to 1$ quick enough as $k \to \infty$. The absolute continuity of μ_ξ means that the asymptotic behavior of elements of the matrix $GP = \|p_{ak}\|$ is "almost the same" as the asymptotic behavior of elements from the matrix $\|g_{ak}\|$. In all other cases we deal with singularly continuously (w.r.t. the two-dimensional Lebesgue measure) distributed random elements ξ.

Lemma 4. *The spectrum S_ξ of the distribution of the random element ξ with independent W^*-symbols coincides with the set $C = C[W^*, \{V_k\}]$, where $V_k \equiv \{v : P_{vk} > 0\}$.*

Proof. Let us show that: 1) $S_\xi \subseteq C$ and 2) $C \subseteq S_\xi$.

1) Let $x \in S_\xi$ gives $\forall\varepsilon$, $P\{\xi \in O_\varepsilon(x)\} > 0$. Then it is easy to find a cylinder $\square_{\gamma_1(x)\ldots\gamma_k(x)} \subset O_\varepsilon(x)$ and $\varepsilon_1 > 0$ such that

$$O_{\varepsilon_1(x)} \subset \square_{\gamma_1(x)\ldots\gamma_k(x)}.$$

Taking into account that $P\{\xi \in O_{\varepsilon_1}(x)\} > 0$ implies

$$P\{\xi \in \square_{\gamma_1(x)\ldots\gamma_k(x)}\} = \prod_{i=1}^{k} p_{\gamma_i(x)i} > 0,$$

we conclude that $P_{\gamma_k(x)k} > 0$, $\forall k \in N$, and consequently $x \in C$.

2) Let $x \in C$ be such that $P_{\gamma_k(x)k} > 0$, $\forall k \in N$. Then $P\{\xi \in \square_{\gamma_1(x)\ldots\gamma_k(x)}\} > 0$. Let us consider any $\varepsilon > 0$ and $O_\varepsilon(x)$. Starting from some k_0 all cylinders $\square_{\gamma_1(x)\ldots\gamma_k(x)}$ belongs to $O_\varepsilon(x)$ if $k \geq k_0$. So we have

$$P\{\xi \in O_\varepsilon(x)\} \geq P\{\xi \in \square_{\gamma_1(x)\ldots\gamma_k(x)}\} > 0,$$

and $x \in S_\xi$. Consequently $S_\xi = C$, which ends the proof. □

4.2. Some fractal properties of random elements of $[0,1]^2$ with independent W^*-symbols.

Let us consider the random element in R^2:

$$\xi = \square_{\xi_1\ldots\xi_k\ldots}$$

where ξ_k are independent random elements with the following distributions

$$\begin{array}{ccccc} \xi_k & \overline{1} & \overline{2} & \ldots & \overline{sr} \\ & p_{1k} & p_{2k} & \cdots & p_{(sr)k} \end{array}$$

Lemma 5. *If there exists some i such that $p_{ik} = 0$ for all k, then the spectrum S_ξ coincides with the set $C[W^*, V]$, where V consists of those digits which correspond to non-zero probabilities.*

The proof is straightforward by observing that: 1) S_ξ consists of those points in the plane for which the probability of any neighborhood is positive; 2) for any ε there exists such a rank that a cylindrical set of $C[W^*, V]$ is completely contained in the ε neighborhood.

Corollary 3. *When the W^*-representation is such that the condition of the theorem 7 are satisfied then the Hausdorff dimension of the spectrum can be computed using the transformation preserving the Hausdorff dimension.*

Once we got a wide class of transformations preserving the Hausdorff-Besicovitch dimension we can use some known results for the simple self-affine construction to compute the dimension of a much wider class of quasi-self-affine sets. In this setting the following result by McMullen [21] (independently T. Bedford) is very useful.

Let us fix two natural numbers m and n ($m \leq n$). Let $K(D)$ be the set in the unit square consisting of those points on the plane which in their m-n-adic (the matrices Q_1^* and Q_2^* have all same columns equal to $1/m$ and $1/n$ respectively) expansion contain only digits from the set D.

Theorem 9. [21]

$$\alpha_0(K(D)) = \log_m \left(\sum_{j=0}^{m-1} a(j)^{\log_n m} \right),$$

where $a(j)$ is the number of elements in D with the second component equal to j.

Example. Let $Q_1^* = ||q_{ik}||$, $Q_2^* = ||q'_{jk}||$, $i \in A_1 = \{0,1,2\}$, $j \in A_2 = \{0,1\}$. Let $q_{0k} = 1/3 - \frac{1}{4k}$, $q_{1k} = 1/3$, $q_{2k} = 1/3 + \frac{1}{4k}$ and $q'_{0k} = q'_{1k} = 1/2$. Consider the random element

$$\xi = \Box_{\xi_1 \ldots \xi_k \ldots},$$

where ξ_k are random elements with the distributions

ξ_k	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$
	1/3	0	0	1/3	1/3	0

The spectrum of the random variable ξ coincides with the set $C[W^*, \{\overline{1}, \overline{4}, \overline{5}\}]$. Using the asymptotics of our matrix Q_1^* and a DP-transformation (of the type of the one given in Theorem 7, note that this DP-transformation is essentially non Lipschitz in this case) which sends the spectrum to a McMullen type carpet we can conclude that

$$\alpha_0(S_\xi) = \log_2 \left(2^{\log_3 2} + 1 \right).$$

Let us also note that we can compute a much wider class of examples using more general dimension results contained in [6].

5. MULTIDIMENSIONAL CLASSIFICATION OF SINGULARLY CONTINUOUS PROBABILITY MEASURES ACCORDING THEIR SPECTRAL PROPERTIES

Let $O_\varepsilon(x)$ be the ε-vicinity of a point $x \in R^n$.

Definition 1. A singularly continuous (w.r.t. the n-dimensional Lebesgue measure λ_n) probability measure μ on $\mathbb{R}^n$ is said to be of GC-type (generalized Cantor type) if there exists a nowhere dense set $E \subset S_\mu$ such that $\mu(A) = 1$ and $\forall x \in E$, $\exists \varepsilon = \varepsilon(x) > 0$: $\lambda_n(S_\mu \cap O_\varepsilon(x)) = 0$.

Example 1.

a) The uniform probability measure on a smooth curve γ (for instance, a circle or a closed segment of the line) is a singularly continuous (w.r.t. λ_2) measure of GC-type (the curve γ is a nowhere dense (in R^2) set of zero two-dimensional Lebesgue measure, and $\forall x \in \gamma, \forall \varepsilon > 0$: $\lambda_2(S_\mu \cap O_\varepsilon(x)) = 0$.

b) Let μ be the probability measure, which is "uniformly distributed" on the classical Sierpinski square (carpet). This measure can be considered as a probability measure with independent W^*-symbols with $Q_1^* = Q_2^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $p_{0k} = p_{1k} = p_{2k} = p_{3k} = p_{5k} = p_{6k} = p_{7k} = p_{8k} = \frac{1}{8}, \quad p_{4k} = 0$. This measure is of GC-type (the spectrum itself can be taken instead of the set E, which has been mentioned in the definition).

c) Let F be a "fat" Sierpinski square (carpet)(it looks like the classical nowhere dense Sierpinski square, but here this set is of positive two-dimensional Lebesgue measure). The set F can be constructed as the spectrum of the following probability measure with independent W^*-symbols: $q_{0k} = q_{2k} = q'_{0k} = q'_{2k} = \frac{1}{2} - \frac{1}{2^{k+1}}, q_{1k} = q'_{1k} = \frac{1}{2^k}, k \in N$ and $p_{0k} = p_{1k} = p_{2k} = p_{3k} = p_{5k} = p_{6k} = p_{7k} = p_{8k} = \frac{1}{8}, \quad p_{4k} = 0$. Geometrically this set can be considered as a result of the known procedure of deleting (from the unit square) the interior parts of the cylinders $\square_4, \square_{i_1 4}, \square_{i_1 i_2 4}, ...$ of different rank with $i_j \in \{0, 1, 2, 3, 5, 6, 7, 8\}$.

Let us order the deleted cylinders (rectangles), and let μ_i be a probability measure, which is uniformly distributed on the diagonal of the $i-$th deleted rectangle. Finally, let

$$\mu = \sum_{i=1}^{\infty} \frac{\mu_i}{2^i}.$$

The spectrum of the measure μ has the following structure:

$$S_\mu = (\bigcup_{i=1}^{\infty} S_{\mu_i}) \bigcup F.$$

So, $\lambda_2(S_\mu) > 0$. Nevertheless this measure is of pure GC-type (the set $\bigcup_{i=1}^{\infty} S_{\mu_i}$ can be taken instead of the set E, which has been mentioned in the definition).

Remark 1. The spectrum of a singularly continuous (w.r.t. λ_n) probability measure of the pure GC-type can be of zero n-dimensional Lebesgue measure as well as of positive such a measure.

Remark 2. If the spectrum of a measure μ is of zero n-dimensional Lebesgue measure, then μ is of pure GC-type in R^n.

Definition 2. A singularly continuous (w.r.t. λ_n) probability measure μ on $\mathbb{R}^n$ is said to be of GP-type if there exists a nowhere dense set $E \subset S_\mu$ such that $\mu(E) = 1$ and $\forall x \in E$, $\forall \varepsilon > 0$: $\lambda_n(S_\mu \cap O_\varepsilon(x)) > 0$.

Example 2.

Let μ be the probability measure with independent W^*-symbols generated by the following matrices Q_1^*, Q_2^*, and $\|p_{ik}\|$: $q_{0k} = q_{2k} = q'_{0k} = q'_{2k} = \frac{1}{2} - \frac{1}{2^{k+1}}, q_{1k} = q'_{1k} = \frac{1}{2^k}, k \in N$ and $p_{0k} = p_{2k} = p_{6k} = p_{8k} = \frac{3}{16}$; $p_{1k} = p_{3k} = p_{5k} = p_{7k} = \frac{1}{16}, \quad p_{4k} = 0$. Directly from Theorem 8 it follows that μ is singularly continuous w.r.t. λ_2. Its spectrum coincides with the "fat" Sierpinski square which has been discussed in example 1 c). The measure μ is of pure GP-type (the spectrum S_μ itself can be taken instead of the set E, which has been mentioned in the definition).

Definition 3. A singularly continuous (w.r.t. the n-dimensional Lebesgue measure λ_n) probability measure μ on $\mathbb{R}^n$ is said to be of GS-type if there exists an open set $A \subset S_\mu$ such that $\mu(A) = 1$.

Remark. Since the maximal open set, which belongs to the spectrum, coincides with the interior part of the spectrum, a singularly continuous (w.r.t. λ_n) probability measure μ on $\mathbb{R}^n$ will be of GS-type if and only if $\mu(int(S_\mu)) = 1$.

Example 3.

a) Let μ be the probability measure with independent W^*-symbols generated by the following matrices Q_1^*, Q_2^*, and $\|p_{ik}\|$: $q_{0k} = q_{1k} = q'_{0k} = q'_{1k} = \frac{1}{2}$, and $p_{0k} = p_{3k} = \frac{1}{6}, p_{1k} = p_{2k} = \frac{1}{3}$, $k \in N$.

Directly from Theorem 8 it follows that μ is singularly continuous w.r.t. λ_2. Its spectrum coincides with the whole unit square. The measure μ is of pure GS-type (the interior part of the spectrum S_μ can be taken instead of the set E, which has been mentioned in the definition). This measure is called "classical" GS-measure *on the unit square.*

b) Let F_0 be the classical Sierpinski square (see example 1 b) for details), and let Δ_i be the i-th square which has been deleted from the unit square during the process of construction of the set F_0. Let μ_i be the "classical" probability GS-measure on the square Δ_i, and let

$$\mu = \sum_{i=1}^{\infty} \frac{\mu_i}{2^i}.$$

It is clear that μ is singularly continuous w.r.t. λ_2. Its spectrum consists of a countable number of squares S_{μ_i} and the nowhere dense set F_0 ($\lambda_2(F_0) = 0$), which belongs to the closure of the union of the above mentioned squares. This measure is of pure GS-type (the set $\bigcup_{i=1}^{\infty} int(S_{\mu_i})$ can be taken instead of the set E, which has been mentioned in the definition).

c) If we put a "fat" Sierpinski square F instead of the set F_0 in the example 3b), we shall also have a singularly continuous probability measure of the pure GS-type. Its spectrum will be of the following structure:

$$S_\mu = (\bigcup_{i=1}^{\infty} S_{\mu_i}) \bigcup F,$$

where S_{μ_i} are rectangles and the Lebesgue measure of the set F can be even greater than the Lebesgue measure of $(\bigcup_{i=1}^{\infty} S_{\mu_i})$.

The family of singularly continuous probability measures on R^n is rather wide and diverse, and there exist, of course, singularly continuous probability measures which are not of the above pure types. Nevertheless, the following theorem establishes the structure of any measure from this family.

Theorem 10. *Any singularly continuous (w.r.t. λ_n) probability measure μ on $\mathbb{R}^n$ can be represented in the following form*

$$\mu = \alpha_1 \mu^{GS} + \alpha_2 \mu^{GC} + \alpha_3 \mu^{GP},$$

where $\alpha_i \geq 0$, $\sum\limits_{i=1}^{3} \alpha_i = 1$ and μ^{GS}, μ^{GC} resp. μ^{GP} are singularly continuous probability measures of $GS-, GC-$ resp. GP-type.

Proof. The proof of the theorem can be naturally split into the proofs of the following two Lemmas.

Lemma 6. *Any singularly continuous (w.r.t. λ_n) probability measure μ on $\mathbb{R}^n$ can be represented in the form*

$$\mu = \beta_1 \mu^{GS} + \beta_2 \mu^{T^*},$$

$\beta_i \geq 0, \beta_1 + \beta_2 = 1$, and μ^{T^*} is a singularly continuous probability measure with a nowhere dense (in R^n) spectrum .

Proof. If the measure μ is of GS-type or the spectrum S_μ is nowhere dense then the above representation is obvious. Now let μ be not of GS-type and let its spectrum S_μ be not nowhere dense. Let $S = intS_\mu$ (S is not empty because S_μ is a perfect set (i.e., it is a closed set without isolated points) as the spectrum of a continuous measure, and S_μ is not nowhere dense by assumption), and we have $\mu(S) \in (0, 1)$. We put $\beta_1 = \mu(S)$ and

$$\mu^{GS}(E) := \frac{1}{\mu(S)} \mu(E \cap S), \ \forall E \in \mathcal{B}(\mathbb{R}^n).$$

The measure μ^{GS} is a probability measure and its property of being singular continuous follows from the singular continuity of μ. The measure μ^{GS} is of GS-type by definition.

Let us consider the set $T = S_\mu \setminus S$. T is nowhere dense because the spectrum S_μ is a perfect set and the set S contains all the interior points of S_μ. Let us set $\beta_2 = \mu(T) = 1 - \mu(S) \in (0, 1)$ and

$$\mu^{T^*}(E) := \frac{1}{\mu(T)}\mu(E \cap T),\ \forall E \in \mathcal{B}(\mathbb{R}^n).$$

The measure μ^{T^*} is a probability measure and its singular continuity also follows directly from the singular continuity of μ. It is clear that $\mu^{T^*}(T) = 1$. Therefore the set $S_{\mu^{T^*}}$, being a subset of the closure of a nowhere dense set T, is nowhere dense.

□

Lemma 7. *Any singularly continuous (w.r.t. λ_n) probability measure μ^{T^*} on $\mathbb{R}^n$ with a nowhere dense spectrum can be represented in the form*

$$\mu^{T^*} = \gamma_1\mu^{GC} + \gamma_2\mu^{GP},$$

where $\gamma_i \geq 0$ and $\gamma_1 + \gamma_2 = 1$.

Proof. Every point of the spectrum $S_\mu^{T^*}$ belongs to one of the following sets

$$T_C = \{x : x \in S_\mu^{T^*}, \exists\varepsilon(x) > 0 : \lambda_n(S_\mu^{T^*} \cap O_\varepsilon(x)) = 0\},$$

$$T_P = \{x : x \in S_\mu^{T^*}, \forall\varepsilon > 0 : \lambda_n(S_\mu^{T^*} \cap O_\varepsilon(x)) > 0\}.$$

It is obvious that $T_C \cap T_P = \emptyset$ and $T_C \cup T_P = S_\mu^{T^*}$.

Let us remark that the set T_P is closed. Indeed, let $\{x_n\}$ be a sequence of points from T_P which converges to a point x_0. Since the set $S_{\mu^{T^*}}$ is closed being the spectrum of a measure, we have $x_0 \in S_{\mu^{T^*}}$. Therefore, x_0 belongs either to the set T_P or to the set T_C. Suppose that $x_0 \in T_C$. Then there exists $\varepsilon(x_0) > 0$ such that $\lambda(O_{\varepsilon(x_0)}(x_0) \bigcap S_{\mu^{T^*}}) = 0$. On the other hand there exists a positive integer N_0 such that $x_n \in O_{\varepsilon(x_0)}(x_0)$ for all $n > N_0$. Let us choose $n > N_0$ and $\varepsilon_1 > 0$ such that $O_{\varepsilon_1}(x_n) \subset O_{\varepsilon(x_0)}(x_0)$. Since $x_n \in T_P$, we have $\lambda(O_{\varepsilon(x_0)}(x_0) \bigcap S_{\mu^{T^*}}) \geq \lambda(O_{\varepsilon_1}(x_n) \bigcap S_{\mu^{T^*}}) > 0$. This contradiction shows that $x_0 \in T_P$, which proves that T_P is a closed set.

Therefore, the set $T_C = S_{\mu^{T^*}} \setminus T_P$ is also a Borel set. Let us show that $\lambda_n(T_C) = 0$. By the definition of the set T_C, for any $x \in T_C$ there exists $\varepsilon(x) > 0$ such that the set $O_{\varepsilon(x)}(x) \bigcap S_{\mu^{T^*}}$ is of zero Lebesgue measure. Therefore, there exists a point $y(x)$ whose coordinates are rationals and a positive rational $\varepsilon^*(y(x))$ such that $x \in O_{\varepsilon^*(y(x))}(y(x)) \subset O_{\varepsilon(x)}(x)$. It is clear that $\lambda_n(O_{\varepsilon^*(y(x))}(y(x)) \bigcap S_{\mu^{T^*}}) = 0$.

Since every point x from the set T_C belongs to the set $O_{\varepsilon^*(y(x))}(y(x)) \bigcap S_{\mu^{T^*}}$, we have

$$T_C \subset \bigcup_{x \in T_C} (O_{\varepsilon^*(y(x))}(y(x)) \bigcap S_{\mu^{T^*}}).$$

Since all coordinates of points $y(x)$ are rational and $\varepsilon^*(y(x))$ is also rational (by the construction), we deduce that the latter union contains an at most countable number of *different* subsets. Each subset from this union is of zero Lebesgue measure (by the construction). Therefore, $\lambda_n(T_C) = 0$ and $\lambda_n(T_P) = \lambda_n(S_{\mu^{T^*}})$.

If $\mu^{T^*}(T_C) = 1$ then μ^{T^*} is of GC-type (by definition) and the representation is valid with $\gamma_1 = 1$, $\gamma_2 = 0$; $\mu^{GC} = \mu^{T^*}$ and one can choose any measure of the GP-type instead of the measure μ^{GP} (see examples before the Theorem).

If $\mu^{T^*}(T_P) = 1$ then μ^{T^*} is of GP-type (by definition), and the representation is valid with $\gamma_1 = 0$, $\gamma_2 = 1$; $\mu^{GP} = \mu^{T^*}$ and one can choose any measure of the GC-type instead of the measure μ^{GC}.

If $0 < \mu^{T^*(T_C)} < 1$ then we the define measures

$$\mu^{GP}(E) := \frac{1}{\mu^{T^*}(T_P)}\mu^{T^*}(E \cap T_P),\ \forall E \in \mathcal{B}(\mathbb{R}^n),$$

$$\mu^{GC}(E) := \frac{1}{\mu^{T^*}(T_C)}\mu^{T^*}(E \cap T_C),\ \forall E \in \mathcal{B}(\mathbb{R}^n).$$

The measures μ^{GP} and μ^{GC} are singulary continuous (because μ^{T^*} is singulary continuous).

Let $\widetilde{T}_P = T_P \bigcap S_{\mu^{GP}}$. It is obvious that $\mu^{GP}(\widetilde{T}_P) = 1$. The set T_P is closed and it is a subset of $S_{\mu^{GP}}$. On the other hand the spectrum $S_{\mu^{GP}}$ is the *minimal* closed support of the measure μ^{GP}. Therefore, $\widetilde{T}_P = S_{\mu^{GP}}$ (the set T_P itself, generally speaking, can contain the set $S_{\mu^{GP}}$ as a proper subset). So, the measure μ^{GP} is of GP-type (the set $\widetilde{T}_P$ can be taken instead of the set E, which has been mentioned in the definition of a measure of GP-type).

Let $\widetilde{T}_C = T_C \bigcap S_{\mu^{GC}}$. It is also obvious that $\mu^{GC}(\widetilde{T})_C = 1$, and $\widetilde{T}_C \subset S_{\mu^{GC}}$. Therefore, μ^{GC} is a measure of GC-type (one can put the set $\widetilde{T}_C$ instead of the set E, which has been mentioned in the definition of a measure of GC-type).

By putting $\gamma_1 = \mu^{T^*}(T_C)$ and $\gamma_2 = \mu^{T^*}(T_P)$ we have $\forall E \in \mathcal{B}(\mathbb{R}^n)$:

$$\mu^{T^*}(E) = \mu^{T^*}(E \cap S_\mu^{T^*}) = \mu^{T^*}(E \cap (T_C \cup T_P)) = \mu^{T^*}(E \cap T_C) + \mu^{T^*}(E \cap T_P) =$$

$$= \gamma_1 \frac{1}{\mu^{T^*}(T_C)}\mu(E \cap T_C) + \gamma_2 \frac{1}{\mu^{T^*}(T_P)}\mu(E \cap T_P) =$$

$$= \gamma_1 \mu^{GC}(E) + \gamma_2 \mu^{GP}(E).$$

□

The proof of the Theorem follows directly from the latter lemmas. □

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