



Measure and Integration Exercises 3

1. Suppose $\Psi : [a, b] \rightarrow \mathbb{R}$ is non-decreasing, and $f : [a, b] \rightarrow \mathbb{R}$ is Ψ -Riemann integrable. Assume that $m < M$, and $m \leq f(x) \leq M$. Let $g : [m, M] \rightarrow \mathbb{R}$ be continuous. Show that the function $g \circ f : [a, b] \rightarrow \mathbb{R}$ is Ψ -Riemann integrable.
2. Let $A, B \subseteq \mathbb{R}^N$. Prove the following.
 - (a) If $|A|_e = 0$, then $|A \cup B|_e = |B|_e$.
 - (b) If $|A \Delta B|_e = 0$, then $|A \cup B|_e = |A|_e = |B|_e = |A \cap B|_e$.
3. Let K_1, K_2 be compact subsets of \mathbb{R}^n such that $K_1 \cap K_2 = \emptyset$. Show that

$$|K_1 \cup K_2|_e = |K_1|_e + |K_2|_e.$$

4. Let F be a closed subset of \mathbb{R}^N . For each $n \geq 1$, let

$$G_n = \{x \in \mathbb{R}^N : |x - y| < \frac{1}{n} \text{ for some } y \in F\}.$$

- (a) Show that G_n is open for each $n \geq 1$.
- (b) Show that $F = \bigcap_{n=1}^{\infty} G_n$.
- (c) Conclude that $\mathcal{F} \subseteq \mathcal{O}_\delta$. Here \mathcal{F} denotes the collection of all closed subset of \mathbb{R}^N , and \mathcal{O}_δ denotes the collections of all subsets of \mathbb{R}^N that can be written as the countable intersection of open sets of \mathbb{R}^N .