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Measure and Integration Exercises 15

1. Let $(E, \mathcal{B}, \mu)$ be a measure space. Show (without using Cauch-Schwartz inequality) that if $f, g \in L^2(\mu)$, then
$$\int_E |fg| \, d\mu \leq ||f||_{L^2(\mu)} \, ||g||_{L^2(\mu)}.$$
This is known as Hölders inequality. (Hint: for any real numbers a, b one has $2|ab| \leq a^2 + b^2$, why?)

2. Let $(E, \mathcal{B}, \mu)$ be a finite measure space. Show that $L^2(\mu) \subseteq L^1(\mu)$. Show that the result is not true in case μ is not a finite measure

3. Let μ and ν be two measures on the measure space $(E, \mathcal{B})$ such that $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{B}$. Show that if f is any non-negative measurable function on $(E, \mathcal{B})$, then $\int_E f \, d\mu \leq \int_E f \, d\nu$. Conclude that if ν is a finite measure, then $L^2(\nu) \subseteq L^1(\nu) \subseteq L^1(\mu)$.

4. Let $(E, \mathcal{B})$ be a measurable space, and μ_1, μ_2 and ν measures on $(E, \mathcal{B})$. Show the following:

   (a) If $\mu_1 \perp \nu$ and $\mu_2 \perp \nu$, then $\mu_1 + \mu_2 \perp \nu$.

   (b) If $\mu_1 \ll \nu$ and $\mu_2 \perp \nu$, then $\mu_1 \perp \mu_2$.

   (c) If $\mu_1 \ll \nu$ and $\mu_1 \perp \nu$, then μ_1 is the zero measure.