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Measure and Integration Exercises 9

1. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda_{\mathbb{R}})$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra over $\mathbb{R}$ and $\lambda_{\mathbb{R}}$ is Lebesgue measure on $\mathcal{B}_{\mathbb{R}}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2^{-k} & \text{if } x \in [k, k+1),\ k \geq 0. \end{cases}$$

   (a) Show that f is measurable, i.e. $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for all $B \in \mathcal{B}_{\mathbb{R}}$.

   (b) Show that $\int_{\mathbb{R}} f \, d\lambda_{\mathbb{R}} = 2$.

2. Let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that $\int_E |f_n| d\mu = \alpha$ for all $n \geq 1$, where $\alpha \in (0, \infty)$. Let

$$A_n = \{x : |f_n(x) - \int_E f_n d\mu| \geq n^2\}.$$

   Show that if $\mu(E) = 1$, then $\mu\left(\limsup_{n\to\infty} A_n\right) = 0$.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $f, g \in L^1(\mu)$, i.e. $\int_E |f| d\mu < \infty$ and $\int_E |g| d\mu < \infty$. Show that $\mu(f \neq g) = 0$ **if and only if** $\int_B f \, d\mu = \int_B g \, d\mu$ for all $B \in \mathcal{B}$.