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Measure and Integration: Extra Exercises Chapter 13

1. Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Let $f : X \to [0, \infty)$, $g : Y \to [0, \infty)$ be $\mathcal{A}/\mathcal{B}(\mathbb{R})$ respectively $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable functions. Define $h : X \times Y \to [0, \infty)$ by $h(x, y) = f(x)g(y)$.

   (i) Show that h is $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.

   (ii) Prove that $\int_{X \times Y} h(x, y)\, d(\mu \otimes \nu)(x, y) = \int_X f(x)\, d\mu(x) \cdot \int_Y g(y)\, d\nu(y)$.

2. Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x, y) = e^{-xt}$) that $\int_{[0,\infty)} (e^{-at} - e^{-bt}) \frac{1}{t} d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure.