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Measure and Integration 2006-Extra exercises

1. Let $(X, \mathcal{B}, \mu)$ be a probability space, i.e. $\mu(X) = 1$. Let $f : X \to [0, 1)$ be a measurable function such that $\mu\left(f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n})\right) = \frac{1}{2^n}$ for $n \geq 1$ and $k = 0, 1, \cdots, 2^n - 1$. Show that $\int_X f^2\, d\mu = \frac{1}{3}$.

2. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f \in L^1(\mu)$ is **strictly** positive. Prove that $(X, \mathcal{B}, \mu)$ is σ-finite.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, f \in \mathcal{L}^p(\mu)$ be such that $f = \mathcal{L}^p(\mu) - \lim_{n\to\infty} f_n$. Prove that for every $\epsilon > 0$, one has

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0.$$

4. Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every non-negative measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\, d\mu = \int_E Fh\, d\nu.$$

Conclude that the result is still true for $F \in \mathcal{L}^1(\mu)$ which is not necessarily non-negative.

5. Let $E = \{(x, y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = y \sin x\, e^{-xy}$.
   (a) Show that f is $\lambda \times \lambda$ integrable on E.
   (b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty \frac{\sin x}{x} \left(\frac{1 - e^{-x}}{x} - e^{-x} \right) dx = \frac{1}{2} \log 2.$$