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Measure and Integration 2006-Selected Solutions Chapter 11

1. (**p.100, exercise 11.1**) Let $(X, \mathcal{A}, \mu)$ be a measure space, and (u_j) a sequence of measrable real valued functions such that $\lim_{j\to\infty} u_j(x) = u(x)$ for all $x \in X$. Suppose that $|u_j| \le g$ for some measurable function g such that $g^p \in \mathcal{L}^1_+$, $p > 0$. Show that $\lim_{j\to\infty} \int |u_j - u|^p \, d\mu = 0$.

   Proof: First notice that for any $a, b \in \mathbb{R}$, one has

   $$|a-b|^p \le (|a|+|b|)^p \le (2\max(|a|,|b|))^p = 2^p \max(|a|^p, |b|^p) \le 2^p(|a|^p + |b|^p).$$

   Applying this fact to our sequence, we see that $|u_j(x) - u(x)|^p \le 2^p g^p(x)$ (note that $|u_j| \le g$ implies $|u| \le g$), and g^p is a non-negative integrable function. Furthermore, $\lim_{j\to\infty} |u_j - u|^p = 0$, hence by Lebesgue Dominated Convergence Theorem,

   $$\lim_{j\to\infty} \int |u_j - u|^p \, d\mu = \int \lim_{j\to\infty} |u_j - u|^p \, d\mu = 0.$$

2. (**p.100, exercise 11.3**) Let (f_k), (g_k) and (G_k) be sequences of integrable functions on a measure space $(X, \mathcal{A}, \mu)$. If

   (i) $\lim_{k\to\infty} f_k(x) = f(x)$, $\lim_{k\to\infty} g_k(x) = g(x)$ and $\lim_{k\to\infty} G_k(x) = G(x)$ for all $x \in X$,

   (ii) $g_k(x) \le f_k(x) \le G_k(x)$ for all $k \ge 1$ and all $x \in X$,

   (iii) $\lim_{k\to\infty} \int g_k \, d\mu = \int g \, d\mu$, $\lim_{k\to\infty} \int G_k \, d\mu = \int G \, d\mu <$ and both $\int g \, d\mu$ and $\int G \, d\mu$ are finite,

   then, $\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$ and $\int f \, d\mu$ is finite.

   Proof: By assumption $0 \le f_k - g_k \to f - g$ and $0 \le G_k - f_k \to G - f$. By Fatou's Lemma we have

   $$\begin{aligned}
   \int (f-g)\, d\mu &= \int \lim_{k\to\infty}(f_k - g_k)\, d\mu \\
   &= \int \liminf_{k\to\infty}(f_k - g_k)\, d\mu \\
   &\le \liminf_{k\to\infty} \int (f_k - g_k)\, d\mu \\
   &\le \liminf_{k\to\infty} \int f_k \, d\mu - \limsup_{k\to\infty} \int g_k \, d\mu \\
   &= \liminf_{k\to\infty} \int f_k \, d\mu - \int g \, d\mu.
   \end{aligned}$$

   Subtracting $\int g \, d\mu (< \infty)$ from both sides of the inequality, we get $\int f \, d\mu \le \liminf_{k\to\infty} \int f_k \, d\mu$. On the other hand,

   $$\begin{aligned}
   \int (G-f)\, d\mu &= \int \lim_{k\to\infty}(G_k - f_k)\, d\mu \\
   &= \int \liminf_{k\to\infty}(G_k - f_k)\, d\mu \\
   &\le \liminf_{k\to\infty} \int (G_k - f_k)\, d\mu \\
   &\le \liminf_{k\to\infty} \int G_k \, d\mu - \limsup_{k\to\infty} \int f_k \, d\mu \\
   &= \int G \, d\mu - \limsup_{k\to\infty} \int f_k \, d\mu.
   \end{aligned}$$

Subtracting $\int G\,d\mu (< \infty)$ from both sides of the inequality we get $\limsup_{k\to\infty}\int f_k\,d\mu \leq \int f\,d\mu \leq \liminf_{k\to\infty}\int f_k\,d\mu$. Thus, $\int f\,d\mu = \lim_{k\to\infty}\int f_k\,d\mu$ and $\int g\,d\mu \leq \int f\,d\mu \leq \int G\,d\mu$, hence $\int f\,d\mu$ is finite.

3. (**p.100, exercise 11.4**) Let $(X, \mathcal{A}, \mu)$ be a measure space, and let (g_n) be a sequence of μ-integrable functions on X such that $\sum_{n=1}^{\infty}\int_{|} g_n|\,d\mu < \infty$. Show that $\sum_{n=1}^{\infty} g_n$ is finite μ a.e, and
$$\int \sum_{n=1}^{\infty} g_n\,d\mu = \sum_{n=1}^{\infty}\int g_n\,d\mu.$$

proof (b): By part Corollary 9.9, $\int\sum_{n=1}^{\infty}|g_n|\,d\mu = \sum_{n=1}^{\infty}\int|g_n|\,d\mu < \infty$, hence $\sum_{n=1}^{\infty}|g_n|$ is μ-integrable. We show that $u = \sum_{n=1}^{\infty}|g_n|$ is finite μ a.e. (see also the proof of Corollary 10.13). Let $N = \{x \in X : u(x) = \infty\}$. Then $N = \bigcap_{n=1}^{\infty}\{u \geq n\}$. Since the sequence of measurable sets $\{u \geq n\}$ is decreasing and by the Markov inequality each has finite measure, then $\mu(N) = \lim_{n\to\infty}\mu(\{u \geq n\}) = \lim_{n\to\infty}\frac{1}{n}\int u\,d\mu = 0$. Thus, $u = \sum_{n=1}^{\infty}|g_n|$ is finite μ a.e. Since $|\sum_{n=1}^{\infty} g_n| \leq \sum_{n=1}^{\infty}|g_n|$, it follows that $\sum_{n=1}^{\infty} g_n$ is finite μ a.e. Let $h_n = \sum_{m=1}^{n} g_m$, then (h_m) converges to $\sum_{n=1}^{\infty} g_n$ μ a.e. Furthermore, $|h_n| \leq \sum_{n=1}^{\infty}|g_n|$, thus by the Dominated Convergence Theorem,
$$\sum_{n=1}^{\infty}\int g_n\,d\mu = \lim_{n\to\infty}\int h_n\,d\mu = \int \lim_{n\to\infty} h_n\,d\mu = \int\sum_{n=1}^{\infty} g_n\,d\mu.$$

4. (**p.100, exercise 11.6**) Give an example of a sequence (u_j) of integrable functions such that $u_j(x) \to u(x)$ for all x where u is an integrable function, but $\lim_{j\to\infty}\int u_j\,d\mu \neq \int u\,d\mu$. Why doesn't this contradict the Lebesgue Dominated Convergence Theorem?

Proof: Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra, and λ the Lebesgue measure. Let $u_j(x) = j\mathbf{1}_{(0,1/j)}(x)$, $j \geq 1$. Clearly, $\lim_{j\to\infty} u_j(x) = 0$ for all $x \in \mathbb{R}$, and
$$\lim_{j\to\infty}\int u_j\,d\lambda = \lim_{j\to\infty} j\lambda((0,1/j)) = \lim_{j\to\infty} j\frac{1}{j} = 1,$$
while
$$\int \lim_{j\to\infty} u_j\,d\lambda = \int 0\,d\lambda = 0.$$

This does not contradict the Lebesgue Dominated Convergence Theorem because the sequence (u_j) is not bounded by an integrable function.

5. (**p.100, exercise 11.8**) Check whether the following functions are Lebesgue integrable:
(i) $u(x) = \dfrac{1}{x}$, $x \in [1,\infty)$, (ii) $v(x) = \dfrac{1}{x^2}$, $x \in [1,\infty)$,
(iii) $w(x) = \dfrac{1}{\sqrt{x}}$, $x \in (0,1]$, (iv) $y(x) = \dfrac{1}{x}$, $x \in (0,1]$.

Proof: The functions in (i) and (iv) are not Lebesgue integrable, while the functions in (ii) and (iii) are Lebesgue integrable. We will prove (i) and (iii) only. The proofs of (ii) and (iv) are similar.

Notice that $u(x) = \frac{1}{x}\mathbf{1}_{[1,\infty)}(x)$, and $u(x) = \sup_n \frac{1}{x}\mathbf{1}_{[1,n)}(x)$. Since the function $\frac{1}{x}\mathbf{1}_{[1,n)}(x)$ is Riemann integrable, then it is also Lebesgue integrable and the Riemann integral equals the Lebesgue integral (Theorem 11.8). Thus, by Beppo-Levi,

$$\int u(x)\,d\lambda(x) = \lim_{n\to\infty}\int \frac{1}{x}\mathbf{1}_{[1,n)}(x)\,d\lambda(x) = \lim_{n\to\infty}\int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty}(\log n - \log 1) = \infty.$$

Thus, u is not Lebesgue integrable.

Now consider the function $w(x) = \dfrac{1}{\sqrt{x}}\mathbf{1}_{(0,1]}$, and notice that $w(x) = \sup_n \frac{1}{\sqrt{x}}\mathbf{1}_{[1/n,1]}$ where $(\frac{1}{\sqrt{x}}\mathbf{1}_{[1/n,1]})$ is an increasing sequence of Riemann integrable functions. Hence, by Beppo-Levi,

$$\int w(x)\,d\lambda(x) = \lim_{n\to\infty}\int \frac{1}{\sqrt{x}}\mathbf{1}_{[1/n,1]},d\lambda(x) = \lim_{n\to\infty}\int_{1/n}^1 \frac{1}{\sqrt{x}}\,dx = \lim_{n\to\infty}(2-2\sqrt{1/n}) = 2 < \infty.$$

Thus, w is Lebesgue integrable.

If all the intervals are replaced by $[1/2,2]$, then all fuctions under consideration (u,v,w,y) are Riemann integrable and therefore Lebesgue integrable.

6. (**p.100, exercise 11.12(i)**) Let λ be the one-dimensional Lebesgue measure. Prove that
$$\int_{(1,\infty)} e^{-x}\ln(x)\,d\lambda(x) = \lim_{k\to\infty}\int_{(1,k)}\left(1-\frac{x}{k}\right)^k \ln(x)\,d\lambda(x).$$

Proof: It is easy to see that for any $0 < x < 1$, one has $\ln(1-x) < -x$ (why?), i.e. $(1-x) < e^{-x}$. Hence, for any $k \ge 1$ we have (notice that $\ln(x) \le x$ on $(1,\infty)$)

$$\mathbf{1}_{(1,k)}(x)\left(1-\frac{x}{k}\right)^k \ln(x) \le \mathbf{1}_{(1,k)}(x)e^{-x}\ln(x) \le \mathbf{1}_{(1,\infty)}(x)e^{-x}\ln(x) \le \mathbf{1}_{(1,\infty)}(x)xe^{-x}.$$

It is easy to see that the function $\mathbf{1}_{(1,\infty)}(x)xe^{-x}$ is Riemann integrable, and hence is Lebesgue integrable. Furthermore, $\lim_{k\to\infty}\mathbf{1}_{(1,k)}(x)\left(1-\frac{x}{k}\right)^k\ln(x) = \mathbf{1}_{(1,\infty)}(x)e^{-x}\ln(x)$, thus by the Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}
\int_{(1,\infty)} e^{-x}\ln(x)\,d\lambda(x) &= \int \mathbf{1}_{(1,\infty)}(x)e^{-x}\ln(x)\,d\lambda(x)\\
&= \int \lim_{k\to\infty}\mathbf{1}_{(1,k)}(x)\left(1-\tfrac{x}{k}\right)^k\ln(x)\,d\lambda(x)\\
&= \lim_{k\to\infty}\int \mathbf{1}_{(1,k)}(x)\left(1-\tfrac{x}{k}\right)^k\ln(x)\,d\lambda(x)\\
&= \lim_{k\to\infty}\int_{(1,k)}(x)\left(1-\tfrac{x}{k}\right)^k\ln(x)\,d\lambda(x).
\end{aligned}$$