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Measure and Integration 2007-More Selected Solutions Chapter 11

1. (**p.100, exercise 11.8**) Check whether the following functions are Lebesgue integrable:
(i) $u(x) = \frac{1}{x},\ x \in [1, \infty)$, (ii) $v(x) = \frac{1}{x^2},\ x \in [1, \infty)$,
(iii) $w(x) = \frac{1}{\sqrt{x}},\ x \in (0, 1]$, (iv) $y(x) = \frac{1}{x},\ x \in (0, 1]$.

Proof: The functions in (i) and (iv) are not Lebesgue integrable, while the functions in (ii) and (iii) are Lebesgue integrable. We will prove (i) and (iii) only. The proofs of (ii) and (iv) are similar.

Notice that $u(x) = \frac{1}{x}\mathbf{1}_{[1,\infty)}(x)$, and $u(x) = \sup_n \frac{1}{x}\mathbf{1}_{[1,n]}(x)$. Since the function $\frac{1}{x}\mathbf{1}_{[1,n]}(x)$ is Riemann integrable, then it is also Lebesgue integrable and the Riemann integral equals the Lebesgue integral (Theorem 11.8). Thus, by Beppo-Levi,

$$\int u(x)\, d\lambda(x) = \lim_{n\to\infty} \int \frac{1}{x}\mathbf{1}_{[1,n]}(x)\, d\lambda(x) = \lim_{n\to\infty} \int_1^n \frac{1}{x}\, dx = \lim_{n\to\infty} (\log n - \log 1) = \infty.$$

Thus, u is not Lebesgue integrable.

Now consider the function $w(x) = \frac{1}{\sqrt{x}}\mathbf{1}_{(0,1]}$, and notice that $w(x) = \sup_n \frac{1}{\sqrt{x}}\mathbf{1}_{[1/n,1]}$ where $(\frac{1}{\sqrt{x}}\mathbf{1}_{[1/n,1]})$ is an increasing sequence of Riemann integrable functions. Hence, by Beppo-Levi,

$$\int w(x)\, d\lambda(x) = \lim_{n\to\infty} \int \frac{1}{\sqrt{x}}\mathbf{1}_{[1/n,1]}, d\lambda(x) = \lim_{n\to\infty} \int_{1/n}^1 \frac{1}{\sqrt{x}}\, dx = \lim_{n\to\infty} (2-2\sqrt{1/n}) = 2 < \infty.$$

Thus, w is Lebesgue integrable.

If all the intervals are replaced by $[1/2, 2]$, then all fuctions under consideration (u, v, w, y) are Riemann integrable and therefore Lebesgue integrable.

2. (**p.100, exercise 11.12(i)**) Let λ be the one-dimensional Lebesgue measure. Prove that

$$\int_{(1,\infty)} e^{-x} \ln(x)\, d\lambda(x) = \lim_{k\to\infty} \int_{(1,k)} \left(1 - \frac{x}{k}\right)^k \ln(x)\, d\lambda(x).$$

Proof: It is easy to see that for any $0 < x < 1$, one has $\ln(1 - x) < -x$ (why?), i.e. $(1 - x) < e^{-x}$. Hence, for any $k \geq 1$ we have (notice that $\ln(x) \leq x$ on $(1, \infty)$)

$$\mathbf{1}_{(1,k)}(x) \left(1 - \frac{x}{k}\right)^k \ln(x) \leq \mathbf{1}_{(1,k)}(x) e^{-x} \ln(x) \leq \mathbf{1}_{(1,\infty)}(x) e^{-x} \ln(x) \leq \mathbf{1}_{(1,\infty)}(x) x e^{-x}.$$

It is easy to see that the function $\mathbf{1}_{(1,\infty)}(x)xe^{-x}$ is Riemann integrable, and hence is Lebesgue integrable. Furthermore, $\lim_{k\to\infty} \mathbf{1}_{(1,k)}(x)\left(1-\frac{x}{k}\right)^k \ln(x) = \mathbf{1}_{(1,\infty)}(x)e^{-x}\ln(x)$, thus by the Lebesgue Dominated Convergence Theorem we have

$$\begin{aligned}
\int_{(1,\infty)} e^{-x}\ \ln(x)\, d\lambda(x) &= \int \mathbf{1}_{(1,\infty)}(x)e^{-x}\ln(x)\, d\lambda(x) \\
&= \int \lim_{k\to\infty} \mathbf{1}_{(1,k)}(x)\left(1-\tfrac{x}{k}\right)^k \ln(x)\, d\lambda(x) \\
&= \lim_{k\to\infty} \int \mathbf{1}_{(1,k)}(x)\left(1-\tfrac{x}{k}\right)^k \ln(x)\, d\lambda(x) \\
&= \lim_{k\to\infty} \int_{(1,k)}(x)\left(1-\tfrac{x}{k}\right)^k \ln(x)\, d\lambda(x).
\end{aligned}$$

3. **(p.100, exercise 15) In the book there are some typos, here is the correct version**. Let X be a non-negative random variable on a probability space $(\Omega, \mathcal{A}, P)$. The function $\Phi_X(t) = \int e^{-tX} dP$, $t \geq 0$, is called *the moment generating function of* X. Suppose $\int X^m\, dP < \infty$.

   (a) Show that for all $0 \leq k \leq m$,

   $$\int X^k\, dP = \lim_{t\downarrow 0}(-1)^k \frac{d^k}{dt^k}\Phi_X(t).$$

   (b) Show that $\Phi_X(t) = \sum_{k=0}^{m-1}(-1)^k t^k \int \frac{X^k}{k!}\, dP + o(t^m)$.

   (c) Show that for all $t \geq 0$,

   $$|\Phi_X(t) - \sum_{k=0}^{m}(-1)^k t^k \int \frac{X^k}{k!}\, dP| \leq \frac{t^m}{m!}\int X^m\, dP.$$

   (d) Suppose that $\int X^k\, dP < \infty$ for all $k \geq 1$. Show that for any $t \geq 0$ such that $\sum_{k=0}^{m}(-1)^k t^k \int \frac{X^k}{k!}\, dP < \infty$, one has $\Phi_X(t) = \sum_{k=0}^{\infty}(-1)^k t^k \int \frac{X^k}{k!}\, dP$.

Proof(a): First notice that for any $k \leq m$,

$$\int X^k\, dP = \int_{\{X>1\}} X^k\, dP + \int_{\{X\leq 1\}} X^k\, dP \leq \int_{\{X>1\}} X^m\, dP + 1 < \infty.$$

So that $\int X^k \in \mathcal{L}^1(P)$, $1 \leq k \leq m$. Further, the function $u(t,\omega) = e^{-tX(\omega)}$ is m times differentiable in t, and

$$|\frac{\partial^k u}{t^k}(t,\omega)| = |(-1)^k X^k(\omega)e^{-tX(\omega)}| \leq X^k \in \mathcal{L}^1(P).$$

It is easily checked that the hypothesis of Theorem 11.5 are satisfied when applied repeatedly to the functions $\frac{\partial^k u}{t^k}$ for $1 \leq k \leq m$. Hence, for each $1 \leq k \leq m$,

$$\frac{d^k\Phi_X}{dt^k}(t) = \int \frac{d^k e^{-tX}}{dt^k}\, dP = \int (-1)^k X^k e^{-tX}\, dP.$$

Therefore, by Theorem 11.4,

$$\lim_{t\downarrow 0}(-1)^k \frac{d^k\Phi_X}{dt^k}(t) = \int X^k\, dP.$$

Proof(b): Since $e^{-tX}, X^k \in \mathcal{L}^1(P)$, for $1 \le k \le m$ and all $t \ge 0$, we have that

$$\Phi_X(t) - \sum_{k=0}^{m-1}(-1)^k t^k \int \frac{X^k}{k!}\, dP = \int (e^{-tX} - \sum_{k=0}^{m-1}(-1)^k t^k \int \frac{X^k}{k!})\, dP < \infty.$$

This implies that $\int \sum_{k=m+1}^{\infty}(-1)^k t^k \int \frac{X^k}{k!})\, dP < \infty$, and hence

$$\int \sum_{k=m+1}^{\infty}(-1)^k t^k \int \frac{X^k}{k!})\, dP = o(t^m).$$

This proves the result.

Proof(c): We first remark that for any $u \ge 0$ and any m, one has

$$|e^{-u} - \sum_{k=0}^{m}\frac{(-u)^k}{k!}| \le \frac{u^m}{m!}.$$

This can be proven easily with the help of induction and the following equality

$$|e^{-u} - \sum_{k=0}^{m}\frac{(-u)^k}{k!}| = |\int_0^u (e^{-y} - \sum_{k=0}^{m-1}\frac{(-y)^k}{k!})\, dy|.$$

Using the above with $u = tX$, we have

$$\begin{aligned} |\Phi_X(t) - \textstyle\sum_{k=0}^{m}(-1)^k t^k \int \frac{X^k}{k!}\, dP| &\le \textstyle\int |e^{-tX} - \sum_{k=0}^{m}(-1)^k t^k \frac{X^k}{k!}|\, dP \\ &\le \textstyle\frac{t^m}{m!}\int X^m\, dP. \end{aligned}$$

Proof(d): Suppose that $\sum_{k=0}^{m}(-1)^k t^k \int \frac{X^k}{k!}\, dP < \infty$, then

$$\lim_{m\to\infty} |(-1)^m t^m \int \frac{X^m}{m!}\, dP| = 0.$$

Hence, by part (c) the result follows.