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Measure and Integration 2007-solutions extra exercises Chapter 13

1. (**Extra exercise 1**) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measure space, and let $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ be the corresponding product measurable space, where $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{A} \times \mathcal{B})$.

   (a) Show that for all $E \in \mathcal{A} \otimes \mathcal{B}$, and for all $x_0 \in X$ and all $y_0 \in Y$, one has $E_{x_0} = \{y \in Y : (x_0, y) \in E\} \in \mathcal{B}$ and $E_{y_0} = \{x \in X : (x, y_0) \in E\} \in \mathcal{A}$.

   (b) Let $f : X \times Y \to \overline{\mathbb{R}}$ be $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\overline{\mathbb{R}})$ measurable. Show that for all $x_0 \in X$ and all $y_0 \in Y$, the functions $f_{x_0} : Y \to \overline{\mathbb{R}}$ and $f_{y_0} : X \to \overline{\mathbb{R}}$ given by $f_{x_0}(y) = f(x_0, y)$ and $f_{y_0}(x) = f(x, y_0)$ are $\mathcal{A}/\mathcal{B}(\mathbb{R})$ respectively $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.

Proof (a): Let

$$\mathcal{C} = \{E \in \mathcal{A} \otimes \mathcal{B} : E_{x_0} \in \mathcal{B}, E_{y_0} \in \mathcal{A} \text{ for all } x_0 \in X, y_0 \in Y\}.$$

To prove part (a), we need to show that $\mathcal{C}$ is a σ-algebra containing $\mathcal{A} \times \mathcal{B}$:
– $\emptyset \in \mathcal{C}$ since $\emptyset_{x_0} = \emptyset \in \mathcal{B}$ and $\emptyset_{y_0} = \emptyset \in \mathcal{A}$.
– Let $E \in \mathcal{C}$, then $(E^c)_{x_0} = (E_{x_0})^c \in \mathcal{B}$ and $(E^c)_{y_0} = (E_{y_0})^c \in \mathcal{A}$, hence $E^c \in \mathcal{C}$.
– Let $E_1, E_2, \ldots \in \mathcal{C}$, then $(\cup_n E_n)_{x_0} = \cup_n (E_{x_0}) \in \mathcal{B}$ and $(\cup_n E_n)_{y_0} = \cup_n (E_{y_0}) \in \mathcal{A}$. Thus, $\cup_n E_n \in \mathcal{C}$. The above show that $\mathcal{C}$ is a σ-algebra. Now let $A \times B \in \mathcal{A} \times \mathcal{B}$, then

$$(A \times B)_{x_0} = \begin{cases} B & x_0 \in A \\ \emptyset & x_0 \notin A \end{cases}$$

so that $(A \times B)_{x_0} \in \mathcal{B}$. Similarly,

$$(A \times B)_{y_0} = \begin{cases} A & y_0 \in B \\ \emptyset & y_0 \notin B \end{cases}$$

so that $(A \times B)_{y_0} \in \mathcal{A}$. Thus $\mathcal{A} \times \mathcal{B} \subset \mathcal{C} \subset \mathcal{A} \otimes \mathcal{B}$ which implies that $\mathcal{A} \otimes \mathcal{B} = \mathcal{C}$. Hence, $E_{x_0} \in \mathcal{B}, E_{y_0} \in \mathcal{A}$ for all $E \in \mathcal{A} \otimes \mathcal{B}$ and all $x_0 \in X, y_0 \in Y$.

Alternatively, notice that $\mathbf{1}_{E_{x_0}}(y) = \mathbf{1}_E(x_0, y)$ and $\mathbf{1}_{E_{y_0}}(x) = \mathbf{1}_E(x, y_0)$. By theorem 13.5, the functions $y \to \mathbf{1}_E(x_0, y) = \mathbf{1}_{E_{x_0}}(y)$ and $x \to \mathbf{1}_E(x, y_0) = \mathbf{1}_{E_{y_0}}(x)$ are measurable, hence $E_{x_0} \in \mathcal{B}, E_{y_0} \in \mathcal{A}$ (see Example 8.5(i) on p. 59).

Proof (b): Let $B \in \mathcal{B}(\overline{\mathbb{R}})$, then $f^{-1}(B) \in \mathcal{A} \otimes \mathcal{B}$. By part (a) we have

$$\begin{aligned} f_{x_0}^{-1}(B) &= \{y \in Y : f_{x_0}(y) = f(x_0, y) \in B\} \\ &= \{y \in Y : (x_0, y) \in f^{-1}(B)\} \\ &= \{y \in Y : y \in (f^{-1}(B))_{x_0} \\ &= (f^{-1}(B))_{x_0} \in \mathcal{B}. \end{aligned}$$

Similarly, $f_{y_0}^{-1}(B) = (f^{-1}(B))_{y_0} \in \mathcal{A}$.

2. (**Extra exercise 2**) Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Let $f : X \to [0, \infty)$, $g : Y \to [0, \infty)$ be $\mathcal{A}/\mathcal{B}(\mathbb{R})$ respectively $\mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable functions. Define $h : X \times Y \to [0, \infty)$ by $h(x, y) = f(x)g(y)$. Show that h is $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.

   Proof. Define $\overline{f} : X \times Y \to [0, \infty)$ by $\overline{f}(x, y) = f(x)$ and $\overline{g} : X \times Y \to [0, \infty)$ by $\overline{g}(x, y) = g(y)$. Then $\overline{f}$ and $\overline{g}$ are $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable since for any $B \in \mathcal{B}(\mathbb{R})$, we have $\overline{f}^{-1}(B) = f^{-1}(B) \times Y \in \mathcal{A} \otimes \mathcal{B}$ and $\overline{g}^{-1}(B) = X \times g^{-1}(B) \in \mathcal{A} \otimes \mathcal{B}$. Now, $h(x, y) = f(x)g(y) = \overline{f}(x, y)\overline{g}(x, y)$ is the product of two $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable functions, hence h is $\mathcal{A} \otimes \mathcal{B}/\mathcal{B}(\mathbb{R})$ measurable.