Measure and Integration 2006-Selected Solutions 13

1. (**Exercise 13.4, p.131**) Denote by λ Lebesgue measure on $(0,1)$. Show that the following iterated integrals exist, but yield different values:

$$\int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(x)d\lambda(y) \neq \int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(y)d\lambda(x).$$

What does this tell about the $(\lambda\times\lambda)$-integral of the function $\frac{x^2-y^2}{(x^2+y^2)^2}$?

Proof: Notice that for each fixed $y\in(0,1)$, the function $x\to\frac{x^2-y^2}{(x^2+y^2)^2}$ is continuous, and is Riemann integrable on $[0,1]$ since

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = -\frac{x}{(x^2+y^2)}\Big|_0^1 = -\frac{1}{1+y^2}.$$

Furthermore, the function $y\to-\frac{1}{1+y^2}$ is continuous and Riemann integrable on $[0,1]$ since

$$\int_0^1 -\frac{1}{1+y^2}dy = -\tan y|_0^1 = -\frac{\pi}{4}.$$

Thus,

$$\int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(x)d\lambda(y) = \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy = -\frac{\pi}{4}.$$

Similar analysis shows that

$$\int_{(0,1)}\int_{(0,1)} \frac{x^2-y^2}{(x^2+y^2)^2}\,d\lambda(y)d\lambda(x) = \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx = \frac{\pi}{4}.$$

Thus the two iterated integrals are not equal. This implies that the function $(x,y)\to\frac{x^2-y^2}{(x^2+y^2)^2}$ is not (Lebesgue) $\lambda\times\lambda$ integrable on $(0,1)\times(0,1)$, otherwise the two integrals would be equal. In fact,

$$\begin{aligned}\int_0^1\int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy\,dx &\geq \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx\\ &= \int_0^1 \frac{1}{2x} = \infty.\end{aligned}$$

2. (**Exercise 13.7, p.131**) Consider $([0,1],\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$, λ is Lebesgue measure and μ is counting measure (i.e. $\mu(A)$ = number of elements in A). Let $\Delta=\{x,y)\in[0,1]\times[0,1]: x=y\}$, show that

$$\int_{[0,1]}\int_{[0,1]} 1_\Delta(x,y)d\lambda(x)d\mu(y) \neq \int_{[0,1]}\int_{[0,1]} 1_\Delta(x,y)d\mu(y)d\lambda(x).$$

Why does not this violate Tonelli's Theorem?

Proof For any $x, y \in [0,1]$, $\Delta_x = \{y \in [0,1] : (x,y) \in \Delta\} = \{x\}$, and $\Delta_y = \{x \in [0,1] : (x,y) \in \Delta\} = \{y\}$. Thus, $\mu(\Delta_x) = \mu(\Delta_y) = 1$ and $\lambda(\Delta_x) = \lambda(\Delta_y) = 0$. Furthermore,

$$1_\Delta(x,y) = 1 \Leftrightarrow 1_{\Delta_x}(y) = 1 \Leftrightarrow 1_{\Delta_y}(x) = 1.$$

Hence,

$$\int_{[0,1]} \int_{[0,1]} 1_\Delta(x,y) d\lambda(x) d\mu(y) = \int_{[0,1]} \lambda(\Delta(y)) d\mu(y) = 0,$$

and

$$\int_{[0,1]} \int_{[0,1]} 1_\Delta(x,y) d\mu(y) d\lambda(x) = \int_{[0,1]} \mu(\Delta(x)) d\lambda(x) = \lambda([0,1]) = 1.$$

The reason why Tonelli's Theorem does not hold is because the measure μ is **not** σ-finite.