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Measure and Integration 2006-Selected Solutions Chapter 3

1. (**Exercise 3.11, p. 21**) According to the book, a family $\mathcal{M} \subset \mathcal{P}(X)$ is a monotone class if it is closed under countable unions and countable intersections.

   (i) Show that for every family $\mathcal{E} \subset \mathcal{P}(X)$, there is a smallest monotone class $m(\mathcal{E})$ containing $\mathcal{E}$.

   (ii) Assume $\emptyset \in \mathcal{E}$, and $E \in \mathcal{E} \Rightarrow E^c \in \mathcal{E}$. Show that $\Sigma = \{B \in m(\mathcal{E}) : B^c \in m(\mathcal{E})\}$ is a σ-algebra.

   (iii) Assume that the assumptions in part (ii) hold, show that $m(\mathcal{E}) = \sigma(\mathcal{E})$.

   Proof (i) Define $m(\mathcal{E}) = \bigcap\{\mathcal{D} : \mathcal{D} \text{ is a monotone class and } \mathcal{E} \subset \mathcal{D}\}$. Since $\mathcal{P}(X)$ is a monotone class containing $\mathcal{E}$, $m(\mathcal{E})$ is well-defined. It remains to show that $m(\mathcal{E})$ is a monotone class. Let $(A_n) \subset m(\mathcal{E})$, then $(A_n) \subset \mathcal{D}$ for each monotone class containing $\mathcal{E}$. Thus, $\cup_n A_n, \cap_n A_n \in \mathcal{D}$ for all $\mathcal{D}$. Therefore, $\cup_n A_n, \cap_n A_n \in m(\mathcal{E})$.

   Proof (ii) Since $\emptyset \in \mathcal{E}$, and $\mathcal{E}$ is closed under complements, we have that $\emptyset, X \in \mathcal{E} \subset m(\mathcal{E})$. This implies that $X \in \Sigma$. From the definition of Σ, it is clear that if $B \in \Sigma$, then $B^c \in \Sigma$. Finally, let $(B_n) \in \Sigma$. Then, $B_n, B_n^c \in \Sigma$ for all n. Since $m(\mathcal{E})$ is closed under countable unions and intersections, it follows that $\cup_n B_n \in m(\mathcal{E})$, and $(\cup_n B_n)^c = \cap_n B_n^c \in m(\mathcal{E})$. Thus, $\cup_n B_n \in \Sigma$, and Σ is a σ-algebra.

   Proof (iii) From part (ii) and the fact that $\sigma(\mathcal{E})$ is also a monotone class containing $\mathcal{E}$, we see that $\mathcal{E} \subset \Sigma \subset m(\mathcal{E}) \subset \sigma(\mathcal{E})$. Since $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$, we have $\sigma(\mathcal{E}) \subset \Sigma \subset m(\mathcal{E})$. Therefore, $\sigma(\mathcal{E}) = m(\mathcal{E})$.

2. (**Exercise 3.12, p. 21**) . The aim of this problem is to show that $\mathcal{B}(\mathbb{R}^n) = \mathcal{M}$, where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra over $\mathbb{R}^n$, and $\mathcal{M}$ is the smallest monotone class containing the open sets $\mathcal{O}^n$. Show that

   (i) $\mathcal{M}$ is well-defined, and $\mathcal{M} \subset \sigma(\mathcal{O}^n)$.

   (ii) If $U \in \mathcal{M}$, then $U^c \in \mathcal{M}$, i.e. $\mathcal{M}$ contains all closed sets.

   (iii) $\{B \in \mathcal{M} : B^c \in \mathcal{M}\}$ is a σ-algebra.

   (iv) $\sigma(\mathcal{O}^n) \subset \{B \in \mathcal{M} : B^c \in \mathcal{M}\} \subset \mathcal{M}$.

   Proof (i) By problem 3.11, $\mathcal{M} = \bigcap\{\mathcal{D} : \mathcal{D} \text{ is a monotone class and } \mathcal{O}^n \subset \mathcal{D}\}$. Since $\sigma(\mathcal{O}^n) = \mathcal{B}(\mathbb{R}^n)$ is also a monotone class containing $\mathcal{O}^n$, we have that $\mathcal{M} \subset \sigma(\mathcal{O}^n)$.

Proof (ii) Let F be a closed set. We show that $F \in \mathcal{M}$ by showing that F is the countable intersection of open sets. To this end, set $U_n = \{y \in \mathbb{R}^n : |x-y| < \frac{1}{n} \text{ for some } x \in F\}$. We first show that U_n is open by showing that U_n^c is closed. Notice that $U_n^c = \{y \in \mathbb{R}^n : |x-y| \geq \frac{1}{n} \text{ for all } x \in F\}$. Now, let (y_n) be a sequence in U_n^c converging to y. We must show that $y \in U_n^c$. Let $\epsilon > 0$, then there exists an integer $M > 0$ such that $|y_m - y| < \epsilon$ for all $m \geq M$. Pick any $m \geq M$, then for all $x \in F$

$$|x-y| \geq |y_m - x| - |y - y_m| \geq \frac{1}{n} - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $|x-y| \geq \frac{1}{n}$ for all $x \in F$. Hence, $y \in U_n^c$. This shows that U_n^c is closed, and hence U_n is open. We claim that $F = \bigcap_{n\in\mathbb{N}} U_n$. Clearly, $F \subseteq U_n$ for all n, hence $F \subset \bigcap_{n\in\mathbb{N}} U_n$. Now suppose $y \in \bigcap_{n\in\mathbb{N}} U_n$, then for each m there exists $x_m \in F$ such that $|x_m - y| < 1/m$. Then, (x_m) is a sequence in F converging to y. Since F is closed, this implies that $y \in F$. Thus $F \subset \bigcap_{n\in\mathbb{N}} U_n$. Therefore, $F = \bigcap_{n\in\mathbb{N}} U_n$, and $F \in \mathcal{M}$.

Proof (iii) Let $\mathcal{A} = \{B \in \mathcal{M} : B^c \in \mathcal{M}\}$. Since $\emptyset, \mathbb{R}^n \in \mathcal{O}^n \subset \mathcal{M}$, it follows that $\mathbb{R}^n \in \mathcal{A}$. From the definition of $\mathcal{A}$, it is clear that if $B \in \mathcal{A}$, then $B^c \in \mathcal{A}$. Finally, let $(B_n) \in \mathcal{A}$. Then, $B_n, B_n^c \in \mathcal{M}$ for all n. Since $\mathcal{M}$ is closed under countable unions and intersections, it follows that $\cup_n B_n \in \mathcal{M}$, and $(\cup_n B_n)^n = \cap_n B_n^c \in \mathcal{M}$. Thus, $\cup_n B_n \in \mathcal{A}$, and $\mathcal{A}$ is a σ-algebra.

Proof (iv) From the definition of $\mathcal{M}$ and parts (ii) and (iii), we see that $\mathcal{O}^n \subset \mathcal{A}$. Hence $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}^n) \subset \mathcal{A} \subset \mathcal{M}$. Together with part (i), we see that $\mathcal{B}(\mathbb{R}^n) = \mathcal{M}$.