Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Boedapestlaan 6
3584 CD Utrecht

Measure and Integration 2007-Selected Solutions Chapter 5

1. (**Exercise 5.8, p.39**) Consider the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ over $\mathbb{R}^n$, and let λ^n be the Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$. For $B \in \mathcal{B}(\mathbb{R}^n)$ and $t > 0$, define $t \cdot B = \{tb = (tb_1, tb_2, \cdots tb_n) : b = (b_1, \cdots, b_n) \in B\}$.

   (i) Show that $t \cdot B \in \mathcal{B}(\mathbb{R}^n)$ for all $B \in \mathcal{B}(\mathbb{R}^n)$, and $t > 0$.

   (ii) Show that $\lambda^n(t \cdot B) = t^n \lambda^n(B)$.

   Proof (i) Let $\mathcal{B}_t = \{B \in \mathcal{B}(\mathbb{R}^n) : t \cdot B \in \mathcal{B}(\mathbb{R}^n)\}$. It is easy to check that $\mathcal{B}_t$ is a σ-algebra containing $\mathcal{I}$, the collection of all right-open rectangles. Since $\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{I})$ is the smallest σ-algebra containing $\mathcal{I}$, then $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{B}_t$ for all $t > 0$. But by definition, $\mathcal{B}_t \subset \mathcal{B}(\mathbb{R}^n)$, thus $\mathcal{B}_t = \mathcal{B}(\mathbb{R}^n)$ for all $t > 0$, i.e. $t \cdot B \in \mathcal{B}(\mathbb{R}^n)$.

   Proof (ii) Define ν on $\mathcal{B}(\mathbb{R}^n)$ by $\nu(B) = \lambda^n(t \cdot B)$. Notice that $\nu(\emptyset) = 0$ and if $B_1, B_2, \cdots,$ is a disjoint sequence in $\mathcal{B}(\mathbb{R}^n)$, then $t \cdot B_1, t \cdot B_2, \cdots,$ is also a disjoint sequence in $\mathcal{B}(\mathbb{R}^n)$. By σadditivity of λ^n, we get

   $$\nu(\cup_n B_n) = \lambda^n(t \cdot \cup_n B_n) = \lambda^n(\cup_n t \cdot B_n) = \sum_n \lambda^n(t \cdot B_n) = \sum_n \nu(B_n).$$

   The above shows that ν is a measure on $\mathcal{B}(\mathbb{R}^n)$. Now, let $I = \prod_{i=1}^n [a_i, b_i)$ be a right-open rectangle. Then,

   $$\nu(I) = \lambda^n(\prod_{i=1}^n [ta_i, tb_i)) = \prod_{i=1}^n (tb_i - ta_i) = t^n \prod_{i=1}^n (b_i - a_i) = t^n \lambda^n(I).$$

   It is easy to see that the set function μ defined on $\mathcal{B}(\mathbb{R}^n)$ by $\mu(B) = t^n \lambda^n(B)$ is a measure that agrees with ν on $\mathcal{I}$. Furthermore, $\mathcal{I}$ is stable under finite intersections and $\prod_{i=1}^n [-k, k)^n$ is an exhausting sequence with finite ν (and hence μ) measure. Thus, by Theorem 5.7 we see that $\nu = \mu$, i.e., $\lambda^n(t \cdot B) = t^n \lambda^n(B)$.

2. (**Exercise 5.9, p.39**) Let $(X, \mathcal{A}, \mu)$ be a finite measure space where $\mathcal{A} = \sigma(\mathcal{G})$ with $\mathcal{G}$ stable under finite intersections. Assume $\phi : X \to X$ is a map with the property that $\phi^{-1}(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. Prove

   $$\mu(G) = \mu(\phi^{-1}(G)) \; \forall G \in \mathcal{G} \implies \mu(A) = \mu(\phi^{-1}(A)) \; \forall A \in \mathcal{A}.$$

Proof Define ν on $\mathcal{A}$ by $\nu(A) = \mu(\phi^{-1}(A))$. Then, $\nu(\emptyset) = 0$ and if $A_1, A_2, \cdots$ is a disjoint sequence in $\mathcal{A}$, then $\phi^{-1}(A_1), \phi^{-1}(A_2), \cdots$ is also a disjoint sequence in $\mathcal{A}$. Hence,

$$\nu(\cup_n A_n) = \mu(\phi^{-1}(\cup_n A_n)) = \mu(\cup_n(\phi^{-1}(A_n))) = \sum_n \mu(\phi^{-1}(A_n)) = \sum_n \nu(A_n).$$

Hence, ν is a measure. Since $\phi^{-1}(X) = X$, it follows that $\nu(X) = \mu(X) < \infty$, and by assumption $\mu(G) = \nu(G)$ for all $G \in \mathcal{G}$. Now, let $\mathcal{G}' = \mathcal{G} \cup \{X\}$. Then, $\mathcal{G}'$ is stable under finite intersections, $\mu(G) = \nu(G)$ for all $G \in \mathcal{G}'$, and $X, X, \cdots$ is an exhaustung sequence in $\mathcal{G}'$ of finite μ and ν measure. Hence, by Theorem 5.7, we have $\mu = \nu$, i.e., $\mu(A) = \mu(\phi^{-1}(A))$ for all $A \in \mathcal{A}$.

3. (**Exercise 5.10, p.39**) Let $(\Omega, \mathcal{A}, P)$ be a probability space, and suppose $\mathcal{G}, \mathcal{H} \subset \mathcal{A}$ are stable under finite intersections. Let $\mathcal{B} = \sigma(\mathcal{G})$, and $\mathcal{C} = \sigma(\mathcal{H})$. Prove

   $$P(B \cap C) = P(B)P(C)\ \forall B \in \mathcal{B},\ C \in \mathcal{C}\ \Leftrightarrow P(B \cap C) = P(B)P(C)\ \forall B \in \mathcal{G},\ C \in \mathcal{H}$$

   Proof The necessity of the condition is trivial since $\mathcal{G} \subset \mathcal{B}$ and $\mathcal{H} \subset \mathcal{C}$. Conversely, suppose $P(B \cap C) = P(B)P(C)\ \forall B \in \mathcal{G},\ C \in \mathcal{H}$. Fix any $C \in \mathcal{H}$. Define μ and ν on $\mathcal{B}$ by $\mu(B) = P(B \cap C)$ and $\nu(B) = P(B)P(C)$. It is easy to check that μ and ν are finite measures on $\mathcal{B}$ agreeing on $\mathcal{G} \cup \{X\}$ and admiiting an axhausting sequence, namely $X, X, \cdots$, of finite μ and ν measure. Hence by Theorem 5.7, we have $\mu = \nu$ on $\mathcal{B}$, i.e. $P(B \cap C) = P(B)P(C)$ for all $B \in \mathcal{B}$. Similarly, if we fix $B \in \mathcal{B}$ and define ρ, τ on $\mathcal{C}$ by $\rho(C) = P(B \cap C)$ and $\tau(C) = P(B)P(C)$, then the same argument as above shows that ρ and τ are equal measures, so that $P(B \cap C) = P(B)P(C)$ for all $C \in \mathcal{C}$. Together with the above we see that $P(B \cap C) = P(B)P(C)$ for all $B \in \mathcal{G}$ and $C \in \mathcal{H}$.