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Measure and Integration 2006-Selected Solutions Chapter 6

1. (**Exercise 6.5(iv), p.46**) Let P be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Prove that P can be written as $P = \mu + \nu$, where μ is a non-atomic measure ($\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$), and ν is purely atomic, i.e., there exist positive real numbers ϵ_j and points $x_j \in \mathbb{R}$ such that for all $A \in \mathcal{B}$, one has $\nu(A) = \sum_{j\in\mathbb{N}} \epsilon_j \delta_{x_j}(A)$, where δ_{x_j} is Dirac measure concentrated at the point x_j.

   Proof If P has no atoms, then $P = \mu$ and $\nu = 0$. Assume P has atoms, and let $A = \{x \in \mathbb{R} : P(\{x\}) > 0\}$ be the set of all atoms of P. For each $n \in \mathbb{N}$, let $A_n = \{x \in A : P(\{x\}) \geq 1/k\}$. Then, $A = \cup_{k\in\mathbb{N}} A_n$. Furthermore, since P is a probability measure, then A_n can have atmost n elements (otherwise A would have measure greater than 1), hence A is countable. Write $A = \{x_1, x_2, \ldots, \}$. Define ν on $\mathcal{B}(\mathbb{R})$ by

   $$\nu(A) = \sum_j P(\{x_j\})\delta_{x_j}(A) = \sum_{x_j\in A} P(\{x_j\}).$$

   Then, ν is a measure (see Example 4.7(iv), p.27), and $\nu(A) = \sum_{x_j\in A} P(\{x_j\}) \leq P(A)$. Now define μ on $\mathcal{B}(\mathbb{R})$ by $\mu(A) = P(A) - \nu(A)$. Then, it is easy to see that μ is a measure, and μ is non-atomic since if $x = x_j$, then $P(\{x_j\} = \nu(\{x_j\})$, and if $x \neq x_j$ for all j, then $\nu(\{x\}) = 0 = P(\{x\})$.

2. (**Exercise 6.7, p.46**) Let $\lambda = \lambda^1$ be Lebesgue measure on $([0,1], \mathcal{B}[0,1])$. Show that for every $\epsilon > 0$ there is a dense open set $U \subset [0,1]$ with $\lambda(U) \leq \epsilon$.

   Proof Let $\epsilon > 0$ and let $\{q_j\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Define the set

   $$U = \cup_j (q_j - \epsilon 2^{-j-1}, q_j + \epsilon 2^{-j-1}) \cap [0,1].$$

   Then, U is open and dense in $[0,1]$ (notice that U contains all the rationals in the unit interval and these are dense in $[0,1]$). Furthermore by σ-subadditivity and monotonicity, one has

   $$\lambda(U) \leq \sum_j \lambda((q_j - \epsilon 2^{-j-1}, q_j + \epsilon 2^{-j-1})) = \sum_j \epsilon/2^j = \epsilon.$$

3. (**Exercise 6.8, p.46**) Let $\lambda = \lambda^1$ be Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R})$. Show that $N \in \mathcal{B}(\mathbb{R}))$ is a null-set (i.e. $\lambda(N) = 0$) if and only if for every $\epsilon > 0$ there is an open set U_ϵ such that $N \subset U_\epsilon$ and $\lambda(U_\epsilon) < \epsilon$.

Proof If for every $\epsilon > 0$ there is an open set U_ϵ such that $N \subset U_\epsilon$ and $\lambda(U_\epsilon) < \epsilon$, then $\lambda(N) \le \epsilon$ for all $\epsilon > 0$, hence $\lambda(N) = 0$.

Before we prove the converse, we will prove the following general fact. Let $B \subset \mathbb{R}$, in the proof of Caratheodory, the outer (Lebesgue) measure of B was defined by $\lambda^*(B) = \inf\{\sum_n \lambda(I_n) : (I_n) \subset \mathcal{I}, B \subset \cup_n I_n\}$, where $\mathcal{I}$ is the collection of half-open intervals. We will show that $\lambda^*(B) = \inf\{\lambda^*(G) : B \subset G, G \text{ open }\}$. By monotonicity of λ^*, it is clear that $\lambda^*(B) \le \lambda^*(G)$ for any G open with $B \subset G$. Hence, $\lambda^*(B) \le \inf\{\lambda^*(G) : B \subset G, G \text{ open }\}$. To prove the other inequality, we first assume that $\lambda^*(B) < \infty$ (otherwise there is nothing to prove). Now, let $\epsilon > 0$. By the definition of the infimum, there exist a cover $(I_n) \subset \mathcal{I}$ such that $B \subset \cup_n I_n$ and $\sum_n \lambda(I_n) \le \lambda^*(B) + \epsilon/2$. Each I_n is a right-open interval, it is easy to see that one can find an open interval I'_n containing I_n and such that $\lambda(I'_n) \le \lambda(I_n) + \epsilon/2^n$. Let $G = \cup_n I'_n$, then G is open and $B \subset G$. Furthermore,

$$\lambda^*(B) \le \sum_n \lambda(I'_n) \le \sum_n \lambda(I_n) + \epsilon/2 \le \lambda^*(B) + \epsilon.$$

Thus, $\inf\{\lambda^*(G) : B \subset G, G \text{ open }\} \le \lambda^*(B) + \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $\inf\{\lambda^*(G) : B \subset G, G \text{ open }\} \le \lambda^*(B)$. Thus, $\lambda^*(B) = \inf\{\lambda^*(G) : B \subset G, G \text{ open }\}$.

Note also that if B is a Borel set, then $\lambda(B) = \lambda^*(B)$. In particular, $\lambda^*(G) = \lambda(G)$ for all open sets G. We apply the above result to our situation. Suppose $N \in \mathcal{B}(\mathbb{R})$ is a null-set. Then,

$$0 = \lambda(N) = \lambda^*(N) = \inf\{\lambda(G) : N \subset G, G \text{ open }\}.$$

From the definition of the infimum, we see that for every $\epsilon > 0$, there is an open set U_ϵ containing N such that $\lambda(U_\epsilon) < \epsilon$.

4. (**Exercise 6.9, p.46**) (**Borel-Cantelli Lemma**) Let $(\Omega, \mathcal{A}, P)$ be a probability space. Prove that if $(A_j) \subset \mathcal{A}$ is a sequence of measurable sets with $\sum_n P(A_n) < \infty$, then $P(\cap_{n=1}^\infty \cup_{j=n}^\infty A_j) = 0$.

   Proof Let $B_n = \cup_{j=n}^\infty A_j$, then clearly B_n is a decreasing sequence in $\mathcal{A}$. Since P is a probability measure, then $P(A) \le 1 < \infty$ for all $A \in \mathcal{A}$. Hence by Theorem 4.4(iii)', we have

$$P(\cap_{n=1}^\infty \cup_{j=n}^\infty A_j) = P(\cap_{n=1}^\infty B_n) = \lim_{n\to\infty} P(B_n).$$

   Since P is σ-subadditive, we have $P(B_n) \le \sum_{j=n}^\infty P(A_j)$. But $\sum_{j=n}^\infty P(A_j)$ is the tail of a convergent series, hence

$$P(\cap_{n=1}^\infty \cup_{j=n}^\infty A_j) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} \sum_{j=n}^\infty P(A_j) = 0.$$