Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Boedapestlaan 6
3584 CD Utrecht

Measure and Integration 2006-Selected Solutions Chapter 10

1. (**p.84, exercise 10.2**) Give an example of a probability space $(\Omega, \mathcal{A}, P)$, and a P integrable function $u \in \mathcal{L}^1(P)$ which is not bounded.

   Proof: Consider the probability space $((0,1), \mathcal{B}((0,1)), \lambda)$, where $\mathcal{B}((0,1))$ is the Borel σ-algebra on $(0,1)$ and λ is the restriction of Lebesgue measure on $(0,1)$. Consider the function u defined on $(0,1)$ by $u = \sum_{n=1}^{\infty} n\mathbf{1}_{[1/2^n, 1/2^{n-1})}$. Clearly u is unbounded, but by Corollary 9.9,

$$\int |u|\, d\lambda = \int u\, d\lambda = \sum_{n=1}^{\infty} n \int \mathbf{1}_{[1/2^n, 1/2^{n-1})}\, d\lambda = \sum_{n=1}^{\infty} n\lambda([1/2^n, 1/2^{n-1})) = \sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

   (in the last equality we used the fact that $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(x-1)^2}$ for $0 < x < 1$). Hence u is λ integrable.

2. (**p.84, exercise 10.5**) Let $(X, \mathcal{A}, \mu)$ be a measure space and $u \in \mathcal{L}^1_{\mathbb{R}}$. Prove that for all $c > 0$ (whenever the expression makes sense)

   (i) $\mu(\{|u| > c\}) \le \frac{1}{c} \int |u|\, d\mu$.

   (ii) $\mu(\{|u| > c\}) \le \frac{1}{c^p} \int |u|^p\, d\mu$ for all $0 < p < \infty$.

   (iii) $\mu(\{|u| \ge c\}) \le \frac{1}{\phi(c)} \int \phi(|u|)\, d\mu$ for an increasing $\phi : \mathbb{R}^+ \to \mathbb{R}^+$.

   (iv) If $u \ge 0$, then $\mu(\{u \ge \alpha \int u\, d\mu\}) \le \frac{1}{\alpha}$ for any $\alpha > 0$.

   Proof (i): By monotonicity of μ and the Markov inequality (prop. 10.12), we have

$$\mu(\{|u| > c\}) \le \mu(\{|u| \ge c\}) \le \frac{1}{c} \int |u|\, d\mu.$$

   Proof (ii): For any $p > 0$, by the Markov inequality and part (i),

$$\mu(\{|u| > c\}) = \mu(\{|u|^p > c^p\}) \le \frac{1}{c^p} \int |u|^p\, d\mu.$$

   Note also that (ii) is a special case of part (iii).

Proof (iii): Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function, then $\mu(\{|u| \geq c\}) = \mu(\{\phi(|u|) \geq \phi(c)\})$. Let $A = \{x : \phi(|u(x)|) \geq \phi(c)\}$, then $A \in \mathcal{A}$ and for any $x \in A$, one has $1 \leq \dfrac{\phi(|u(x)|)}{\phi(c)}$. By the above, $\mu(\{|u| \geq c\}) = \mu(A) = \int \mathbf{1}_A \, d\mu$, thus

$$\mu(\{|u| \geq c\}) = \int \mathbf{1}_A \, d\mu \leq \int \frac{\phi(|u(x)|)}{\phi(c)} \mathbf{1}_A \, d\mu \leq \int \frac{\phi(|u(x)|)}{\phi(c)} d\mu = \frac{1}{\phi(c)} \int \phi(|u|) \, d\mu.$$

Proof (iv): Let $A = \{x : u(x) \geq \alpha \int u \, d\mu\}$. Then for $x \in A$ one has $1 \leq \dfrac{u(x)}{\alpha \int u \, d\mu}$. Hence,

$$\mu(\{u \geq \alpha \int u \, d\mu\}) = \int \mathbf{1}_A \, d\mu \leq \int \frac{u(x)}{\alpha \int u \, d\mu} \mathbf{1}_A \, d\mu \leq \int \frac{u(x)}{\alpha \int u \, d\mu} \, d\mu = \frac{1}{\alpha}.$$

3. (**p.84, exercise 10.7**) Let $(X, \mathcal{A}, \mu)$ be a measure space and let (A_j) be a sequence of disjoint measurable sets. Show that

$$u\mathbf{1}_{\cup_j A_j} \in \mathcal{L}^1(\mu) \iff u\mathbf{1}_{A_n} \in \mathcal{L}^1(\mu) \text{ for all } n \text{ and } \sum_{j=1}^{\infty} \int_{A_j} |u| \, d\mu < \infty.$$

Proof: Suppose $u\mathbf{1}_{\cup_j A_j} \in \mathcal{L}^1(\mu)$, then for any n,

$$|u\mathbf{1}_{A_n}| = |u|\mathbf{1}_{A_n} \leq |u|\mathbf{1}_{\cup_j A_j} = |u\mathbf{1}_{\cup_j A_j}|.$$

By Theorem 10.3(iv), we have that $u\mathbf{1}_{A_n} \in \mathcal{L}^1(\mu)$. Since the sequence (A_j) is pairwise disjoint, then $\mathbf{1}_{\cup_j A_j} = \sum_j \mathbf{1}_{A_j}$ and $|u|\mathbf{1}_{\cup_j A_j} = \sum_j |u|\mathbf{1}_{A_j}$. By Corollary 9.9 we have

$$\sum_{j=1}^{\infty} \int_{A_j} |u| \, d\mu = \int \sum_{j=1}^{\infty} |u|\mathbf{1}_{A_j} \, d\mu = \int |u|\mathbf{1}_{\cup_j A_j} \, d\mu < \infty.$$

Conversely, suppose $u\mathbf{1}_{A_n} \in \mathcal{L}^1(\mu)$ for all n and $\sum_{j=1}^{\infty} \int_{A_j} |u| \, d\mu < \infty$. Then by Corollary 9.9, $u\mathbf{1}_{\cup_j A_j} = \sum_j u\mathbf{1}_{A_j}$ is measurable and

$$\int |u|\mathbf{1}_{\cup_j A_j} \, d\mu = \int \sum_{j=1}^{\infty} |u|\mathbf{1}_{A_j} \, d\mu = \sum_{j=1}^{\infty} \int |u|\mathbf{1}_{A_j} \, d\mu = \sum_{j=1}^{\infty} \int_{A_j} |u| \, d\mu < \infty.$$

Hence, $u\mathbf{1}_{\cup_j A_j} \in \mathcal{L}^1(\mu)$.

4. (**p.84, exercise 10.9**)Let $(\Omega, \mathcal{A}, P)$ be a probability space. Show that for $u \in \mathcal{M}(\mathcal{A})$

$$u \in \mathcal{L}^1(P) \iff \sum_{j=0}^{\infty} P(\{|u| \geq j\}) < \infty.$$

Proof: Notice that if $u \in \mathcal{M}(\mathcal{A})$, then

$$|u(x)| = \sum_{j=0}^{\infty} |u(x)| \mathbf{1}_{\{j \leq |u| < j+1\}}(x).$$

We first show that if $u \in \mathcal{M}(\mathcal{A})$, then for each $x \in \Omega$, one has

$$\sum_{j=0}^{\infty} j \mathbf{1}_{\{j \leq |u| < j+1\}}(x) = \sum_{j=0}^{\infty} \mathbf{1}_{\{j \leq |u|\}}(x).$$

For any $N \geq 0$,

$$\begin{aligned} \textstyle\sum_{j=0}^{N} j \mathbf{1}_{\{j \leq |u| < j+1\}}(x) &= \textstyle\sum_{j=0}^{N} j \left(\mathbf{1}_{\{j \leq |u|\}}(x) - \mathbf{1}_{\{j+1 \leq |u|\}}(x) \right) \\ &= \textstyle\sum_{j=0}^{N} \mathbf{1}_{\{j \leq |u|\}}(x) - N \mathbf{1}_{\{N+1 \leq |u|\}}(x). \end{aligned}$$

Since $|u(x)| < \infty$ for all $x \in \Omega$, then $\lim_{N \to \infty} N \mathbf{1}_{\{N+1 \leq |u|\}}(x) = 0$. Taking the limit in the above equations we get

$$\sum_{j=0}^{\infty} j \mathbf{1}_{\{j \leq |u| < j+1\}}(x) = \sum_{j=0}^{\infty} \mathbf{1}_{\{j \leq |u|\}}(x).$$

Now suppose $u \in \mathcal{L}^1(P)$, then

$$\begin{aligned} \textstyle\sum_{j=0}^{\infty} P(\{|u| \geq j\}) &= \textstyle\sum_{j=0}^{\infty} \int \mathbf{1}_{\{|u| \geq j\}} \, dP \\ &= \textstyle\int \sum_{j=0}^{\infty} \mathbf{1}_{\{|u| \geq j\}} \, dP \\ &= \textstyle\int \sum_{j=0}^{\infty} j \mathbf{1}_{\{j \leq |u| < j+1\}} \, dP \\ &\leq \textstyle\int \sum_{j=0}^{\infty} |u| \mathbf{1}_{\{j \leq |u| < j+1\}} \, dP \\ &= \textstyle\int |u| \, dP < \infty. \end{aligned}$$

Conversely, suppose $\sum_{j=0}^{\infty} P(\{|u| \geq j\}) < \infty$, then

$$\begin{aligned} \textstyle\int |u| \, dP &= \textstyle\int \sum_{j=0}^{\infty} |u| \mathbf{1}_{\{j \leq |u| < j+1\}} \, dP \\ &\leq \textstyle\int \sum_{j=0}^{\infty} (j+1) \mathbf{1}_{\{j \leq |u| < j+1\}} \, dP \\ &\leq 1 + \textstyle\int \sum_{j=1}^{\infty} (2j) \mathbf{1}_{\{j \leq |u| < j+1\}} \, dP \\ &\leq 1 + 2 \textstyle\sum_{j=0}^{\infty} P(\{|u| \geq j\}) < \infty \end{aligned}$$