Universiteit Utrecht
Mathematisch Instituut

Universiteit Utrecht

Boedapestlaan 6
3584 CD Utrecht

Measure and Integration Solutions 7

1. Suppose E is a set, $\mathcal{C}$ a π-system over E and $\mathcal{B} = \sigma(E;\mathcal{C})$ (the smallest σ-algebra over E containing $\mathcal{C}$). Let μ and ν be two measures on $(E,\mathcal{B})$ such that (i) $\mu(E) = \nu(E) < \infty$, and (ii) $\mu(C) = \nu(C)$ for all $C \in \mathcal{C}$. Let $\mathcal{H} = \{A \in \mathcal{B} : \mu(A) = \nu(A)\}$.

   (a) Show that $\mathcal{H}$ is a λ-system over E.

   (b) Show that $\mathcal{B} = \mathcal{H}$, and conclude that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$.

   Proof(a) We need to verify properties (**a**)-(**d**) on page 34.

   — It is clear that $E \in \mathcal{H}$.

   — If $A_1, A_2 \in \mathcal{H}$ with $A_1 \cap A_2 = \emptyset$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) = \nu(A_1) + \nu(A_2) = \nu(A_1 \cup A_2)$. Thus, $A_1 \cup A_2 \in \mathcal{H}$.

   — Let $A_1, A_2 \in \mathcal{H}$ with $A_1 \subseteq A_2$. Since $\mu(A_1) = \nu(A_1) < \infty$, then

   $$\mu(A_2 \setminus A_1) = \mu(A_2) - \mu(A_1) = \nu(A_2) - \nu(A_1) = \nu(A_2 \setminus A_1).$$

   therefore, $A_2 \setminus A_1 \in \mathcal{H}$.

   — If $A_1 \subseteq A_2 \subseteq \ldots \in \mathcal{H}$, then by Theorem 3.1.6 it follows that

   $$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu(\cup_{n=1}^{\infty} A_n).$$

   Thus, $\cup_{n=1}^{\infty} A_n \in \mathcal{H}$, and therefore $\mathcal{H}$ is a λ-system over E.

   Proof(b) Notice that $\mathcal{C} \subseteq \mathcal{H} \subseteq \mathcal{B}$, and $\mathcal{C}$ is a π-system. By Lemma 3.1.3, $\mathcal{B} = \sigma(E;\mathcal{C})$ is the smallest λ-system containg $\mathcal{C}$, hence $\mathcal{B} \subseteq \mathcal{H}$. This implies that $\mathcal{B} = \mathcal{H}$, therefore $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$.

2. Let $(E,\mathcal{B},\mu)$ be a measure space, and $\overline{\mathcal{B}}^{\mu}$ be the completion of the σ-algebra $\mathcal{B}$ with respect to the measure μ. We denote by $\overline{\mu}$ the extension of the measure μ to the σ-algebra $\overline{\mathcal{B}}^{\mu}$. Suppose $f : E \to E$ is a function such that $f^{-1}(B) \in \mathcal{B}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$, where $f^{-1}(B) = \{x \in E : f(x) \in B\}$. Show that $f^{-1}(\Gamma) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\Gamma)) = \overline{\mu}(\Gamma)$ for all $\Gamma \in \overline{\mathcal{B}}^{\mu}$.

   Proof: Let $\Gamma \in \overline{\mathcal{B}}^{\mu}$, then there exist $A, B \in \mathcal{B}$ such that $A \subseteq \Gamma \subseteq B$, $\mu(B \setminus A) = 0$ and $\overline{\mu}(\Gamma) = \mu(A)$. Then, $f^{-1}(A), f^{-1}(B) \in \mathcal{B}$ satisfy $f^{-1}(A) \subseteq f^{-1}(\Gamma) \subseteq f^{-1}(B)$ and $\mu(f^{-1}(B) \setminus f^{-1}(A)) = \mu(f^{-1}(B \setminus A)) = \mu(B \setminus A) = 0$. Thus, $f^{-1}(\Gamma) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\Gamma)) = \mu(f^{-1}(A) = \mu(A) = \overline{\mu}(\Gamma)$.

3. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\{A_n\}$ a sequence in $\mathcal{B}$. Define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m,$$

and

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m.$$

(a) Prove that $\mu(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mu(A_n)$.

(b) Suppose that $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$. Prove that $\mu(\limsup_{n\to\infty} A_n) \geq \limsup_{n\to\infty} \mu(A_n)$.

(c) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(\limsup_{n\to\infty} A_n) = 0$. (This is known as the Borel-Cantelli Lemma).

Proof (a): Let $B_n = \bigcap_{m=n}^{\infty} A_n$, then $B_1 \subseteq B_2 \subseteq \dots$ are measurable and $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Thus,

$$\mu(\liminf_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(B_n) = \liminf_{n\to\infty} \mu(B_n) \leq \liminf_{n\to\infty} \mu(A_n).$$

Proof (b): Let $C_n = \bigcup_{m=n}^{\infty} A_n$, then $C_1 \supseteq C_2 \supseteq \dots$ are measurable and $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} C_n$. Since $\mu(C_1) = \mu(\bigcup_{n=1}^{\infty} A_n) < \infty$, then

$$\mu(\limsup_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(C_n) = \limsup_{n\to\infty} \mu(C_n) \geq \limsup_{n\to\infty} \mu(A_n).$$

Proof (c): Notice that since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(A_m) = 0$. Thus,

$$\mu(\limsup_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(\bigcup_{m=n}^{\infty} A_n) \leq \lim_{n\to\infty} \sum_{m=n}^{\infty} \mu(A_m) = 0.$$

4. Let $\mathcal{C} = \{(a, \infty) : a \in \mathbb{R}\}$, and let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra over $\mathbb{R}$.

(a) Show that $\mathcal{B}_{\mathbb{R}} = \sigma(E; \mathcal{C})$.

(b) Let $(E, \mathcal{F}, \mu)$ be a **finite** measure space. Suppose $f : E \to \mathbb{R}$ satisfies $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{B}_{\mathbb{R}}$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra over $\mathbb{R}$. Define μ_f on $\mathcal{B}_{\mathbb{R}}$ by $\mu_f(A) = \mu(f^{-1}(A))$ for all $A \in \mathcal{B}_{\mathbb{R}}$. Show that μ_f is a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Proof (a): Notice that $\sigma(E; \mathcal{C}) \subseteq \mathcal{B}_{\mathbb{R}}$ since each element of $\mathcal{C}$ is open. We now show that $\sigma(E; \mathcal{C})$ contains all the open intervals. For any $a \in \mathbb{R}$,

$$[a, \infty) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) \in \sigma(E; \mathcal{C}).$$

Thus, for any $a \leq b \in \mathbb{R}$,

$$(a, b) = (a, \infty) \setminus [b, \infty) \in \sigma(E; \mathcal{C}).$$

Now, by Lemma 2.1.9, any open set G in $\mathbb{R}$ is a countable disjoint union of open intervals, hence $G \in \sigma(E; \mathcal{C})$. Since, $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra over $\mathbb{R}$ containing the open sets, it follows that $\mathcal{B}_{\mathbb{R}} \subseteq \sigma(E; \mathcal{C})$. Therefore, $\mathcal{B}_{\mathbb{R}} = \sigma(E; \mathcal{C})$.

Proof (b): Notice that $\mu_f(\emptyset) = 0$, and if $A_1, A_2, \dots \in \mathcal{B}_{\mathbb{R}}$ are pairwise disjoint, then $f^{-1}(A_1), f^{-1}A_2, \dots \in \mathcal{F}$ are pairwise disjoint, and $f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}A_n$. Hence,

$$\mu_f\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} f^{-1}A_n\right) = \sum_{n=1}^{\infty} \mu(f^{-1}A_n) = \sum_{n=1}^{\infty} \mu_f(A_n).$$

So μ_f is countably additive and μ_f is a measure on $\mathcal{B}_{\mathbb{R}}$.