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Measure and Integration solutions of extra problems

1. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$ and λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \\ 2^{-k} & \text{if } x \in [k, k+1),\ k \in \mathbb{Z},\ k \geq 0. \end{cases}$$

   (a) Show that f is measurable, i.e. $f^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for all $B \in \mathcal{B}(\mathbb{R})$.

   (b) Determine the values of $\lambda(\{f > 1\})$, $\lambda(\{f < 1\}$ and $\lambda(\{1/4 \leq f < 1\}$.

Proof(a): It is enough to show that $f^{-1}((-\infty, a]) \in \mathcal{B}_{\mathbb{R}}$ for all $a \in \mathbb{R}$. Now,

$$f^{-1}((-\infty, a]) = \begin{cases} \emptyset & \text{if } a < 0 \\ \\ (-\infty, 0] \cup [k+1, \infty) & \text{if } \frac{1}{2^{k+1}} \leq a < \frac{1}{2^k},\ k \geq 0 \\ \\ \mathbb{R} & \text{if } a \geq 1. \end{cases}$$

In all cases one sees that $f^{-1}((-\infty, a]) \in \mathcal{B}_{\mathbb{R}}$. Thus, f is measurable.

Proof(b):

$$\lambda(\{f > 1\}) = \lambda(\emptyset) = 0.$$

$$\lambda(\{f < 1\} = \sum_{k=1}^{\infty} \lambda(\{f = 2^{-k}\}) = \sum_{k=1}^{\infty} \lambda([k, k+1)) = \infty.$$

$$\lambda(\{1/4 \leq f < 1\}) = \lambda(\{f = 1/2\}) + \lambda(\{f = 1/4\}) = 2.$$

2. Let $(X, \mathcal{B}, \mu)$ be a measure space, and $(G_n)_n \subset \mathcal{B}$ such that $\mu(G_n \cap G_m) = 0$ for $m \neq n$. Show that $\mu(\bigcup_n G_n) = \sum_n \mu(G_n)$.

Proof: Let $A_1 = G_1$, $B_1 = \emptyset$. For $n \geq 2$, set $A_n = G_n \setminus \bigcup_{m=1}^{n-1} G_m$ and $B_n = G_n \cap \bigcup_{m=1}^{n-1} G_m = \bigcup_{m=1}^{n-1} (G_n \cap G_m)$. Then,

– $G_n = A_n \cup B_n$ for all $n \geq 1$,

– $A_n \cap A_m = \emptyset$ for $m \neq n$,

– $\mu(B_n) = 0$ for all $n \geq 1$ (since $\mu(G_n \cap G_m) = 0$ for $n \neq m$), hence $\mu(G_n) = \mu(A_n)$ for all $n \geq 1$,

– $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} G_n$: clearly the left handside is a subset of the right handside. Now, let $x \in \bigcup_{n=1}^{\infty} G_n$, then $x \in G_n$ for some n. Let n_0 be the smallest positive integer such that $x \in G_{n_0}$, then $x \in A_{n_0} \subseteq \bigcup_{n=1}^{\infty} A_n$.

Hence,

$$\mu(\bigcup_{n=1}^{\infty} G_n) = \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \mu(G_n).$$

3. Let $(X, \mathcal{B}, \nu)$ be a measure space, and suppose $X = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\nu(E_n) < \infty$ for all $n \geq 1$. Define μ on $\mathcal{B}$ by $\mu(B) = \sum_{n=1}^{\infty} 2^{-n}\nu(B \cap E_n)/(\nu(E_n) + 1)$.

   (a) Prove that μ is a finite measure on $(X, \mathcal{B})$.

   (b) Let $B \in \mathcal{B}$. Prove that $\mu(B) = 0$ **if and only if** $\nu(B) = 0$.

Proof (a): Clearly $\mu(\emptyset) = 0$, and

$$\mu(X) = \sum_{n=1}^{\infty} 2^{-n}\nu(E_n)/(\nu(E_n) + 1) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty.$$

Now, let (C_n) be a disjoint sequence in $\mathcal{B}$. Then,

$$\begin{aligned}
\mu(\textstyle\bigcup_{m=1}^{\infty} C_m) &= \sum_{n=1}^{\infty} 2^{-n}\nu((\bigcup_{m=1}^{\infty} C_m) \cap E_n)/(\nu(E_n) + 1) \\
&= \sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{\infty} \nu(C_m \cap E_n)/(\nu(E_n) + 1) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-n}\nu(C_m \cap E_n)/(\nu(E_n) + 1) \\
&= \sum_{m=1}^{\infty} \mu(C_m).
\end{aligned}$$

Thus, μ is a finite measure.

Proof (b): Suppose that $\nu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n, hence $\mu(B) = 0$. Conversely, suppose $\mu(B) = 0$, then $\nu(B \cap E_n) = 0$ for all n. Since $X = \bigcup_{n=1}^{\infty} E_n$ (disjoint union), then

$$\nu(B) = \nu(B \cap \bigcup_{n=1}^{\infty} E_n) = \nu(\bigcup_{n=1}^{\infty}(B \cap E_n)) = \sum_{n=1}^{\infty} \nu(B \cap E_n) = 0.$$

4. Let $(E, \mathcal{B}, \mu)$ be a measure space, and $\overline{\mathcal{B}}^{\mu}$ be the completion of the σ-algebra $\mathcal{B}$ with respect to the measure μ (see exercise 4.13, p.29). We denote by $\overline{\mu}$ the extension

of the measure μ to the σ-algebra $\overline{\mathcal{B}}^{\mu}$. Suppose $f : E \to E$ is a function such that $f^{-1}(B) \in \mathcal{B}$ and $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$. Show that $f^{-1}(\overline{B}) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\overline{B})) = \overline{\mu}(\overline{B})$ for all $\overline{B} \in \overline{\mathcal{B}}^{\mu}$.

Proof: Let $\overline{B} \in \overline{\mathcal{B}}^{\mu}$, then there exist $A, B \in \mathcal{B}$ such that $A \subseteq \overline{B} \subseteq B$, $\mu(B \setminus A) = 0$ and $\overline{\mu}(\overline{B}) = \mu(A)$. Then, $f^{-1}(A)$, $f^{-1}(B) \in \mathcal{B}$ satisfy $f^{-1}(A) \subseteq f^{-1}(\overline{B}) \subseteq f^{-1}(B)$ and $\mu(f^{-1}(B) \setminus f^{-1}(A)) = \mu(f^{-1}(B \setminus A)) = \mu(B \setminus A) = 0$. Thus, $f^{-1}(\overline{B}) \in \overline{\mathcal{B}}^{\mu}$ and $\overline{\mu}(f^{-1}(\overline{B})) = \mu(f^{-1}(A) = \mu(A) = \overline{\mu}(\overline{B})$.

5. Let X be a set, and $\mathcal{C} \subseteq \mathcal{P}(X)$. Consider $\sigma(\mathcal{C})$, the smallest σ-algebra over X containing $\mathcal{C}$, and let $\mathcal{D}$ be the collection of sets $A \in \sigma(\mathcal{C})$ with the property that there exists a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ (depending on A) such that $A \in \sigma(\mathcal{C}_0)$.

   (a) Show that $\mathcal{D}$ is a σ-algebra over X.

   (b) Show that $\mathcal{D} = \sigma(\mathcal{C})$.

   Proof (a): Clearly $\emptyset \in \mathcal{D}$ since $\emptyset$ belongs to every σ-algebra. Let $A \in \mathcal{D}$, then there is a countable collection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $A \in \sigma(\mathcal{C}_0)$. But then $A^c \in \sigma(\mathcal{C}_0)$, hence $A^c \in \mathcal{D}$. Finally, let $\{A_n\}$ be in $\mathcal{D}$, then for each n there exists a countable collection $\mathcal{C}_n \subseteq \mathcal{C}$ such that $A_n \in \sigma(\mathcal{C}_n)$. Let $\mathcal{C}_0 = \bigcup_n \mathcal{C}_n$, then $\mathcal{C}_0 \subseteq \mathcal{C}$, and $\mathcal{C}_0$ is countable. Furthermore, $\sigma(\mathcal{C}_n) \subseteq \sigma(\mathcal{C}_0)$, and hence $A_n \in \sigma(\mathcal{C}_0)$ for each n which implies that $\bigcup_n A_n \in \sigma(\mathcal{C}_0)$. Therefore, $\bigcup_n A_n \in \mathcal{D}$ and $\mathcal{D}$ is a σ-algebra.

   Proof (b): By definition $\mathcal{D} \subseteq \sigma(\mathcal{C})$. Also, $\mathcal{C} \subseteq \mathcal{D}$ since $C \in \sigma(\{C\})$ for every $C \in \mathcal{C}$. Since $\sigma(\mathcal{C})$ is the smallest σ-algebra over X containg $\mathcal{C}$, then by part (a) $\sigma(\mathcal{C}) \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \sigma(\mathcal{C})$.

6. Let E be a set, and $\mathcal{A}$ an algebra over E, i.e. $\mathcal{A}$ contains the empty set, is closed under complements and **finite** unions. Let $\mu : \mathcal{A} \to [0, 1]$ be a probability measure $\mathcal{A}$, i.e. a function satisfying

   (I) $\mu(E) = 1 = 1 - \mu(\emptyset)$,

   (II) if $A_1, A_2, \cdots, \in \mathcal{A}$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then

   $$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

   (a) Show that if $\{A_n\}$ and $\{B_n\}$ are increasing sequences in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$, then $\lim_{n\to\infty} \mu(A_n) \leq \lim_{n\to\infty} \mu(B_n)$.

   (b) Let $\mathcal{G}$ be the collection of all subsets G of E such that there exists an increasing sequence $\{A_n\}$ in $\mathcal{A}$ with $G = \bigcup_{n=1}^{\infty} A_n$. Define $\overline{\mu}$ on $\mathcal{G}$ by

   $$\overline{\mu}(G) = \lim_{n\to\infty} \mu(A_n),$$

   where $\{A_n\}$ is an increasing sequence in $\mathcal{A}$ such that $G = \bigcup_{n=1}^{\infty} A_n$. Show the following.

      (i) $\overline{\mu}$ is well defined.

(ii) If $G_1, G_2 \in \mathcal{G}$, then $G_1 \cup G_2, G_1 \cap G_2 \in \mathcal{G}$ and

$$\overline{\mu}(G_1 \cup G_2) + \overline{\mu}(G_1 \cap G_2) = \overline{\mu}(G_1) + \overline{\mu}(G_2).$$

Proof (a): By Theorem 4.4, one has that if $\{D_n\}$ is an increasing sequence in $\mathcal{A}$ such that $\bigcup_n D_n \in \mathcal{A}$, then $\mu(\bigcup_n D_n) = \lim_{n\to\infty} \mu(D_n)$. Suppose that $\{A_n\}$ and $\{B_n\}$ are increasing sequences in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$. For each $m \geq 1$, $\{A_m \cap B_n : n \geq 1\}$ is an increasing sequence in $\mathcal{A}$ and $A_m = A_m \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty}(A_m \cap B_n) \in \mathcal{A}$. Thus, for each $m \geq 1$,

$$\mu(A_m) = \lim_{n\to\infty} \mu(A_m \cap B_n) \leq \lim_{n\to\infty} \mu(B_n).$$

Taking the limit as $m \to \infty$, we get $\lim_{m\to\infty} \mu(A_m) \leq \lim_{n\to\infty} \mu(B_n)$.

Proof (b)(i): Let $G \in \mathcal{G}$. If $\{A_n\}$ and $\{B_n\}$ are two increasing sequences in $\mathcal{A}$ such that $G = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Then, by part (a) $\lim_{m\to\infty} \mu(A_m) = \lim_{n\to\infty} \mu(B_n)$. This shows that $\overline{\mu}$ is well defined on $\mathcal{G}$.

Proof (b)(ii): Let $G_1, G_2 \in \mathcal{G}$, there exist increasing sequences $\{A_n\}$, $\{B_n\}$ in $\mathcal{A}$ such that $G_1 = \bigcup_{n=1}^{\infty} A_n$ and $G_1 = \bigcup_{n=1}^{\infty} B_n$. Then, $\{A_n \cup B_n\}$, $\{A_n \cap B_n\}$ are increasing sequences in $\mathcal{G}$ such that $G_1 \cup G_2 = \bigcup_{n=1}^{\infty}(A_n \cup B_n)$ and $G_1 \cap G_2 = \bigcup_{n=1}^{\infty}(A_n \cap B_n)$. Thus, $G_1 \cup G_2, G_1 \cap G_2 \in \mathcal{G}$. By definition of $\overline{\mu}$,

$$\begin{aligned}
\overline{\mu}(G_1 \cup G_2) &= \lim_{n\to\infty} \mu(A_n \cup B_n) \\
&= \lim_{n\to\infty} (\mu(A_n) + \mu(B_n) - \mu(A_n \cap B_n)) \\
&= \overline{\mu}(G_1) + \overline{\mu}(G_2) - \overline{\mu}(G_1 \cap G_2).
\end{aligned}$$